Theory of Risky Choice

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(Lecture 15, Micro Theory I)

Theory of Risky Choice

- We analyzed preferences, utility and choices
- Apply them to study risk and uncertainty
  - Preference for probabilities
  - Expected Utility
- Discuss Experimental Anomalies
  - Allais paradox and Ellsberg paradox
  - Bayes' Rule paradoxes: Soft vs. Hard probability, Game show paradox (Monty Hall problem)
  - Rabin paradox

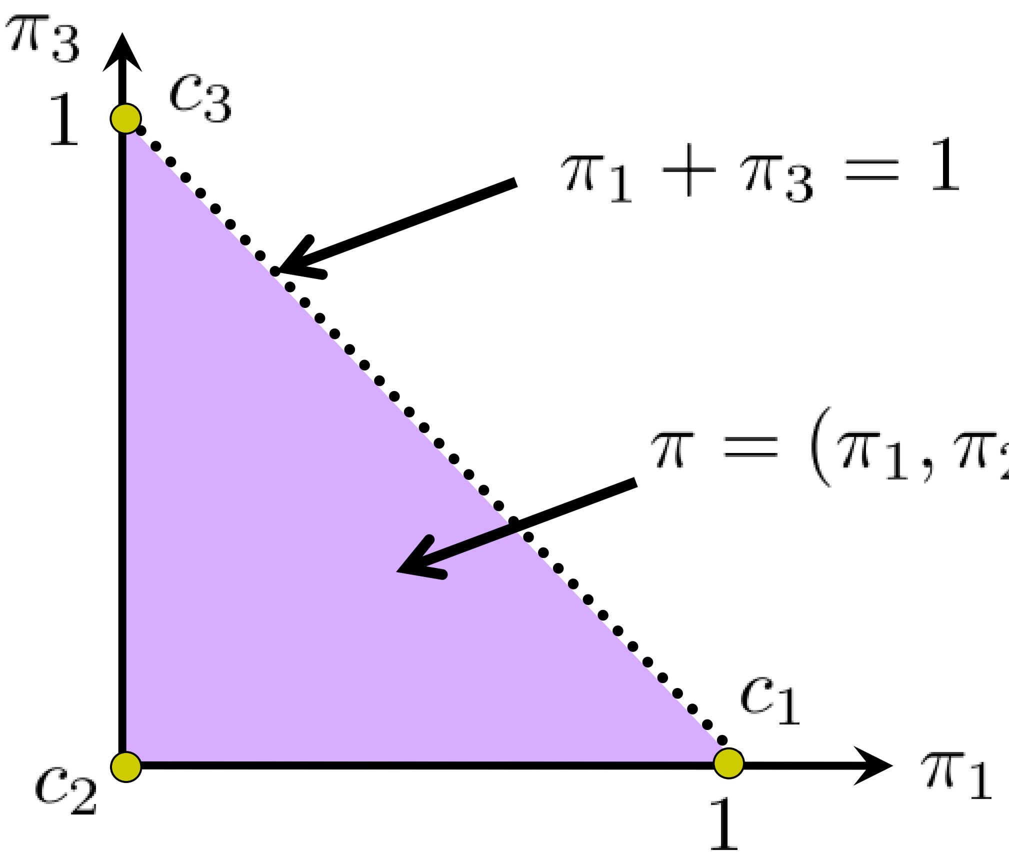
States and Probabilities

- Consequence c_s happens in state $s = 1, \cdots, S$
- Assign (subjective) probability π_s to state s
- A prospect $(\pi; c) = \big((\pi_1, \cdots, \pi_S); (c_1, \cdots, c_S)\big)$
  - People have preferences for these prospects
- Under the Axioms of Consumer Choice, exists continuous $U(\pi; c)$ representing these pref.
- If we fix consequences; focus on probabilities

$$U(\pi; c) = U(\pi) = U(\pi_1, \pi_2, \pi_3)$$

States and Probabilities

- Assume $c_1 \succ c_2 \succ c_3$, show all possible probabilities on 2D: $\pi = (\pi_1, \pi_2, \pi_3)$

