112-2 Introduction to Quantitative Methods: Final Exam

June 11, 2024 (9:10–12:10) **TA: Po-Hsien Kang**

Please answer each of the questions and clearly explain your answers. The total score is 100.

1. (15pts) Let $a > 1$. We assume that $a^{1/n}$ is already a well-defined notion in the following context for $n \in \mathbb{N}$, which denotes the unique positive solution of $x^n = a$.

   (i) (3pts) If m, n, p, q are integers, $n > 0,\ q > 0$, and $r = m/n = p/q$, prove that
   $$(a^m)^{\frac{1}{n}} = (a^p)^{\frac{1}{q}}.$$

   (ii) (3pts) Prove that $a^{r+s} = a^r a^s$ if r and s are rational.

   (iii) (4pts) If x is real, define $A(x)$ to be the set of all numbers a^t, where t is rational and $t \le x$. Prove that
   $$a^r = \sup A(r)$$
   when r is rational. Hence it makes sense to define
   $$a^x = \sup A(x)$$
   for every real x.

   (iv) (5pts) Prove that $a^x a^y = a^{x+y}$ for all $x, y \in \mathbb{R}$.

2. (10pts) Let $\{E_n\}, n = 1, 2, 3, \cdots$, be a sequence of countable sets, and put $S = \bigcup_{n=1}^{\infty} E_n$. Prove that S is countable.

3. (15pts) State and prove the Heine-Borel Theorem.

4. (15pts) Suppose $a_n > 0$, $s_n = a_1 + a_2 + \cdots + a_n$ and $\sum_{n=1}^{\infty} a_n$ diverges.

   (i) (7pts) Prove that
   $$\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n} \text{ diverges.}$$

   (ii) (8pts) Prove that
   $$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \ge 1 - \frac{s_N}{s_{N+k}}$$
   and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges.

5. (15pts) Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}^1$.

   (i) (10pts) Prove that f is bounded on E.

   (ii) (5pts) Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

6. (15pts) Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that
   $$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$
   whenever $0 < |t - x| < \delta,\ a \le t \le b,\ a \le x \le b$.

7. (15pts) Suppose $a \in \mathbb{R}$, and f is twice differentiable on (a, ∞). Let $|f(x)|, |f'(x)|, |f''(x)|$ be bounded and M_0, M_1, M_2 are their least upper bounds respectively. Prove that

$$M_1^2 \leq 4M_0M_2.$$