Theory of the Firm: Return to Scale and IO

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(Lecture 10, Micro Theory I)

Producers vs. Consumers

- Chapter 2-3 focus on Consumers (and exchange between consumers)
- Now focus on transformation of commodities
  - Raw material, inputs → final (intermediate) product
  - Depending on technology
- Example: “Fair Trade” coffee shop on campus
  - Inputs: Coffee beans, labor, cups, fair trade brand
  - Output: Fair trade coffee
  - Technology: Coffee machine (+ FT workshops?)

Why do we care about this?

- Besides exchanging endowments, economics is also about producing goods and services
- Efficiency: Produce at the lowest possible cost
- Consider yourself as a study machine, producing good grades (in micro theory!)
- What are your inputs? What are the outputs?
- How do you determine the amount of study hours used to study micro theory?
- Are you maximizing your happiness?

Things We Don't Discuss: Scope of the Firm

- Example: Fair Trade coffee shop on campus
- Could the coffee shop buy a new coffee machine?
  - Can choose technology in the LR
- Can the coffee shop buy other shops to form a chain (like Starbucks?)
  - Choose scale economy in the VLR?
- Why can't the firm buy up all other firms in the economy?
  - "Theory of the Firm" in Modern IO

Things We Don't Discuss: Internal Structure of the Firm

- Example: Fair Trade coffee shop on campus
- How does the owner monitor employees?
  - Check if workers are handing out coffee for free?
- Does the owner hire managers to do this?
  - Workers → Managers → Owner (board of directors)
- How does internal structure affect the productivity of the firm?
  - "Team Production" or "Principal-Agent" in Modern IO
- Here we simply assume firms maximize profit

Production Set

- Output: $q = (q_1, \cdots, q_m)$
- Input: $z = (z_1, \cdots, z_m)$
- Production Plan in Production Set: $(z, q) \in \gamma^f$

  $(z, q) \geq 0$ Feasible if output q is feasible given input z
- Set of Feasible Output: $Q(z)$
- Output-efficient: Being on the boundary of $Q(z)$
- Single output Example: $q = F(z)$
  - Production Function: $F(.)$

Production Set

- Example 1: Cobb-Douglas Production Function

$$q = A z_1^{\alpha_1} \cdot \cdots \cdot z_n^{\alpha_n}$$

- Example 2: CES Production Function

$$q = \left(\sum_{j=1}^{n} a_j z_j^{\frac{\sigma - 1}{\sigma}} \right)^{\frac{\sigma}{\sigma - 1}} , a, \sigma > 0, \sigma \neq 1$$

Production Set

- Production Set: Multiple Output
  - Set of input-outputs satisfying certain constraints

$$\gamma^f = \{(z, q) | h_i(z, q) \geq 0, i = 1, \cdots, m\}$$

- Convex if each constraint is quasi-concave (having convex upper-contour sets)

- Example 3: Multi-Product Production Set

$$\gamma^f = \left\{(z, q_1, q_2) | z_1 - q_1^2 - q_2^2 \geq 0\right\}$$

Production Set for Studying

- Output 1: Micro score, Output 2: Macro score
- Input 1: Hour of Self-Study
- Input 2: Hour of Group Discussion
- Input 3: Brain Power (Cognitive Load)
- Production Set for Studying:

$$\gamma^f = \{(z_1, z_2, z_3, q_1, q_2) | \\ z_1 + z_2 + z_3 \leq 24 - 8, \\ q_1 + 10q_2 - z_3 * z_1 * z_2 \leq 0\}$$

Net Output Reformulation

- Production Plan: $y^f = (y_1^f, \cdots, y_n^f)$
- Net output: $y_i^f > 0$ Net input: $y_j^f < 0$
- Profit: $p \cdot y = \underbrace{\sum_{i, y_i > 0} p_i \cdot y_i}_{revenue} - \underbrace{\sum_{j, y_j < 0} p_j \cdot (-y_j)}_{cost}$
- Why is this a better approach?
  - Account for intermediate goods
  - Allow firms to switch to consumers
  - Also convenient in math…

(Classical) Theory of the Firm

- Port consumer theory if firms are price-taking
  - Seen this in 4.2
- Other cases:
  - Monopoly (4.5)
  - Oligopoly (IO or next semester micro)
- What determines the scope of the firm?
  - Scale Economy!

Definition: Returns to Scale

- Constant Returns to Scale

γ is **CRS** if for all $y \in \gamma$, and any $\lambda > 0$, $\lambda y \in \gamma$.