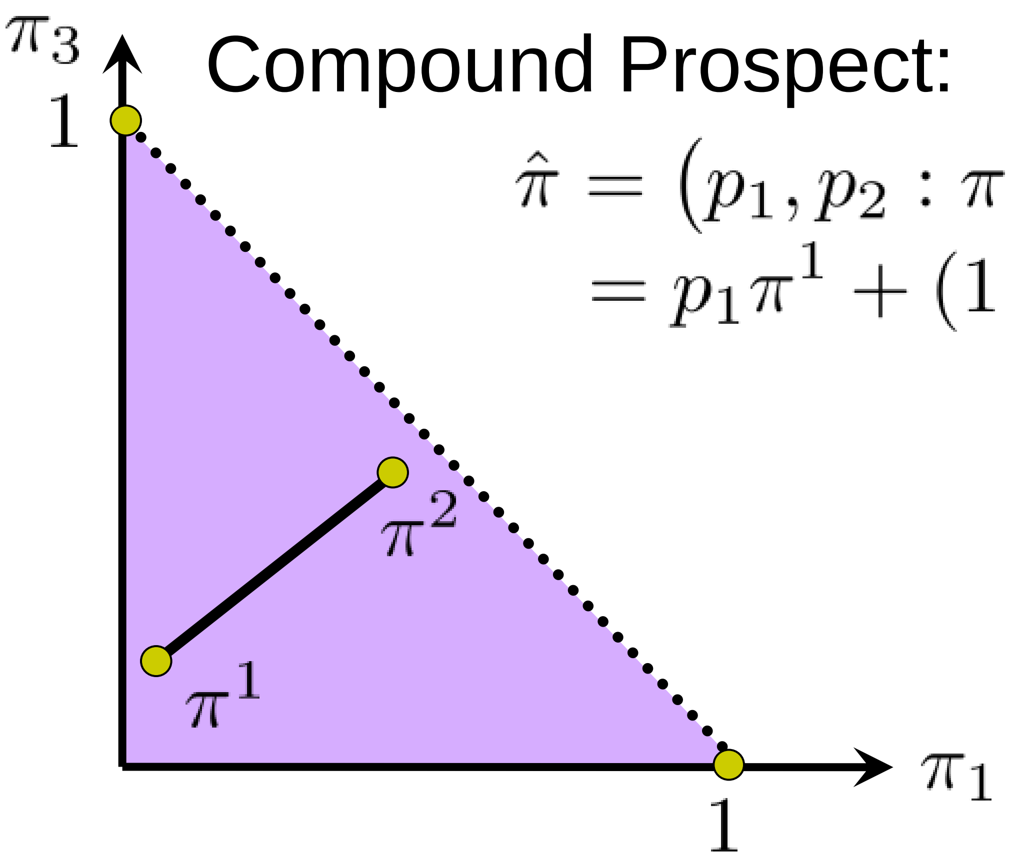


Compound Prospect (Compound Lottery)

- If I offer you $\pi^1 = (\pi_1^1, \pi_2^1, \pi_3^1)$ with prob. p_1, and
- $\pi^2 = (\pi_1^2, \pi_2^2, \pi_3^2)$ with probability $p_2 = 1 - p_1$

