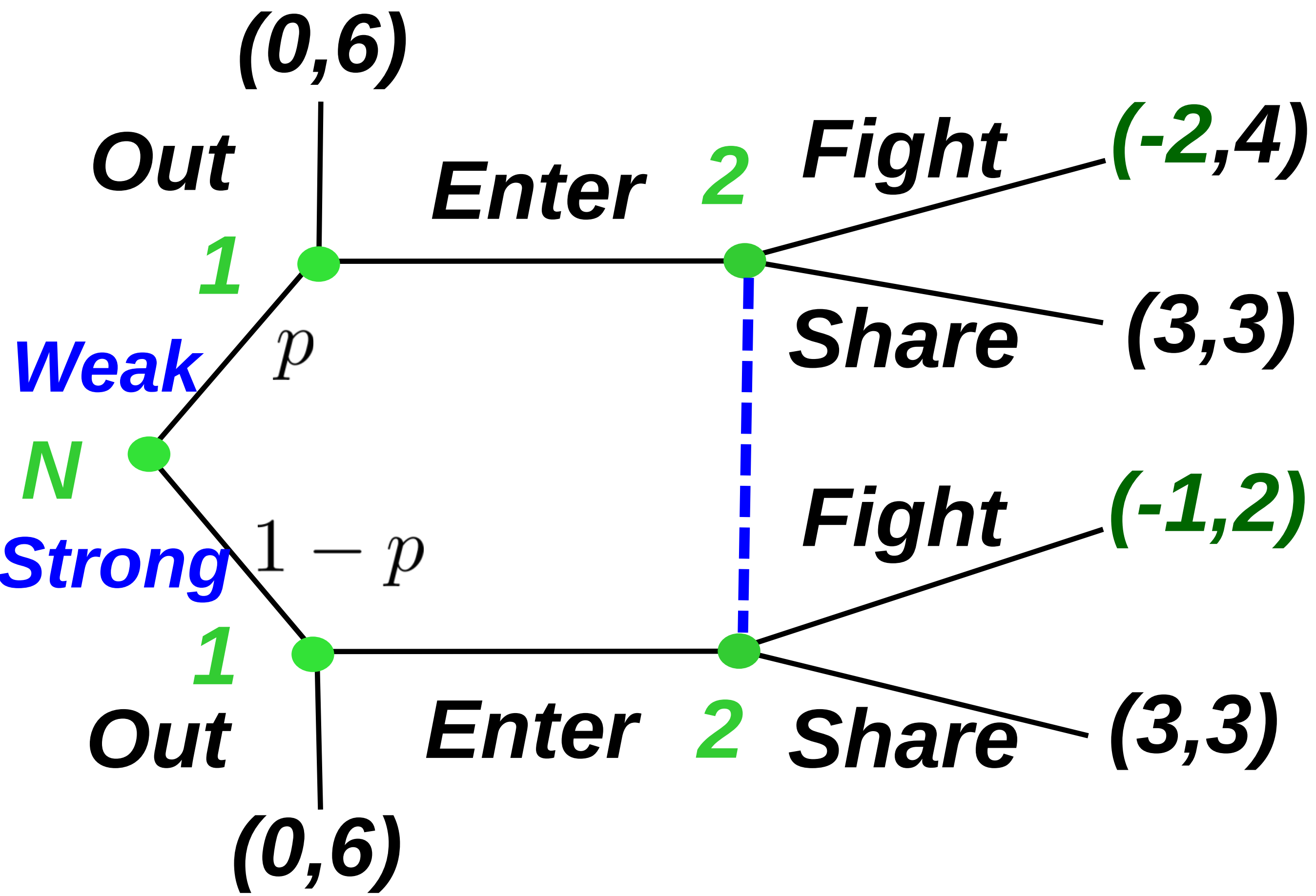
Games with Private Information
資訊不透明賽局

Joseph Tao-yi Wang

2010/10/15

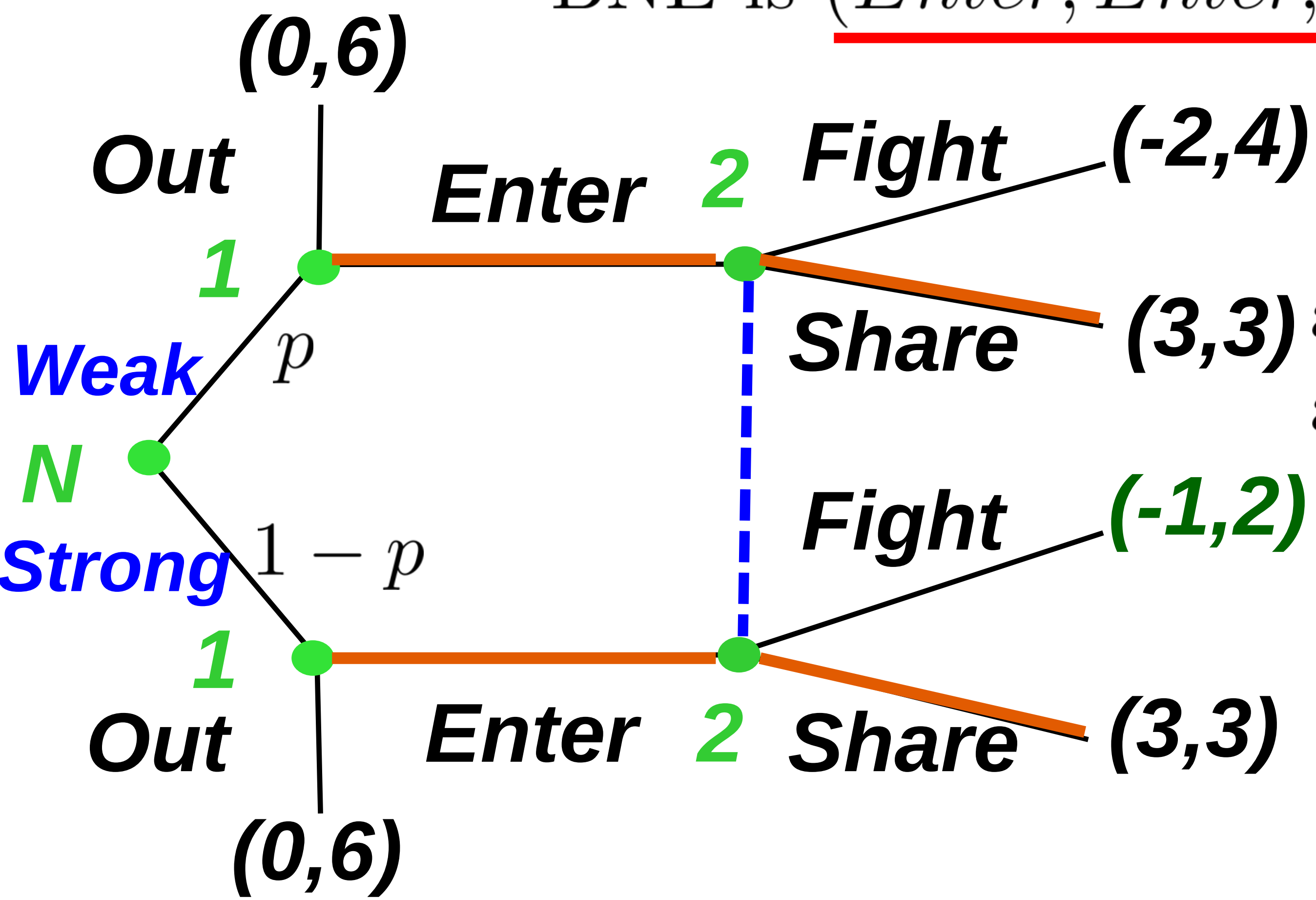
(Lecture 9, Micro Theory I-2)

Market Entry Game with Private Information



BNE when $p < 1/2$:
(*Enter*, *Enter*, *Share*)

