1. (32 pts) State and prove *Mean Value Theorem*.

Mean Value Theorem: If $f:[a,b]\to\mathbb{R}$ is continuous, and is differentiable on (a,b), then $\exists c\in(a,b)$ s.t. $f(b)-f(a)=f'(c)(b-a)$.

pf. Let $g(x)=f(x)-(x-a)\dfrac{f(b)-f(a)}{b-a}$, then $g(a)=g(b)=f(a)$.

By Rolle's thm, $\exists c\in(a,b)$ s.t. $g'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}=0$

$\Rightarrow f'(c)=\dfrac{f(b)-f(a)}{b-a}$, $f'(c)(b-a)=f(b)-f(a)$.

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2. (28 pts) Differentiate $f(x) = x^n$ with careful $\epsilon - \delta$ argument.

$$\frac{d}{dx} x^n = \lim_{\delta \to 0} \frac{(x+\delta)^n - x^n}{\delta} = \lim_{\delta \to 0} nx^{n-1} + \delta\left(\binom{n}{2}x^{n-2} + \dots + \delta^{n-1}\right) \overset{①}{=} nx^{n-1}$$

Since $\delta\left(\binom{n}{2}x^{n-2} + \dots + \delta^{n-1}\right) =: f(\delta)$ is continuous and $\to \infty$ as $n \to \infty$

$\forall \varepsilon > 0$, by IVT, $\exists \delta' > 0$ s.t. $f(\delta') = \frac{\varepsilon}{2} < \varepsilon$. Hence ① holds. $\boxtimes$

3. (28 pts) Discuss the continuity for the Dirichlet function.

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \text{ and } (p,q)=1 \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

- f is discontinuous at all $x \overset{\frac{p}{q}}{\in} \mathbb{Q}$: $\forall 0<\varepsilon<\frac{1}{q}$, $\forall \delta>0$, $\exists y \in \mathbb{R}/\mathbb{Q}$ s.t. $y \in B_\delta(x)$, but $|f(x)-f(y)| = |\frac{1}{q}-0| = \frac{1}{q} > 0$.

- f is continuous at all $x \in \mathbb{R}/\mathbb{Q}$: $\forall \varepsilon>0$, $\exists N>0$ s.t. $\varepsilon < \frac{1}{N}$.

$\Rightarrow \exists \delta>0$ s.t. $\underbrace{\{\frac{x}{1}, \frac{x}{2}, \frac{x}{3}, \dots \frac{x}{N-1}, x \in \mathbb{Z}\}}_{S} \cap B_\delta(x) = \emptyset$.

(Since for fixed $a>0$, finite elements of S are in $B_a(x)$).

Hence $|f(x)-f(y)| < \frac{1}{N} < \varepsilon$, $\forall y$ s.t. $|x-y|<\delta$. #

4. (26 pts) Describe and prove *the quotient rule* for differentiation.

$\forall$ function g s.t. $g(x) \neq 0$ for all x, $\left(g \cdot \frac{1}{g}\right)' = 0$

$\Rightarrow g' \cdot \frac{1}{g} + g \cdot \left(\frac{1}{g}\right)' = 0 \Rightarrow \left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$.

$\Rightarrow \left(\frac{f}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' = \frac{f'g - fg'}{g^2}$.

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5. (24 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function, prove that f has at most countably infinite discontinuities.

Let $S = \{ x \mid f \text{ is discontinuous at } x \}$

$S_d = \{ D_x \mid D_x = (f(x^-), f(x^+)),\ f \text{ is discontinuous at } x \}$.

$S_r = \{ r_x \mid \text{for each } D_x \in S_d, \text{ choose unique representative } r_x \in D_x \cap \mathbb{Q} \}$

Then $S \hookrightarrow\!\!\!\!\rightarrow S_d \hookrightarrow\!\!\!\!\rightarrow S_r \hookrightarrow \mathbb{Q}$, so S is countable. #