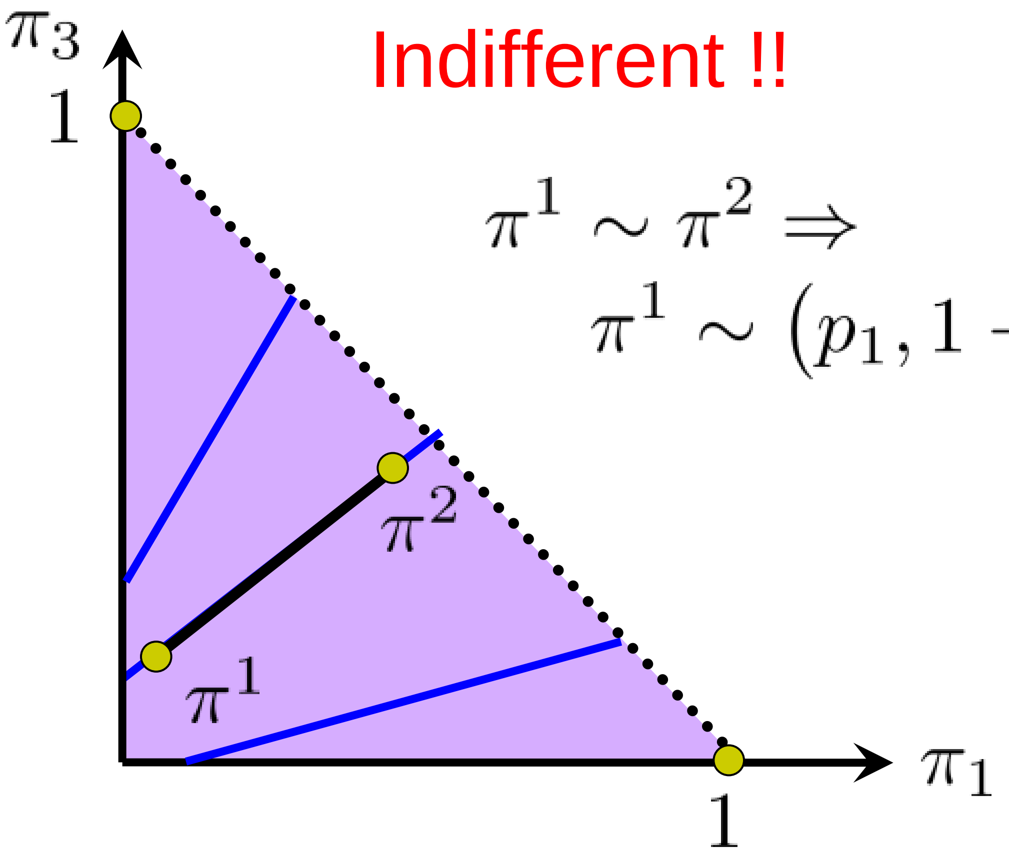
Compound Prospect:

$$\hat{\pi} = (p_1, p_2 : \pi^1, \pi^2) = p_1\pi^1 + (1 - p_1)\pi^2$$



Linear Indifference Curves

- If you are indifferent between π^1 and π^2
- How would you feel about randomizing them?



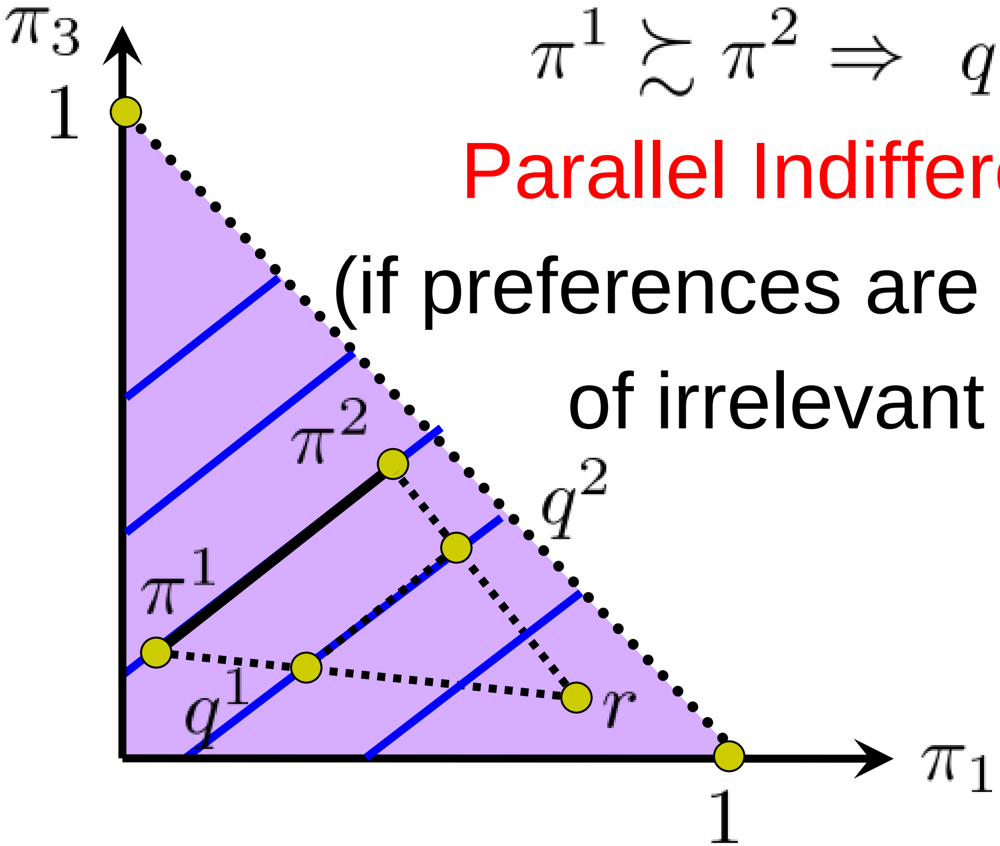
Indifferent !!