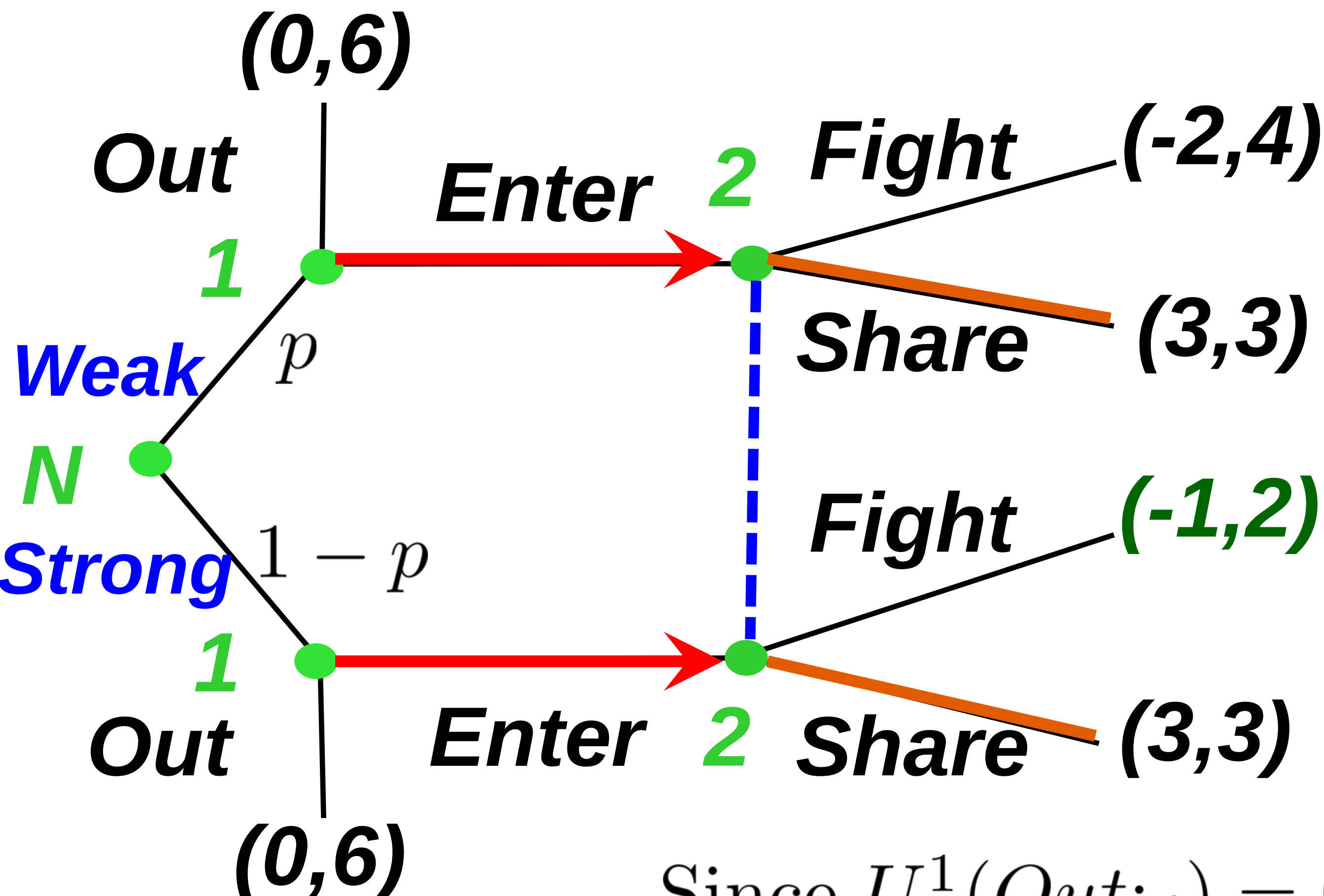
BNE is $(Enter, Enter, Share)$



(action of 1 if W
action of 1 if S,
action if $Enter$)

BNE when $p < 1/2$: (*Enter*, *Enter*, *Share*)

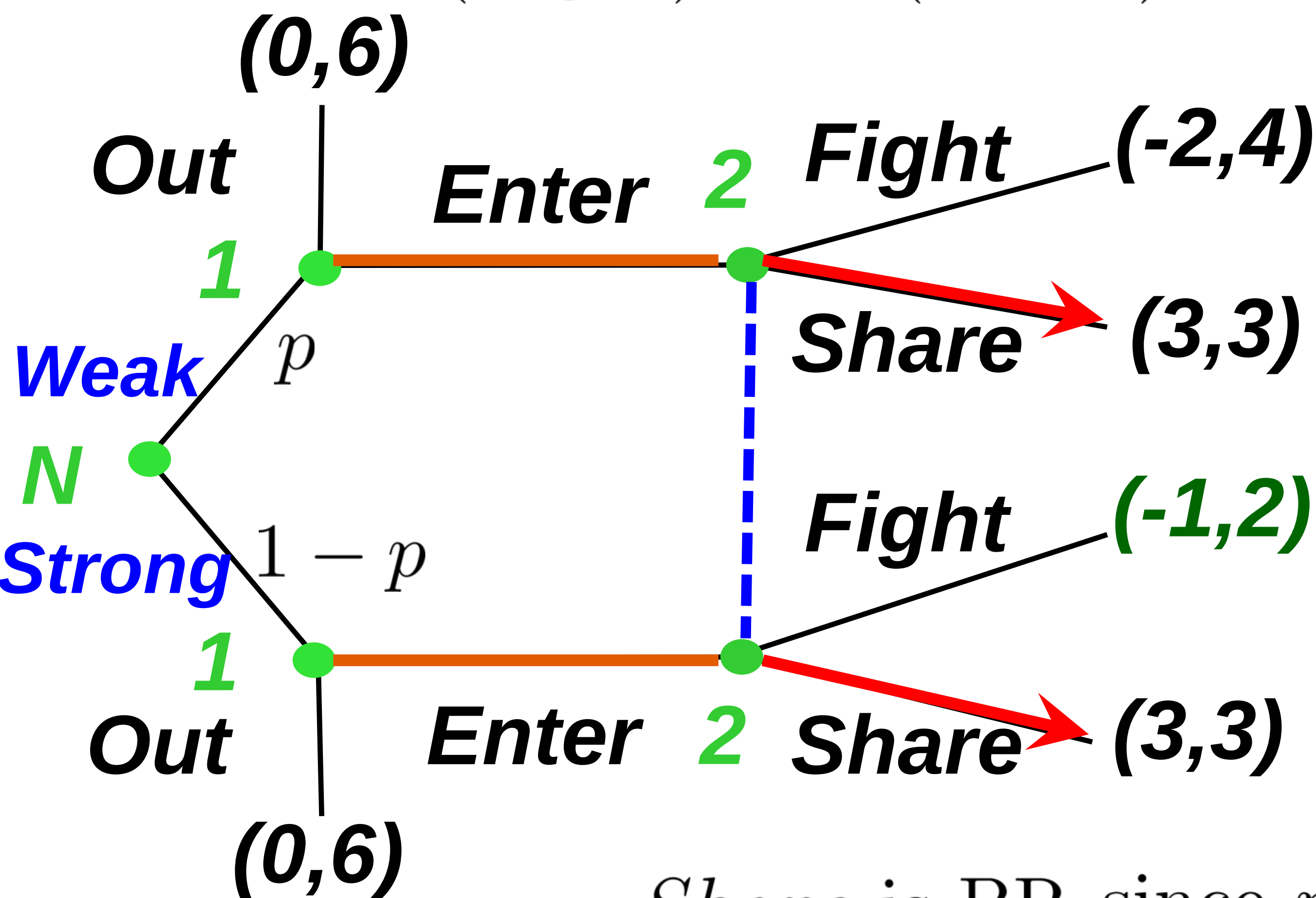
$$U^1(Enter; Weak) = U^1(Enter; Strong) = 3$$



Since $U^1(Out; \cdot) = 0$, $Enter$ is BR

BNE when $p < 1/2$: (*Enter*, *Enter*, *Share*)

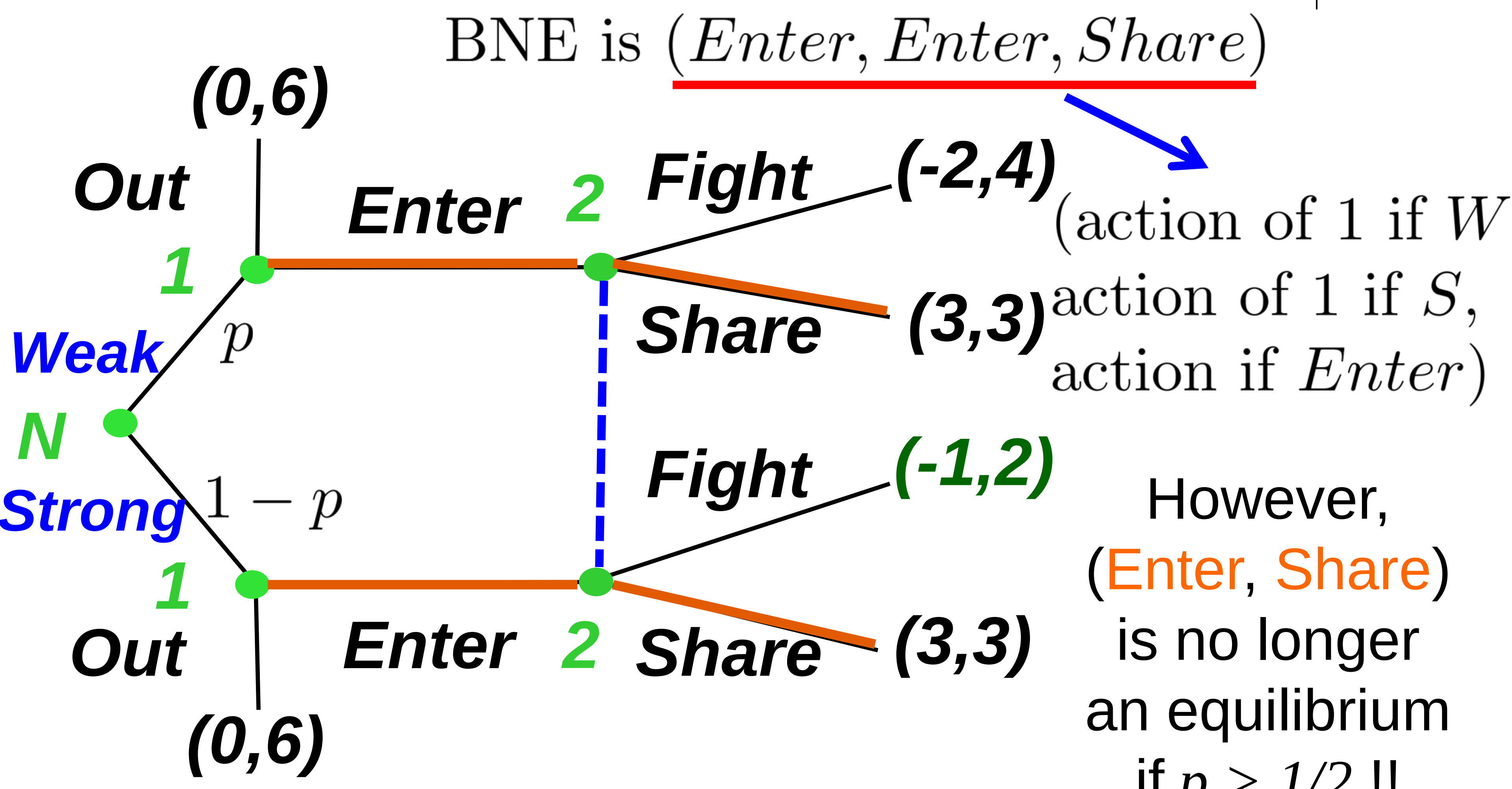
$$U^2(Fight) - U^2(Share) = 4p + 2(1-p) - 3$$
$$= (2+2p) - 3$$
$$= 2p - 1$$



