Introduction to Real Analysis, Final Exam

Total Score is 138. You can lose 2 points for free.

Problems (10 pts each)

1. Give formal definitions to the following statements.

   (i) (X, d) is a metric space.

   (ii) $f : X \to Y$ is uniformly continuous.

2. Countable or Uncountable? (Rigorous proofs are needed)

   (i) The set of complex numbers $\mathbb{C}$.

   (ii) The Cantor set.

   (iii) The set of roots of integer-coefficient polynomials.

3. Calculate the following values.

   (i) $\lim_{x\to 0} f(x, kx^2)$, where $f(x, y) = \dfrac{x^2y}{x^4 + y^2}$. ($\epsilon$-$\delta$ argument is needed.)

   (ii) $\lim_{(x,y)\to(0,0)} f(x, y)$, where $f(x, y) = \dfrac{x^2y}{x^4 + y^2}$.

   (iii) $\dfrac{d}{dx}\dfrac{x\sin(x)}{x^2 + 3}$ at $x = 2$.

4. Prove that a Cauchy sequence that has a convergent subsequence is convergent.

5. Sketch open balls with radius 1 centered in the origin point in

   (i) Euclidean metric.

   (ii) Discrete metric, where $d(x, y) = 1$, $\forall x, y \in E$ if $x \neq y$.

   (iii) d_1 metric, where $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $\forall (x_1, y_1),\ (x_2, y_2) \in \mathbb{R}^2$.

6. Discuss the convergence of the following series.

(i) $\sum_n \frac{\sin n}{n^2}$.

(ii) $\sum_n \frac{n^2 z^n}{5^n}$ with different $z > 0$.

7. Prove that $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open for every open set $V \subseteq Y$.

8. Prove that continuous function preserves compactness.

9. State and prove *Intermediate Value Theorem*.

10. Discuss which function sequence uniformly converges.

    (i) $f_n(x) = x^n$ on $(0, 1)$.

    (ii) $f_n(x) = x^n(1 - x)$ on $(0, 1)$.

11. Consider a person who faces a two-commodity market, say Commodity x and Commodity y. Suppose the utility function is not bounded above. Now, he wants to make a purchasing decision that makes his utility $\geq U$, where U is a positive constant. Show that there is a bundle that minimizes his cost.

Challenging Problems (Try to solve them to get A^+, 10 pts each)

1. Suppose $\{a_n\} \in \mathbb{R}$ is bounded but not converges. Prove that there exists two subsequences of $\{a_n\}$ that converge to different limits.

2. We say that (M, d) is an ultrametric space if (M, d) is a metric space and

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \forall x, y, z \in M.$$

   Prove that every triangle in an ultrametric space is isosceles and every open ball is closed. (Hint: An isosceles is a triangle with two sides of equal length.)

3. Let I be an interval. f is a continuous injection from I to $\mathbb{R}$. Prove that $f^{-1} : f(I) \to I$ is continous.

Extremely Hard Problem (0pt, just in case you get bored.)

Suppose $f : M \to M$, where M is compact. Given that for each x, y, $d(f(x), f(y)) \geq d(x, y)$, prove that f is a surjection.