$$\pi^1 \sim \pi^2 \Rightarrow$$
$$\pi^1 \sim (p_1, 1 - p_1 : \pi^1, \pi^2)$$

When Would Indifference Curves Become Parallel?

- For $q^1 = (1-\lambda, \lambda : \pi^1, r)$, $q^2 = (1-\lambda, \lambda : \pi^2, r)$
- Then,

$$\pi^1 \sim \pi^2 \Rightarrow \; q^1 \sim q^2$$
$$\pi^1 \succsim \pi^2 \Rightarrow \; q^1 \succsim q^2$$



Parallel Indifference Curves

(if preferences are independent of irrelevant alternatives)

Independence Axiom(s)

- (IA) If $\pi^1 \succsim \pi^2$, then for any prospect r and probabilities $p_1, p_2 > 0, p_1 + p_2 = 1$

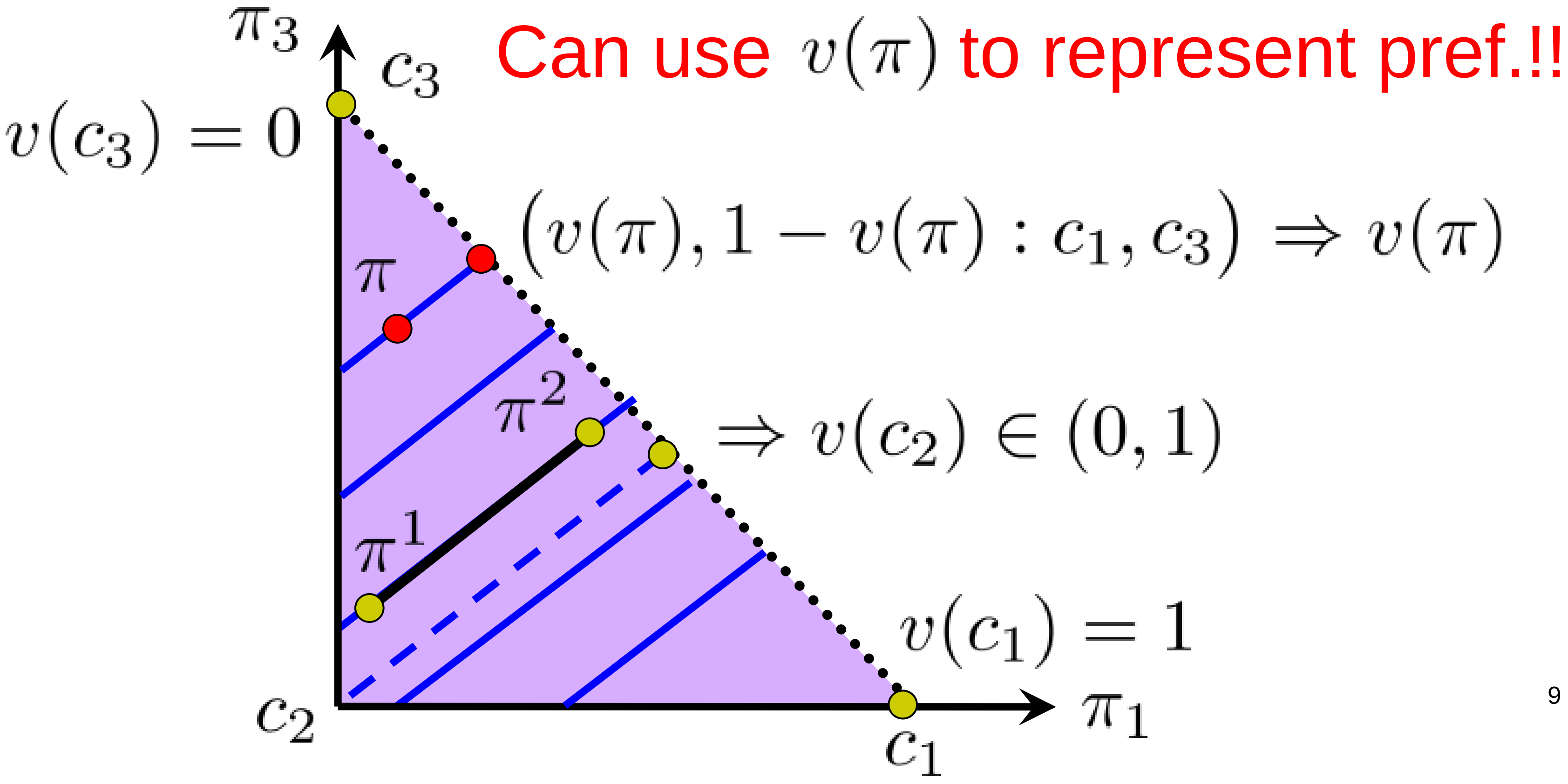
$$q^1 = (p_1, p_2 : \pi^1, r) \succsim (p_1, p_2 : \pi^2, r) = q^2$$