No Information Update!

$\underline{Share \text{ is BR}}$ since $p < 1/2$

BNE when $p < 1/2$:
(Enter, Enter, Share)

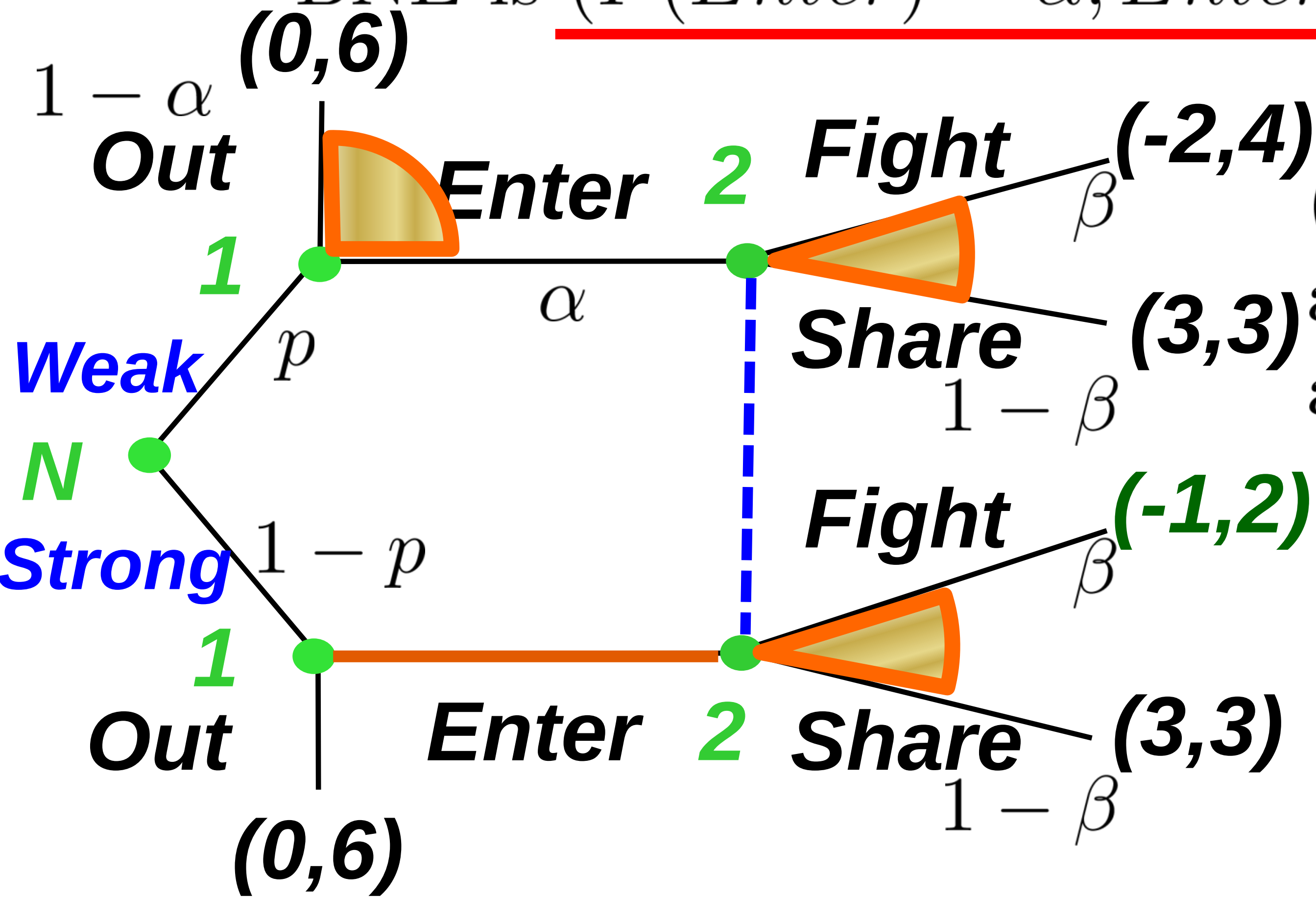


However, (Enter, Share) is no longer an equilibrium if $p > 1/2$!!

BNE when *p > 1/2*: (Strong Entrant *Enter*; Others Mix)

BNE is $(P(Enter) = \alpha, Enter, P(Fight) = \beta)$

(action of 1 if W,
action of 1 if S,
action if $Enter$)

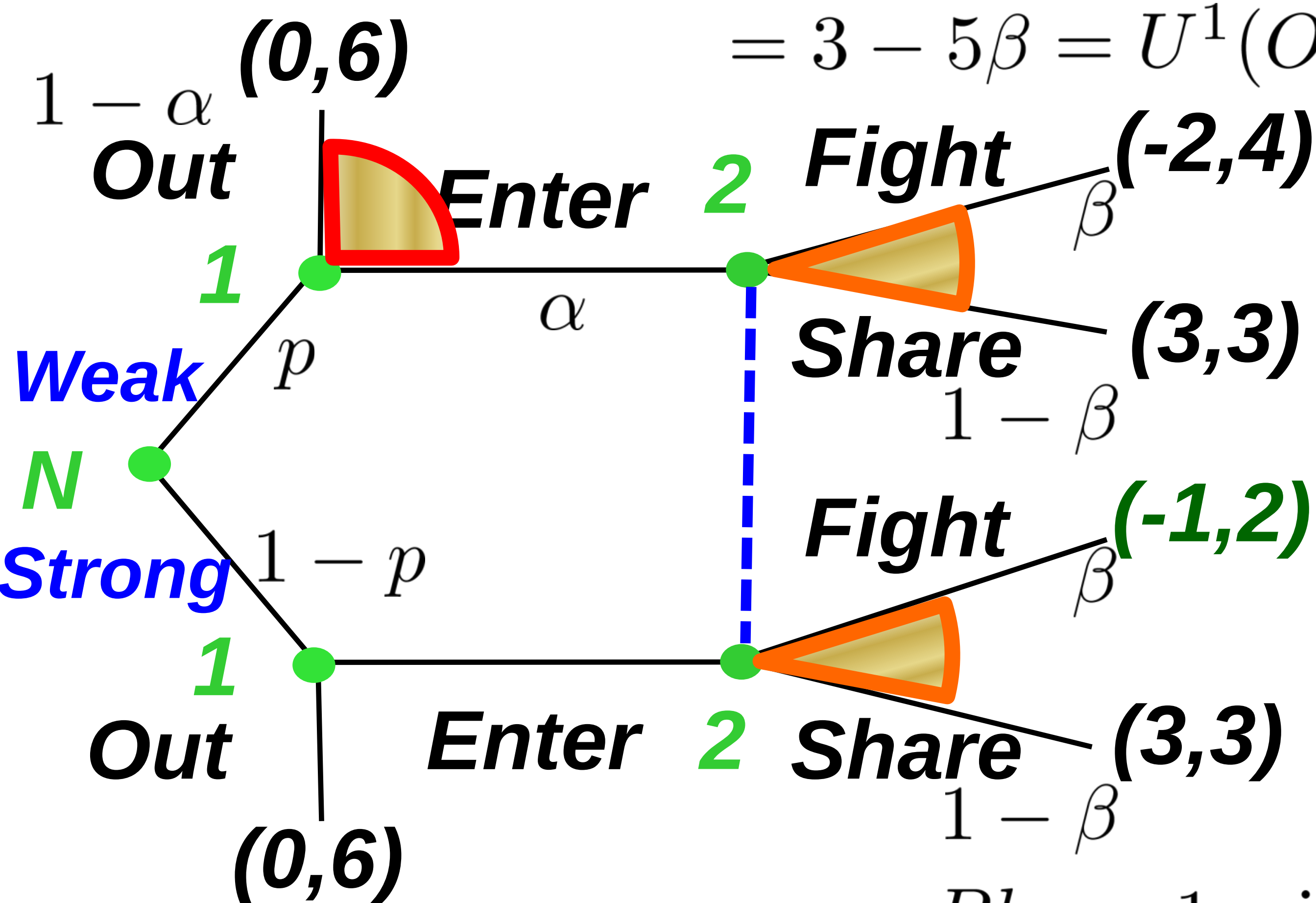


BNE when *p > 1/2*: (Strong Entrant *Enter*; Others Mix)