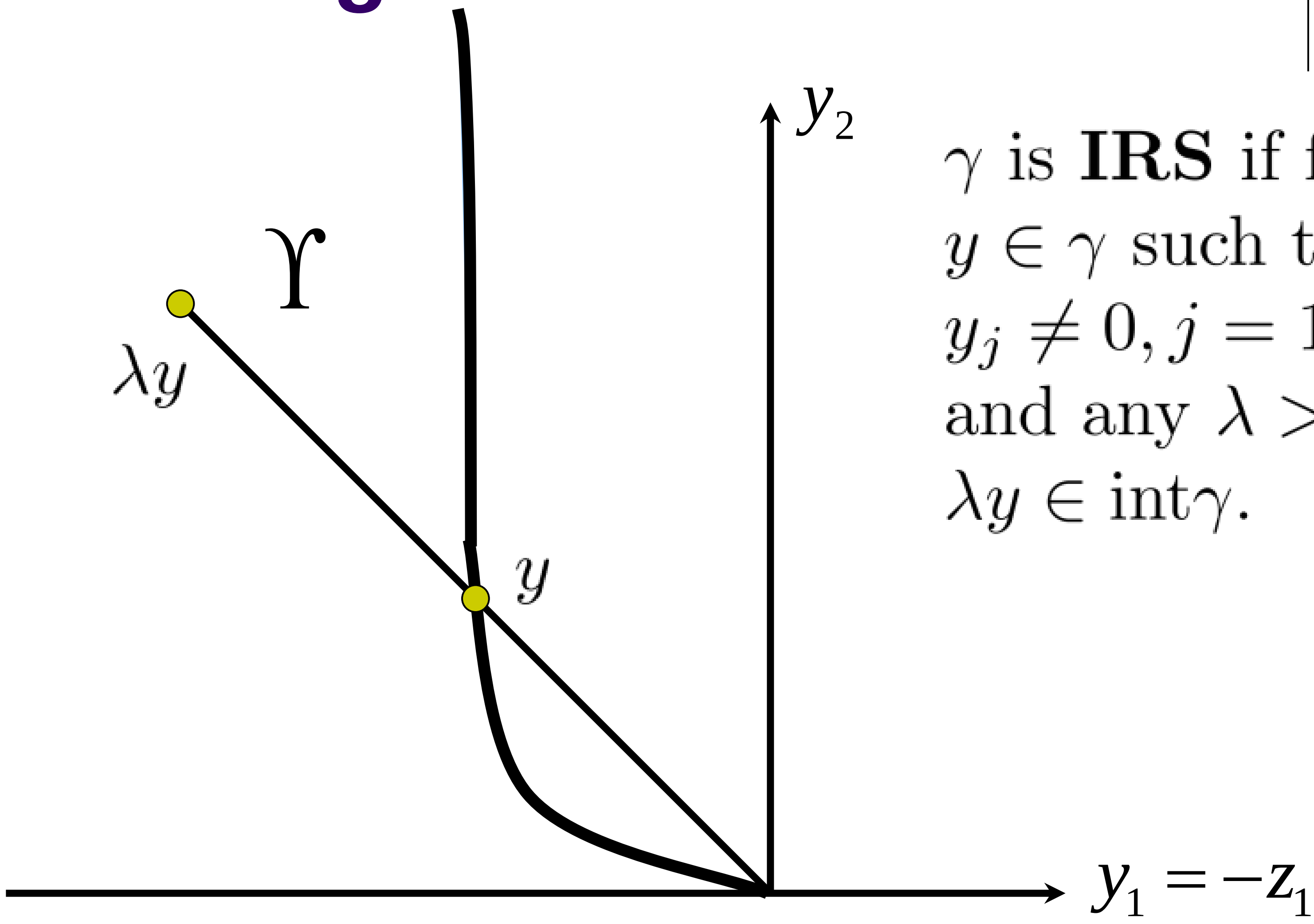
- Increasing Returns to Scale

γ is **IRS** if for $y \in \gamma$ such that $y_j \neq 0, j = 1 \sim n$, and any $\lambda > 1$, $\lambda y \in \text{int}\gamma$.