- (IA') If $\pi^m \succsim \hat{\pi}^m, m = 1, \cdots, M$, then for any probability vector $p = (p_1, \cdots, p_M)$

$$(p_1, \cdots, p_M : \pi^1, \cdots, \pi^M) \succsim (p_1, \cdots, p_M : \hat{\pi}^1, \cdots, \hat{\pi}^M)$$

Expected Utility

- For any prospect π, consider (on $\pi_1 + \pi_3 = 1$):
- Extreme lottery $\big(v(\pi), 0, 1 - v(\pi)\big) \sim \pi$



Expected Utility

- In general, for any prospect $p = (p_1, \cdots, p_S)$
- The consumer is indifferent between p and playing the extreme lottery

$$\left(\sum_{s=1}^{S} p_s v(c_s), 0, \cdots, 0, 1 - \sum_{s=1}^{S} p_s v(c_s) \right)$$

- Hence, we can represent her preferences with the above expected win probabilities
  - Expected Utility!!

Expected Utility Rule

- Assume (IA'), then
- Preferences over prospects

$$(p; c) = (p_1, \cdots, p_S; c_1, \cdots, c_S)$$

- Can be represented by the Von Neumann-Morgenstern utility function

$$u(p, c) = \sum_{s=1}^{S} p_s v(c_s)$$

- Proof:

Expected Utility Rule

- Proof: S consequences, best is c^*, worse is c_*
- Can assign probability for extreme lotteries:

$$e^s \equiv \big(v(c_s), 1 - v(c_s) : c^*, c_*\big) \sim c_s$$

- (IA') implies $(p; c) \sim (p_1, \cdots, p_S : e_1, \cdots, e_S)$

$$\sim \big(u(p,c), 1 - u(p,c) : c^*, c_*\big)$$

where $u(p,c) = \sum_{s=1}^{S} p_s v(c_s)$

- (by reducing compound prospects)

Experimental Anomalies

- Allais Paradox
- Ellsberg Paradox
- Bayes' Rule Paradoxes
  - Soft vs. Hard Probabilities
  - Game Show Paradox
- Rabin Paradox

Allais Paradox

- Consider four prospects:

A. $1 million for sure
B. 90% chance $5 million (& 10% chance zero)
C. 10% chance $1 million (& 90% chance zero)
D. 9% chance $5 million (& 91% chance zero)

- Among A and B, you choose…
- Among C and D, you choose…
- Is this consistent with Expected Utility???