$$U^1(Enter; Weak) = \beta \cdot (-2) + (1-\beta) \cdot 3$$

$$= 3 - 5\beta = U^1(Out; Weak) = 0$$

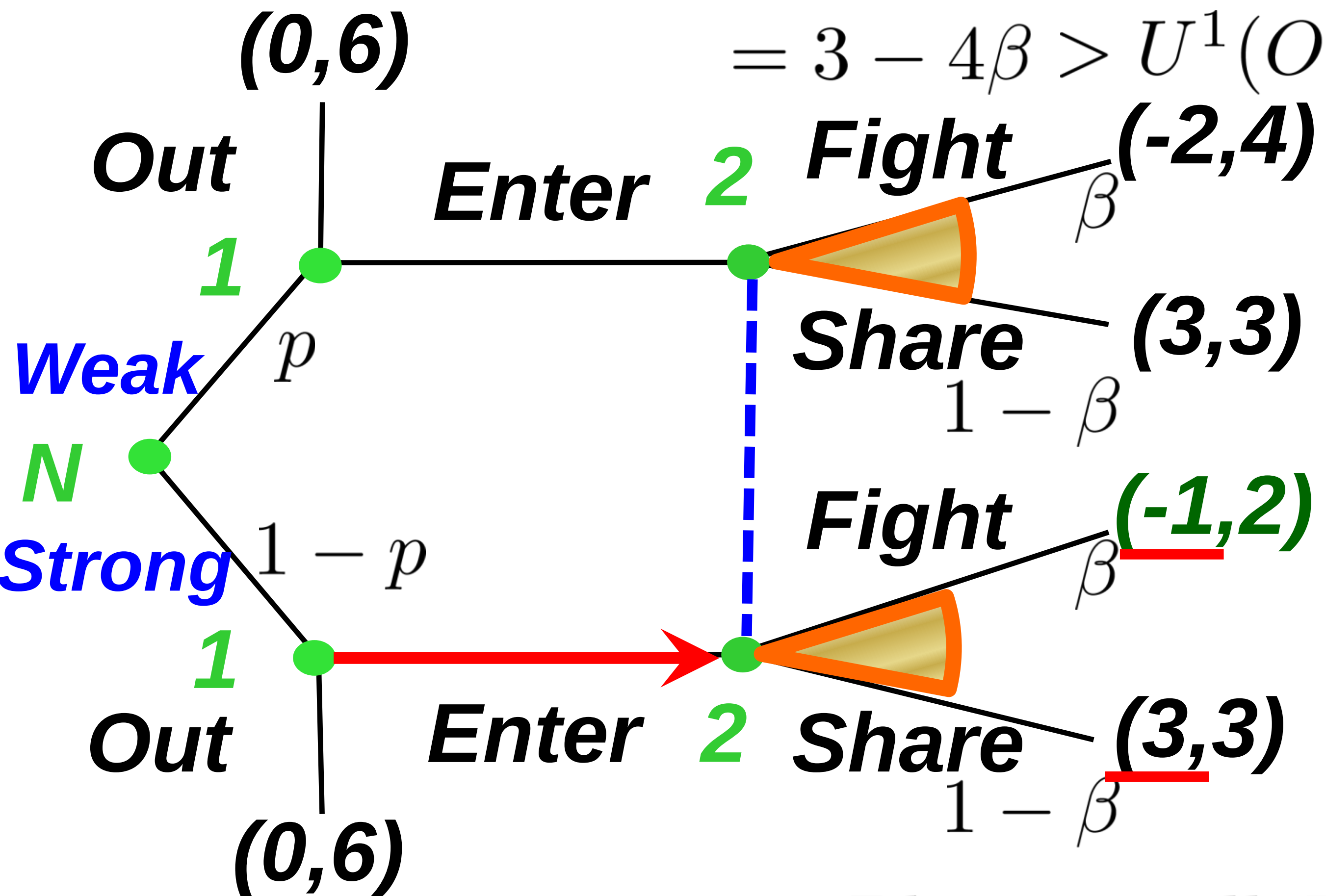
$$\text{if } \beta = \frac{3}{5}$$



$Player1$ will mix if $Weak$

BNE when $p > 1/2$: (Strong Entrant *Enter*; Others Mix)

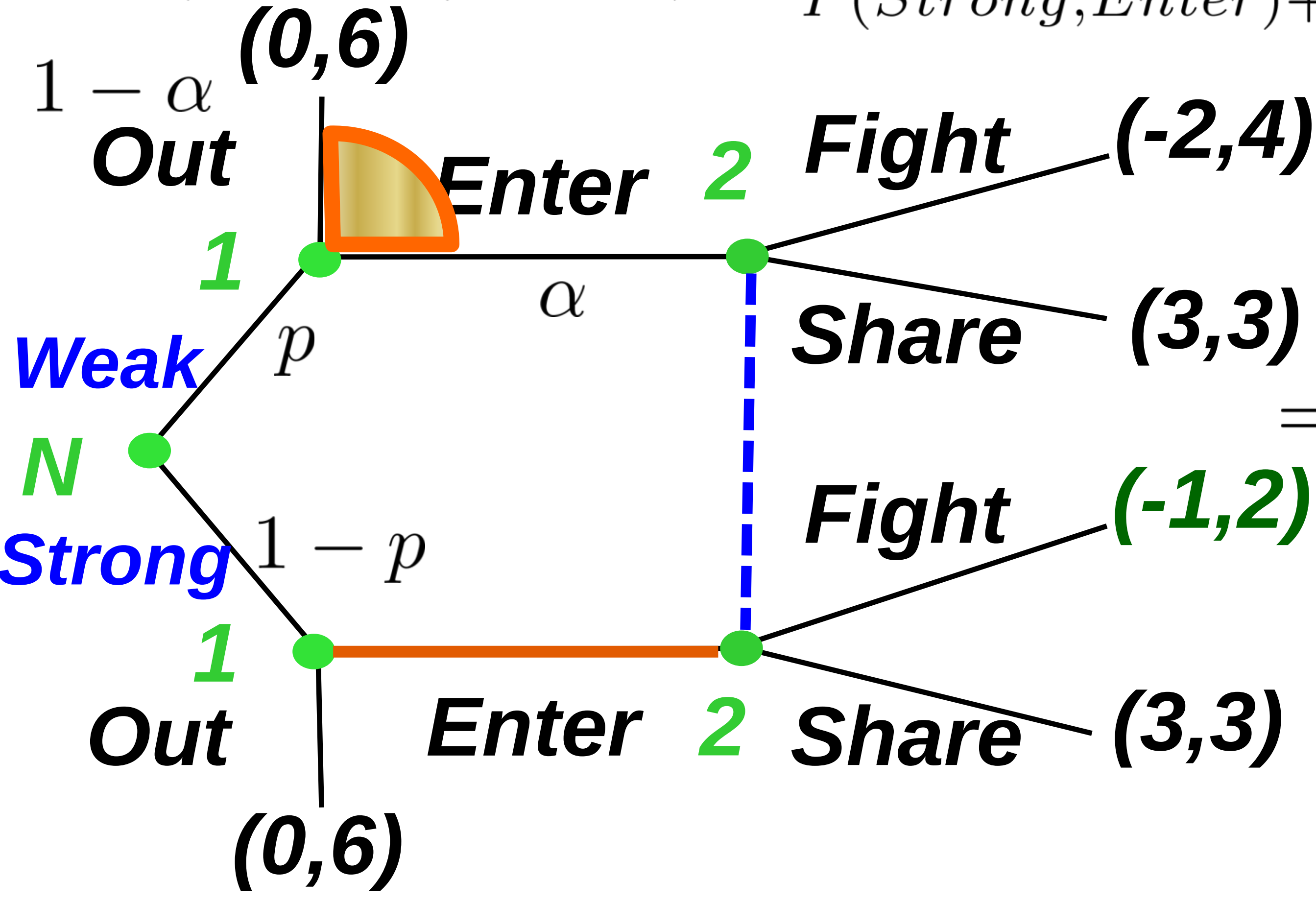
$$\text{If } \beta = \frac{3}{5}, U^1(Enter; Strong) = \beta \cdot (-1) + (1-\beta) \cdot 3$$
$$= 3 - 4\beta > U^1(Out; Strong) = 0$$