- Decreasing Returns to Scale

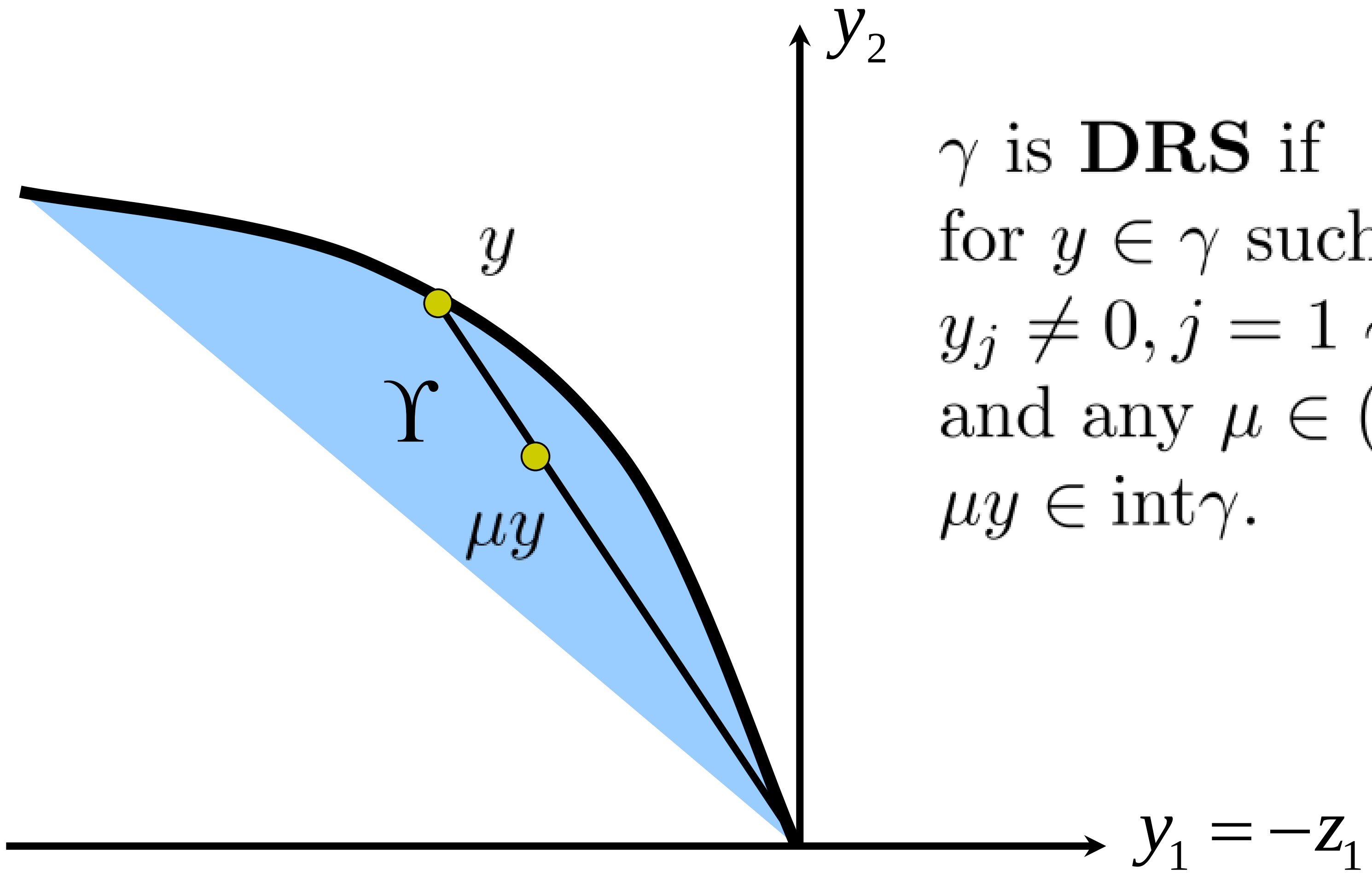
γ is **DRS** if for $y \in \gamma$ such that $y_j \neq 0, j = 1 \sim n$, and any $\mu \in (0, 1)$, $\mu y \in \text{int}\gamma$.

Increasing Returns to Scale



γ is **IRS** if for $y \in \gamma$ such that $y_j \neq 0, j = 1 \sim n$, and any $\lambda > 1$, $\lambda y \in \text{int}\gamma$.

Decreasing Returns to Scale



γ is **DRS** if
for $y \in \gamma$ such that
$y_j \neq 0, j = 1 \sim n$,
and any $\mu \in (0, 1)$,
$\mu y \in \text{int}\gamma$.

Why do we care about this?

- Link to single output CRS, IRS, DRS
- IRS: $\lambda > 1 \Rightarrow F(\lambda z) > \lambda F(z)$
- DRS: $\lambda > 1 \Rightarrow F(\lambda z) < \lambda F(z)$
- CRS: $F(\lambda z) = \lambda F(z)$
  - Recall: Homothetic Preferences…
- Can you double your study hours, group discussion and brain power to double your score?