Allais Paradox

- state 1: $5 million; state 2: $1 million; state 3: zero

- The four prospects become:

A. $1 million for sure – (0, 1, 0)
B. 90% chance $5 million – (0.90, 0, 0.10)
C. 10% chance $1 million – (0, 0.10, 0.90)
D. 9% chance $5 million – (0.09, 0, 0.91)

- (IA) suggests you should order “A and B” the same as “C and D”. Did you?

Allais Paradox * 1,000

A. $1 billion for sure
B. 90% chance $5 billion (& 10% chance zero)
C. 10% chance $1 billion (& 90% chance zero)
D. 9% chance $5 billion (& 91% chance zero)

- Among A and B, you choose…
- Among C and D, you choose…
- Are your answers (still) consistent with Expected Utility? Why or why not?

Ellsberg Paradox

- One urn: 30 Black balls, and 60 “other balls”
  - Other balls could be either Red or Green

1. One ball is drawn. You win $100 if the ball is (a) Black or (b) Green. You pick…?
2. Now you win $50 if the ball is “either Red or another color you choose.” Would you choose (a) Black or (b) Green?

- What did you choose? Did it violate EU?

Ellsberg Paradox

1. One ball is drawn. You win $100 if the ball is (a) Black or (b) Green.
- Picking Black = Believe <30 Green balls
2. Now you win if "either Red or another color." You choose (a) Black or (b) Green?
- Picking Green = Believe >30 Green balls
- Since it is the same urn, this is inconsistent!
  - Can this be due to hedging (risk aversion)?
  - Maybe, but can fix this by paying only 1 round…

Bayes' Rule Paradoxes: Soft vs. Hard Probabilities

- Two urns, each contain 100 balls.

1. Urn 1 has 60 Yellow balls.
2. Urn 2 has 75 or 25 Yellow balls with equal chance.

- You win a prize if you draw a Yellow ball.
- A ball is drawn from Urn 2 and it is Yellow.
- Which Urn should you choose?
- Did you do Bayesian updating correctly?

Bayes' Rule Paradoxes: Soft vs. Hard Probabilities

- Prior to draw, Pr (draw a Y) = 0.5. After:
- Pr(Y | 75-Y) = 0.75, Pr(Y | 25-Y) = 0.25.
- Pr(75-Y | Y) = 0.5 * 0.75 / 0.5 = 0.75
- Pr (25-Y | Y) = 1- 0.75 = 0.25
- Pr (draw another Y | Y) =

  Pr(75-Y | Y) * Pr(Y | 75-Y) +

  Pr(25-Y | Y) * Pr(Y | 75-Y)

  = 0.75 * 0.75 + 0.25 * 0.25 = 0.625 > 0.6
- So you should pick Urn 2!! (Did you do that?)

Bayes' Rule Paradoxes: Game Show Paradox

One door hides the prize (a car).
Remaining two doors hides a goat (non-prize).

Suppose you choose door number 1…

Game Show Paradox (Monty Hall Problem)

Door 3 is opened for you…

Obviously the car is not behind door 3…

Would you want to switch to door 2?

Depends on how door is opened…

- **Rule to open one door:**

  The Host must open one "other" door without the prize. If he has a choice between more than one door, he will randomly open one of the possible (goat) doors.

- The Game Show Paradox is also known as the Monty Hall Problem, named after the name of the TV show host "Monty Hall"

Game Show Paradox Plus: The (Generalized) Monty Hall Problem

One door hides the prize (a car).

Remaining two doors hides a goat (non-prize).

Door 3 is transparent (and you see the goat)

Suppose you choose door number 1…

Game Show Paradox Plus: The (Generalized) Monty Hall Problem

Door 3 is opened for you…

Obviously the car is not behind door 3 (and you knew that already)…

Would you want to switch to door 2?

If You Picked the Right Door (50%)

Host randomizes between door 2 and 3 (50-50)

If host opens door 2… (Prob.=50%*50%)

You should definitely not switch!