$Player 1$ will $Enter$ if $Strong$

BNE when $p > 1/2$: (Strong Entrant *Enter*; Others Mix)

$$P(Strong|Enter) = \frac{P(Strong, Enter)}{P(Strong, Enter) + P(Weak, Enter)}$$

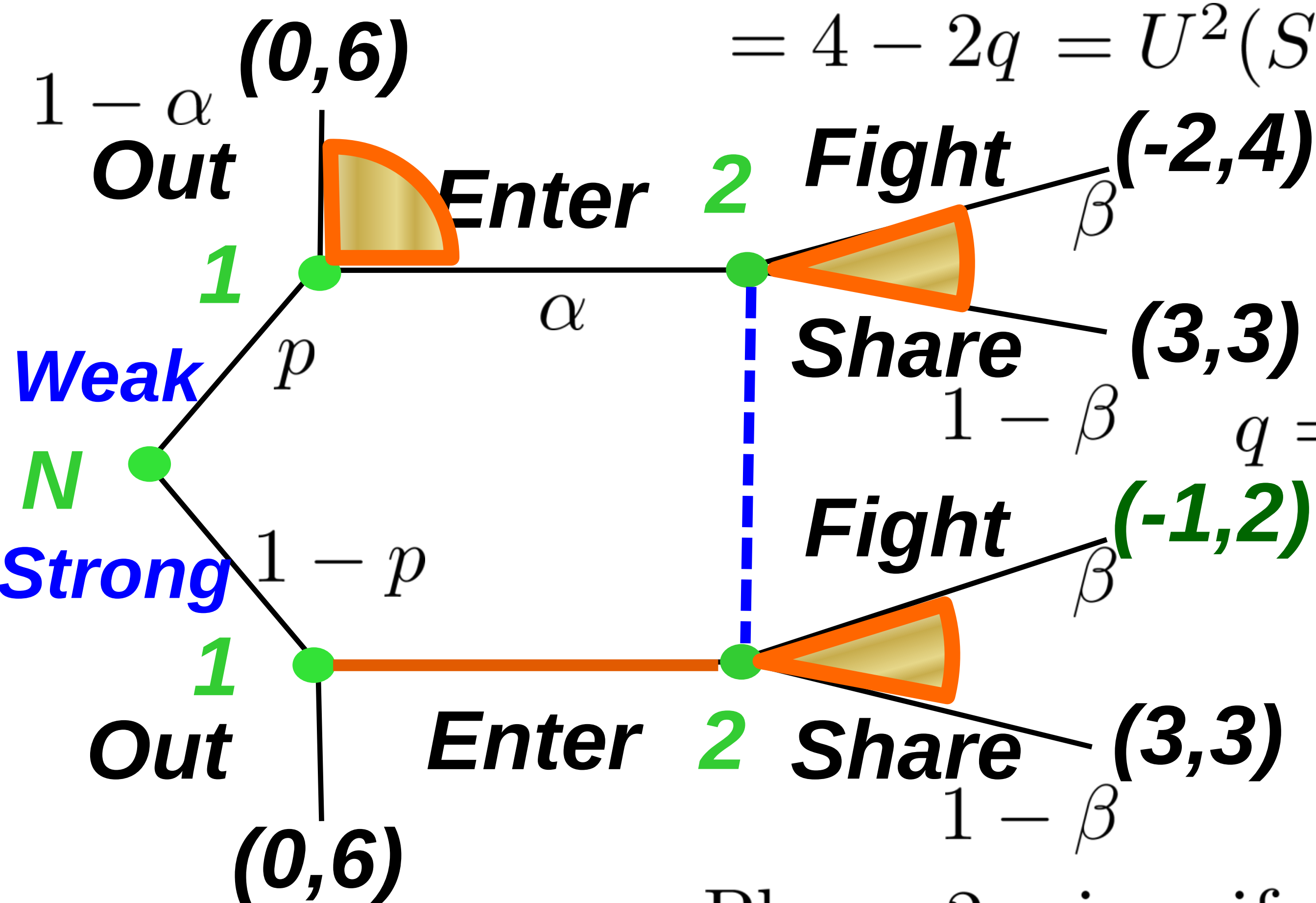


$$= \frac{(1-p)}{p \cdot \alpha + (1-p)}$$

BNE when $p > 1/2$: (Strong Entrant *Enter*; Others Mix)