Lemma 4.3-1: Constant Gradient Along a Ray

- Suppose F exhibits CRS
- Differentiable for all $z >> 0$
- Then, for all $z >> 0$, $\frac{\partial F}{\partial z}(\lambda z) = \frac{\partial F}{\partial z}(z)$

- Proof:
- CRS implies $F(\lambda z) = \lambda F(z)$
- Differentiating by $\hat{z}_j$: $\frac{\partial F}{\partial z_j}(\lambda \hat{z}) \cdot \lambda = \lambda \frac{\partial F}{\partial z_j}(\hat{z})$

Indeterminacy Property of Identical CRS Firm Industry

$$F(z^1 + z^2) = F(z^1) + F(z^2) \text{ if } z^1 = kz^2$$

- Proof: z^1 and z^2 are proportional,
- Then they are both proportional to their sum
- I.e. $z^1 = \theta(z^1 + z^2), z^2 = (1-\theta)(z^1 + z^2)$
- Then, CRS implies

$$\begin{aligned} F(z^1) + F(z^2) &= F(\theta(z^1 + z^2)) + F((1-\theta)(z^1 + z^2)) \\ &= \theta F(z^1 + z^2) + (1-\theta)F(z^1 + z^2) \\ &= F(z^1 + z^2) \end{aligned}$$

Proposition 4.3-2: Super-additivity
Proposition 4.3-3: Concavity

- If F is strictly quasi-concave and exhibits CRS,
- Then F is super-additive. I.e.

$$F(x+y) \geq F(x) + F(y) \text{ for all } x+y >> 0$$

- Moreover, inequality is strict unless $x = \theta y$
  - Always strictly better off to combine inputs
- Proposition 4.3-3: Concavity

$$F((1-\lambda)z^0 + \lambda z^1) \geq (1-\lambda)F(z^0) + \lambda F(z^1)$$

  - (Inequality is strict unless $x = \theta y$)
- Proof: Apply Proposition 4.3-2 and done.

Proof of Proposition 4.3-2: Super-additivity

- Consider (x^0, y^0) (not proportional)
- Consider the firm problem with 2 plants:

$$\max_{x,y} \left\{ F(x) + F(y) | x + y \leq x^0 + y^0 \right\}$$

- Unique solution $(\hat{x}, \hat{y})$ (by strict quasi-concavity)

$$\mathcal{L} = F(x) + F(y) - \lambda \cdot [x + y - x^0 - y^0]$$

- FOC requires $\dfrac{\partial F(\hat{x})}{\partial x_i} = \lambda_i, \quad \dfrac{\partial F(\hat{y})}{\partial y_i} = \lambda_i$
  - $\hat{x} = \theta \hat{y}$ since F is CRS (homothetic/redial parallel)

Proof of Proposition 4.3-2: Super-additivity

- Knowing $\hat{x} = \theta \hat{y}$, and $x + y = x^0 + y^0$

$$(\hat{x}, \hat{y}) = \left(\frac{1}{1+\theta}(x^0 + y^0), \frac{\theta}{1+\theta}(x^0 + y^0) \right)$$

- Uniquely solves (by strict quasi-concavity)

$$\max_{x,y} \left\{ F(x) + F(y) | x + y \leq x^0 + y^0 \right\}$$

- Hence, (by uniqueness and CRS)

$$F(x^0) + F(y^0) < F(\hat{x}) + F(\hat{y})$$
$$= F\left(\frac{1}{1+\theta}(x^0 + y^0) \right) + F\left(\frac{\theta}{1+\theta}(x^0 + y^0) \right)$$
$$= F(x^0 + y^0)$$

Scale Elasticity of Output

- Scale parameter rises from *1*→*λ*
- Proportional increase in output increases by:

$$\frac{q(\lambda)-q(1)}{q(1)}\cdot\frac{1}{\lambda-1}=\frac{F(\lambda z)-F(z)}{F(z)}\cdot\frac{1}{\lambda-1}$$

$$(\Delta\lambda=\lambda-1) \qquad =\frac{F(z+\Delta\lambda z)-F(z)}{\Delta\lambda z}\cdot\frac{z}{F(z)}$$

- Take limit *λ*→*1*:

$$\mathcal{E}\Big(F(\lambda z),\lambda\Big)\Big|_{\lambda=1}=\frac{\lambda}{F(z)}\cdot\frac{\partial}{\partial\lambda}F(\lambda z)\Big|_{\lambda=1}$$