If You Picked the Right Door (50%)

Host randomizes between door 2 and 3

If host opens door 3… (Prob=50%*50%)

You should still not switch (but you don't know)

If You Picked the Wrong Door (50%)

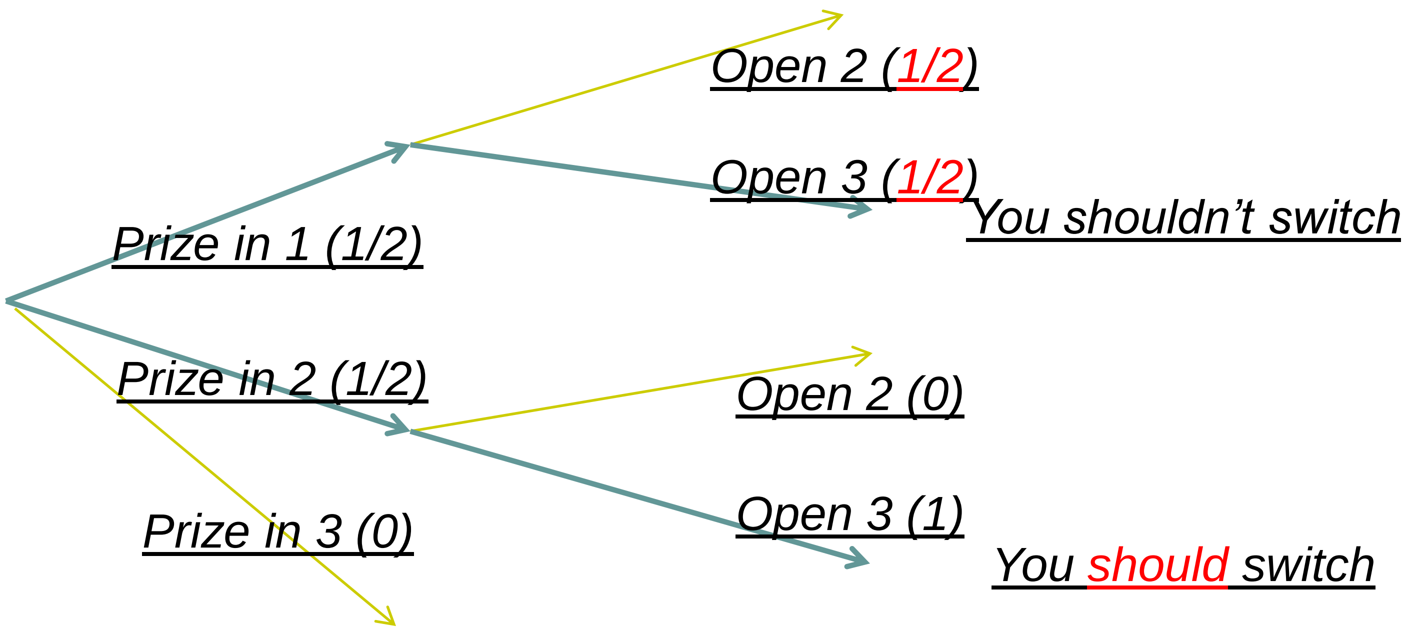
Host cannot open door 2 (contains car)

See host opening door 3… (Prob.=50%*100%)

You should switch (but you don't know)

Bayesian Solution (GMHP)

Door #3 is transparent…



$$\mathrm{P(Winning\ if\ you\ choose\ to\ switch)} = \frac{\frac{1}{2}\times 1}{\frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{2}}$$

Rabin Paradox: Which Cells Will You Accept?

Payoff if Green Ball	Number of Green Balls (out of 100)					Payoff if Red Ball
100	52	55	60	66	70	-100
1000	13	20	33	46	57	-100
5000	7	18	33	46	57	-100
25000	7	18	33	46	57	-100

Summary of 7.1

- Preferences over prospects
- Indifference Curves
  - Linear: “Reduction of Compound Lotteries”
  - Parallel: “Independent of Irrelevant Alternatives”
- Expected Utility
- Anomalies: Allais paradox, Ellsberg paradox, Bayes’ Rule paradoxes (Soft vs. Hard prob. and Monty Hall Problem) and Rabin paradox
- Homework: Riley - Exercise 7.1-2~4