$$U^2(Fight) = (1-q)\cdot 4 + q\cdot 2$$
$$= 4 - 2q = U^2(Share) = 3$$

if $q = \frac{1}{2}$

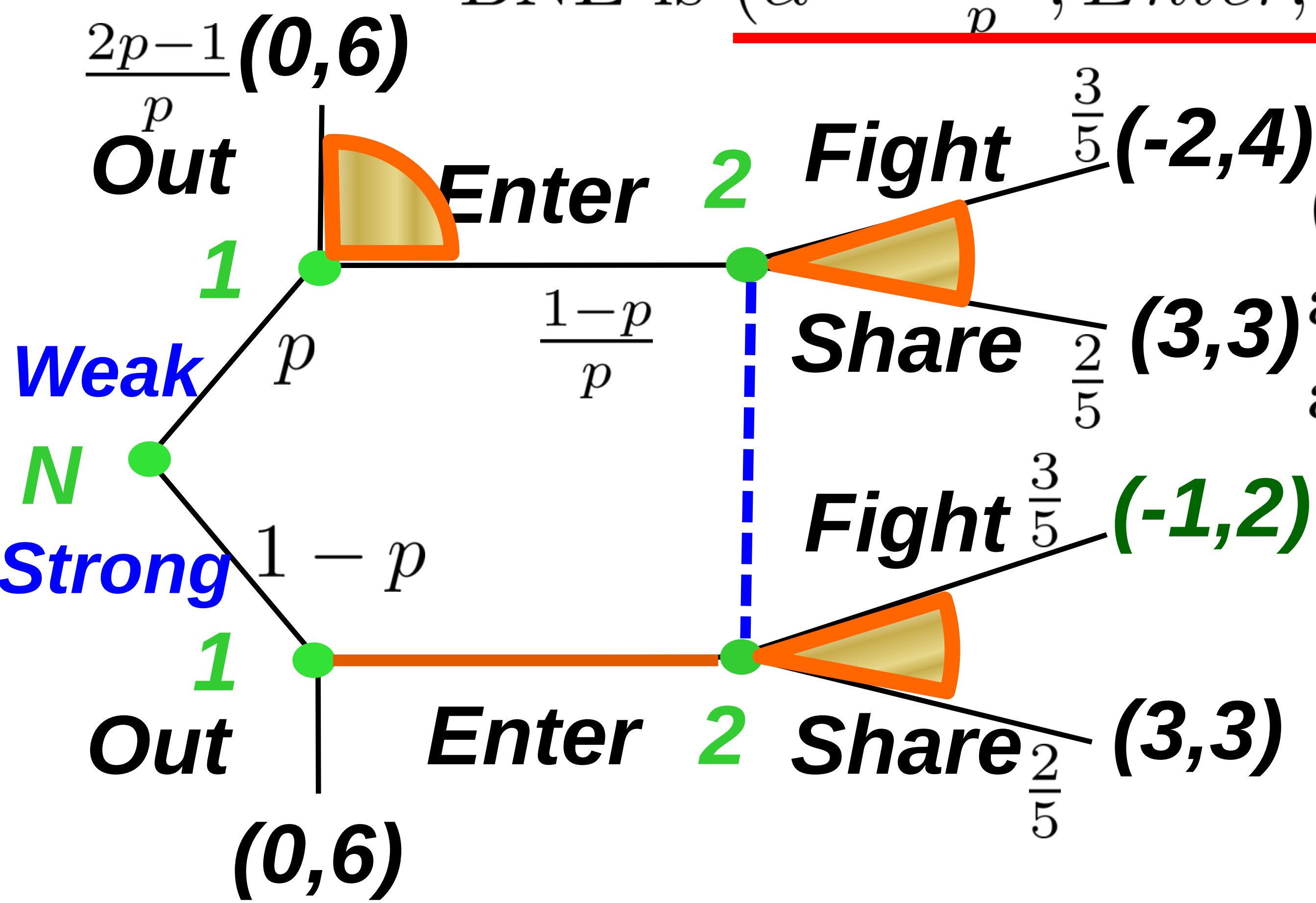


$$q = \frac{(1-p)}{p\cdot\alpha + (1-p)}$$

Player 2 mixes if $\alpha = \frac{1-p}{p}$

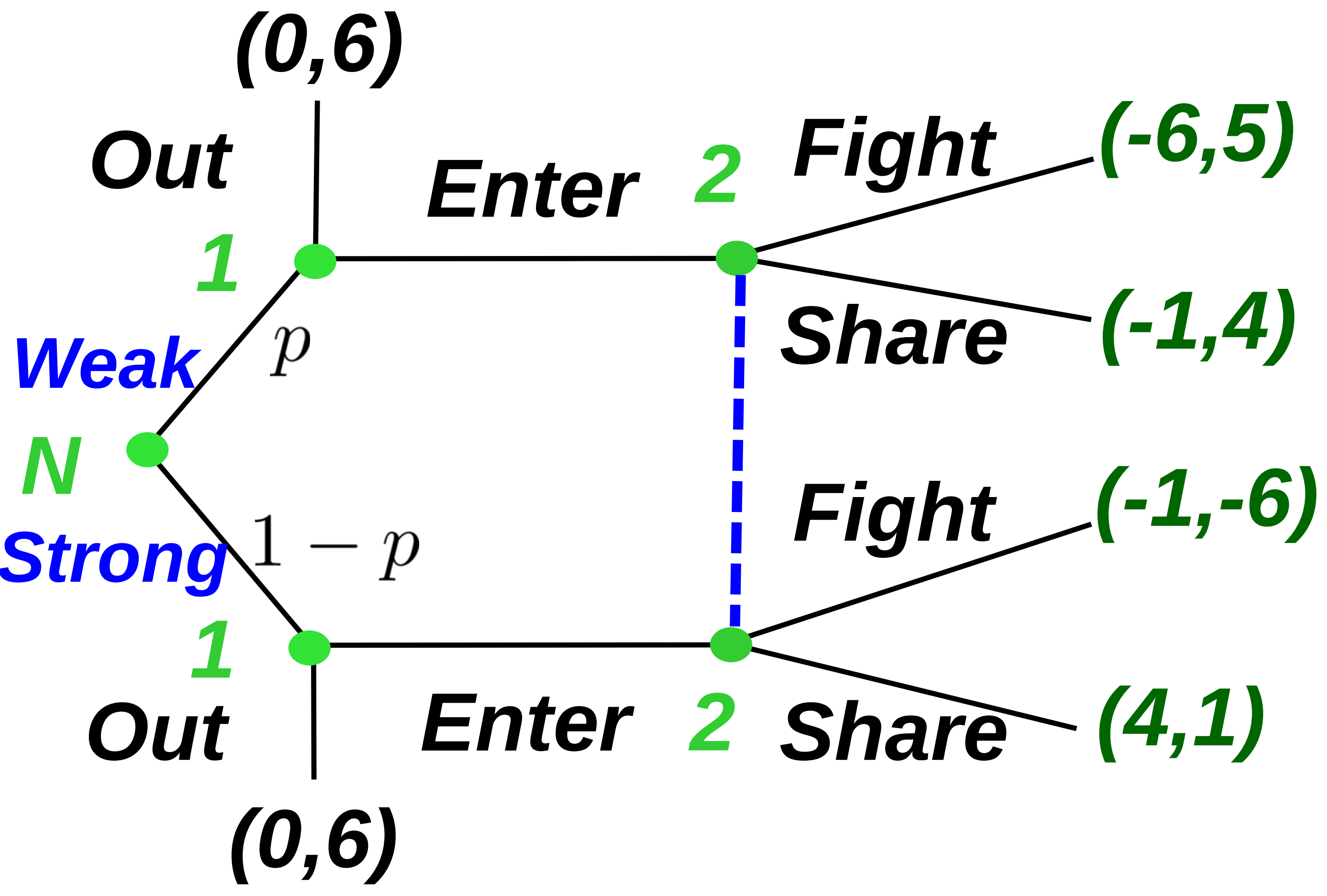
BNE when $p > 1/2$: (Strong Entrant *Enter*; Others Mix)

$$\text{BNE is } \left(\alpha = \frac{1-p}{p}, Enter, \beta = \frac{3}{5}\right)$$

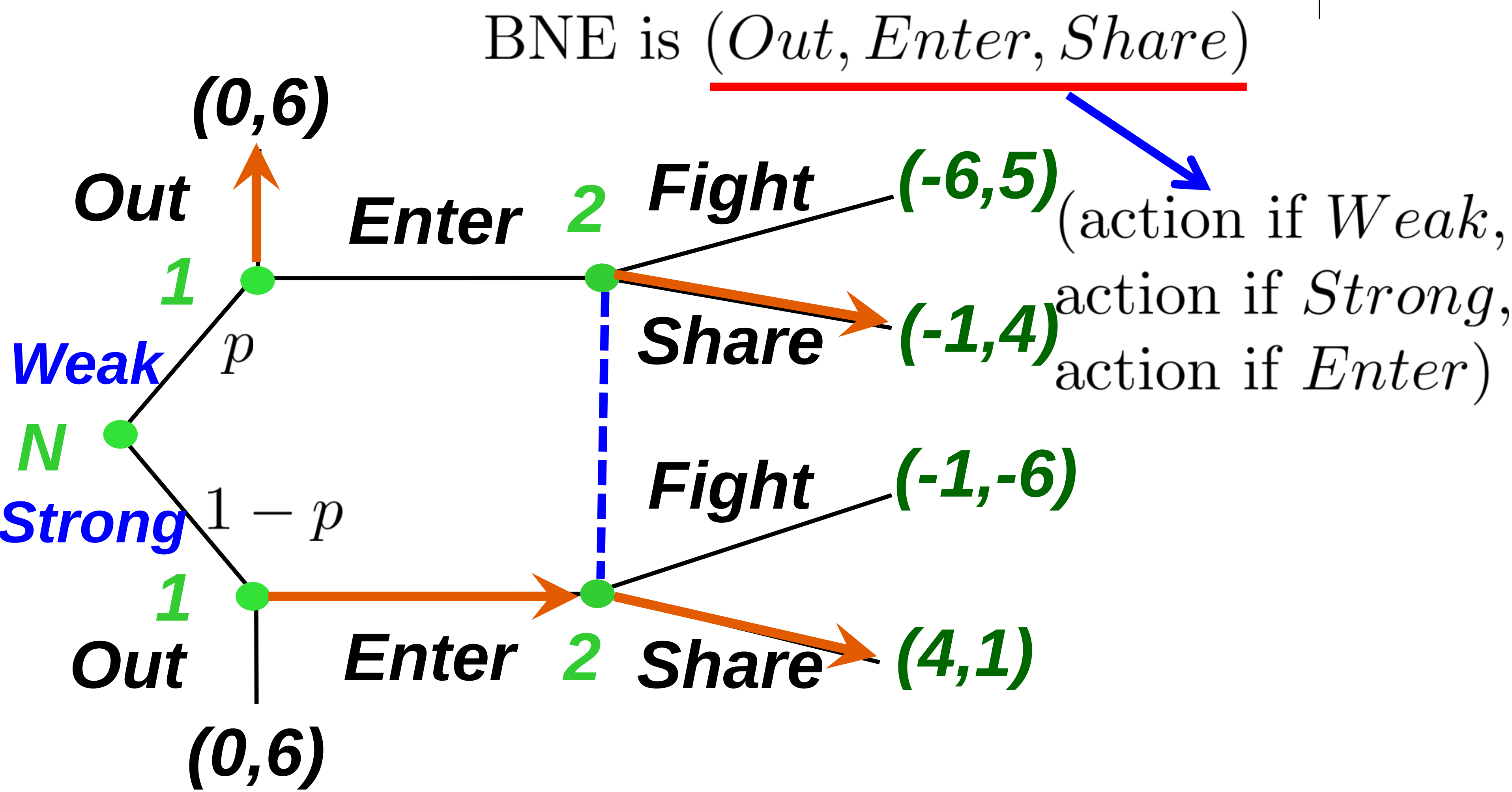


(action of 1 if W, action of 1 if S, action if $Enter$)