Scale Elasticity of Output

- DRS:

$$\mathcal{E}\Big(F(\lambda z),\lambda\Big)\Big|_{\lambda=1} \leq \lim_{\lambda\to 1}\frac{\lambda F(z)-F(z)}{F(z)}\cdot\frac{1}{\lambda-1}=1$$

- IRS:

$$\mathcal{E}\Big(F(\lambda z),\lambda\Big)\Big|_{\lambda=1} \geq \lim_{\lambda\to 1}\frac{\lambda F(z)-F(z)}{F(z)}\cdot\frac{1}{\lambda-1}=1$$

- CRS: (You know…)

Local Returns to Scale

- Firms typically exhibit IRS at low output levels
  - Indivisibility in entrepreneurial setup/monitoring
- But DRS at high output levels
  - Large managerial burden for conglomerates
- Local Returns to Scale

$$\mathcal{E}\Big(F(\lambda z), \lambda\Big) = \frac{\lambda}{F(\lambda z)} \cdot \frac{\partial}{\partial \lambda} F(\lambda z) = \frac{\lambda z \cdot \frac{\partial F}{\partial z}(\lambda z)}{F(\lambda z)}$$

- since

$$\frac{\partial}{\partial \lambda} F(\lambda z) = \sum_{i=1}^{n} z_i \cdot \frac{\partial F}{\partial z_i}(\lambda z) = z \cdot \frac{\partial F}{\partial z}(\lambda z)$$

Local Returns to Scale

- Local Returns to Scale

$$\mathcal{E}\Big(F(\lambda z), \lambda\Big)\bigg|_{\lambda=1} = \frac{z \cdot \dfrac{\partial F}{\partial z}(\lambda z)}{F(\lambda z)}\Bigg|_{\lambda=1}$$

- IRS:

$$\mathcal{E}\Big(F(\lambda z), \lambda\Big)\bigg|_{\lambda=1} = \frac{z \cdot \dfrac{\partial F}{\partial z}(z)}{F(z)} > 1$$

- DRS

$$\mathcal{E}\Big(F(\lambda z), \lambda\Big)\bigg|_{\lambda=1} = \frac{z \cdot \dfrac{\partial F}{\partial z}(z)}{F(z)} < 1$$

Proposition 4.3-4: AC vs. MC

- If z minimizes cost for output q,
- Then,
- $AC(q)/MC(q) = \mathcal{E}\Big(F(\lambda z), \lambda\Big)\Big|_{\lambda=1}$
- In other words,
- IRS: $AC(q) > MC(q)$
- DRS: $AC(q) < MC(q)$
  - (You should have noticed this from Principles)

Proposition 4.3-4: AC vs. MC

- Proof: $C(q,r) = \min_z \{r \cdot z | q \leq F(z)\}$
$$\mathcal{L} = -r \cdot z + \lambda(F(z) - q)$$
- FOC requires
$$\frac{\partial \mathcal{L}}{\partial z_i} = -r_i + \lambda \frac{\partial F}{\partial z_i} \leq 0 \text{ with equality if } z_i > 0$$
- Or,
$$-r_i z_i + \lambda z_i \frac{\partial F}{\partial z_i} = 0$$
- Hence,
$$C(q,r) = r \cdot z = \lambda z \cdot \frac{\partial F}{\partial z}$$

Proposition 4.3-4: AC vs. MC

- Proof (continued):

$$C(q,r) = r \cdot z = \lambda z \cdot \frac{\partial F}{\partial z}$$

$$\mathcal{L} = -r \cdot z + \lambda(F(z) - q)$$

- By Envelope Theorem, $MC(q) = \frac{\partial C}{\partial q} = \lambda$
- Thus,

$$AC(q) = C(q,r)/q = \lambda \cdot \frac{z \cdot \frac{\partial F}{\partial z}}{F(z)}$$

$$= MC(q) \cdot \frac{z \cdot \frac{\partial F}{\partial z}}{F(z)}$$

Summary of 4.1, 4.3

- The Neoclassical Firm: Maximizes Profit
  - Scope of a Firm? (Theory of the Firm)
  - Internal Structure of a Firm? (modern IO)
- Global Returns to Scale: CRS, IRS, DRS
  - Super-additive, concavity
  - Scale Elasticity of Output
- Local Returns to Scale
  - AC vs. MC
- Homework: J/R – 3.4, 3.6, 3.11, Riley - 4.3-3, 4