Modified Market Entry Game: New Payoffs if Enter…

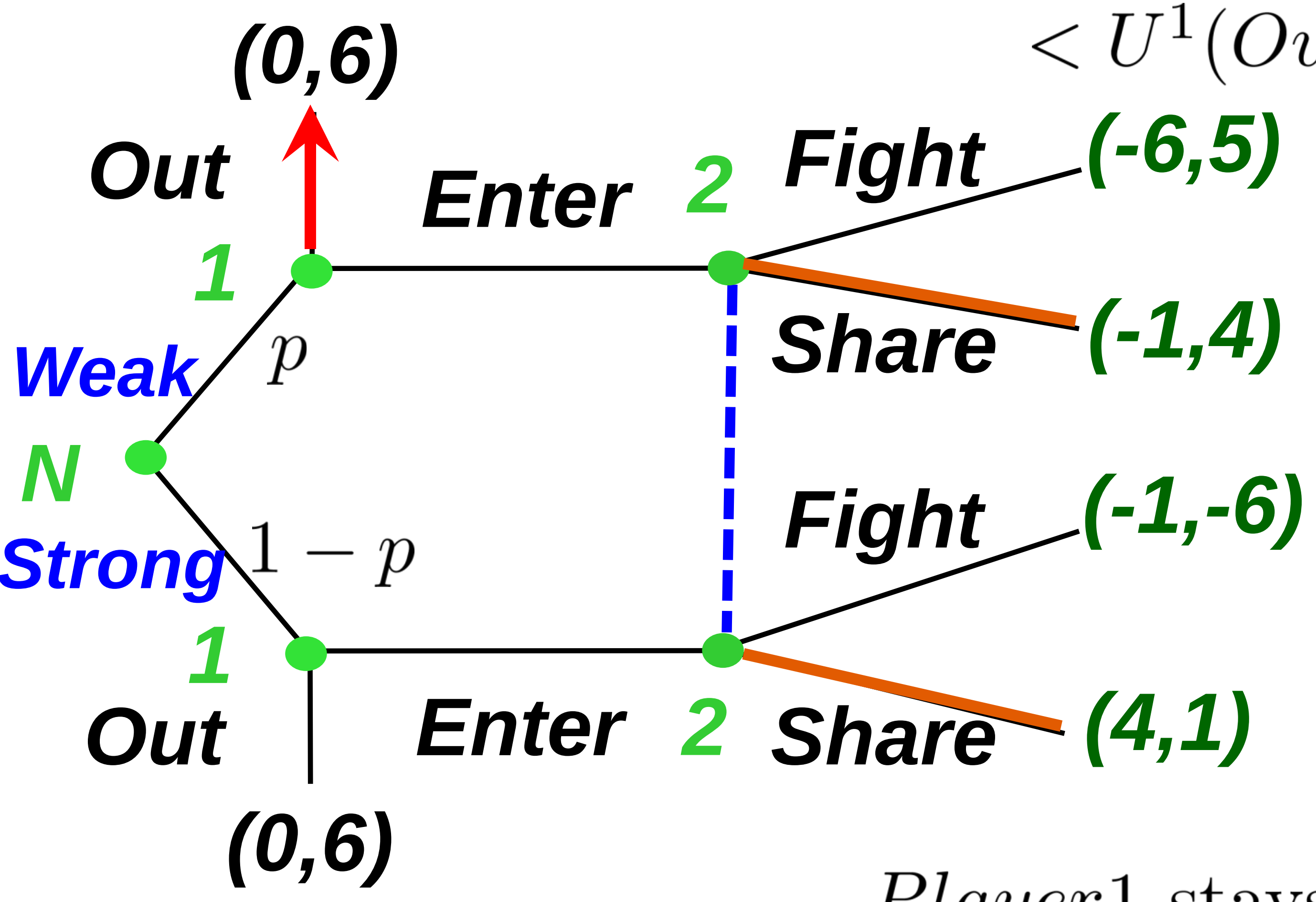


Separating Equilibrium: (Strong-*Enter*; Weak-*Out*)



Separating Equilibrium: (Strong-*Enter*; Weak-*Out*)

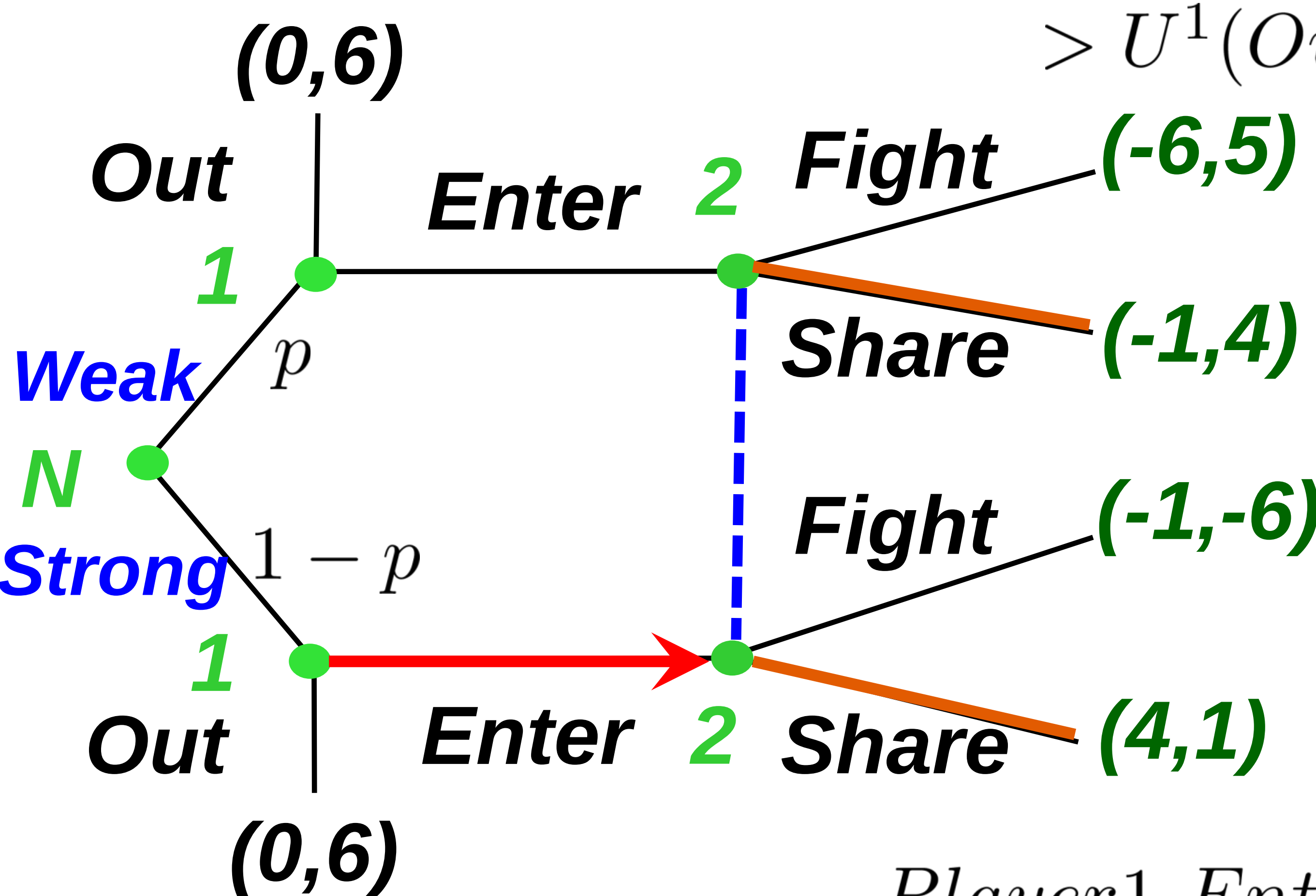
$$U^1(Enter; Weak) = -1 < U^1(Out; Weak) = 0$$



$Player 1$ stays Out if $Weak$

Separating Equilibrium: (Strong-*Enter*; Weak-*Out*)

$$U^1(Enter; Strong) = 4$$
$$> U^1(Out; Strong) = 0$$

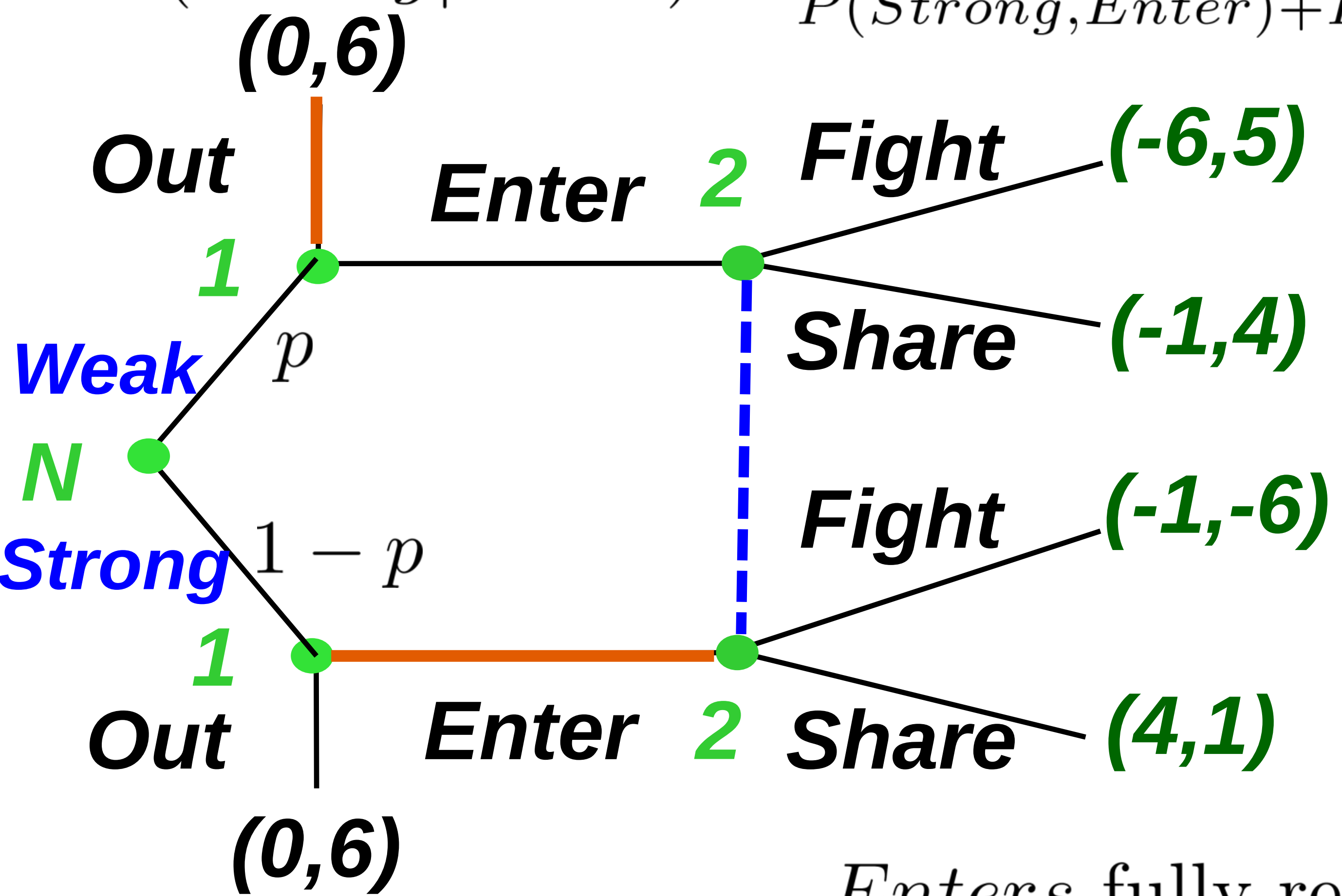


$Player 1\ Enters$ if $Strong$

Separating Equilibrium: (Strong-*Enter*; Weak-*Out*)

$$P(Strong|Enter) = \frac{P(Strong,Enter)}{P(Strong,Enter)+P(Weak,Enter)}$$

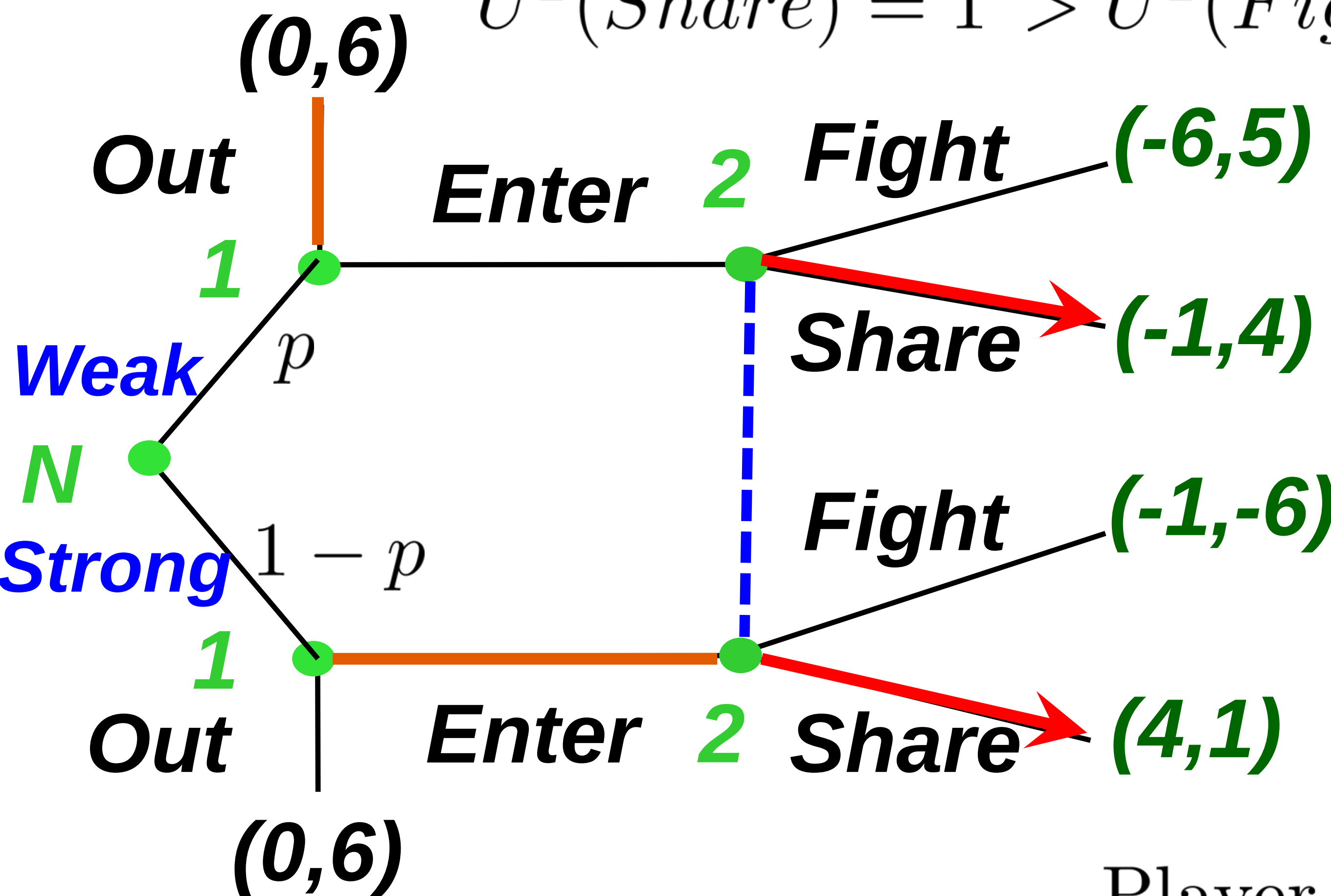
$$= 1$$



$Enter s$ fully reveals 1's type!

Separating Equilibrium: (Strong-*Enter*; Weak-*Out*)

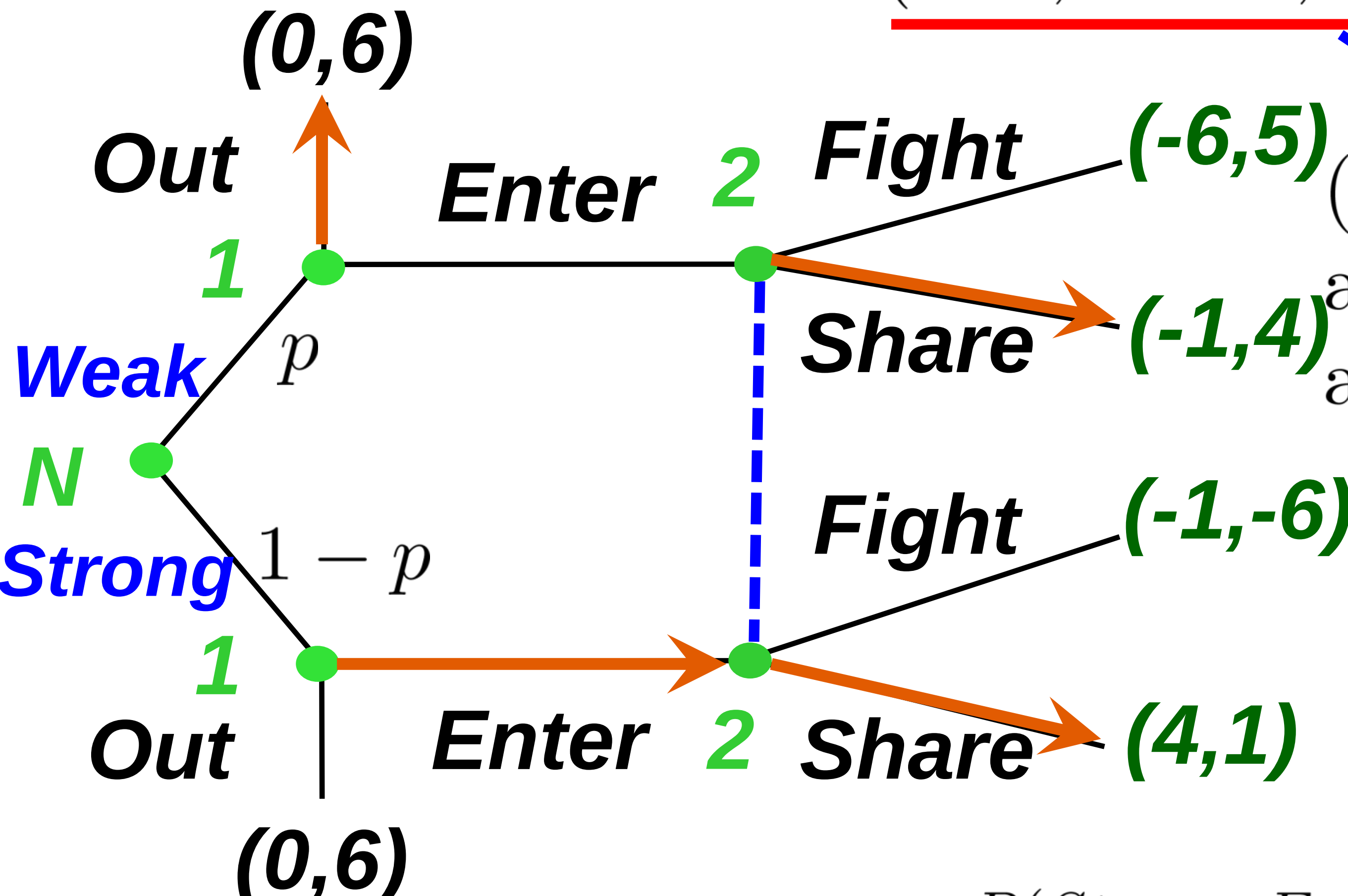
$$U^2(Share) = 1 > U^2(Fight) = -6$$



Player 2 will *Share*

Separating Equilibrium: (Strong-*Enter*; Weak-*Out*)

BNE is $(Out, Enter, Share)$



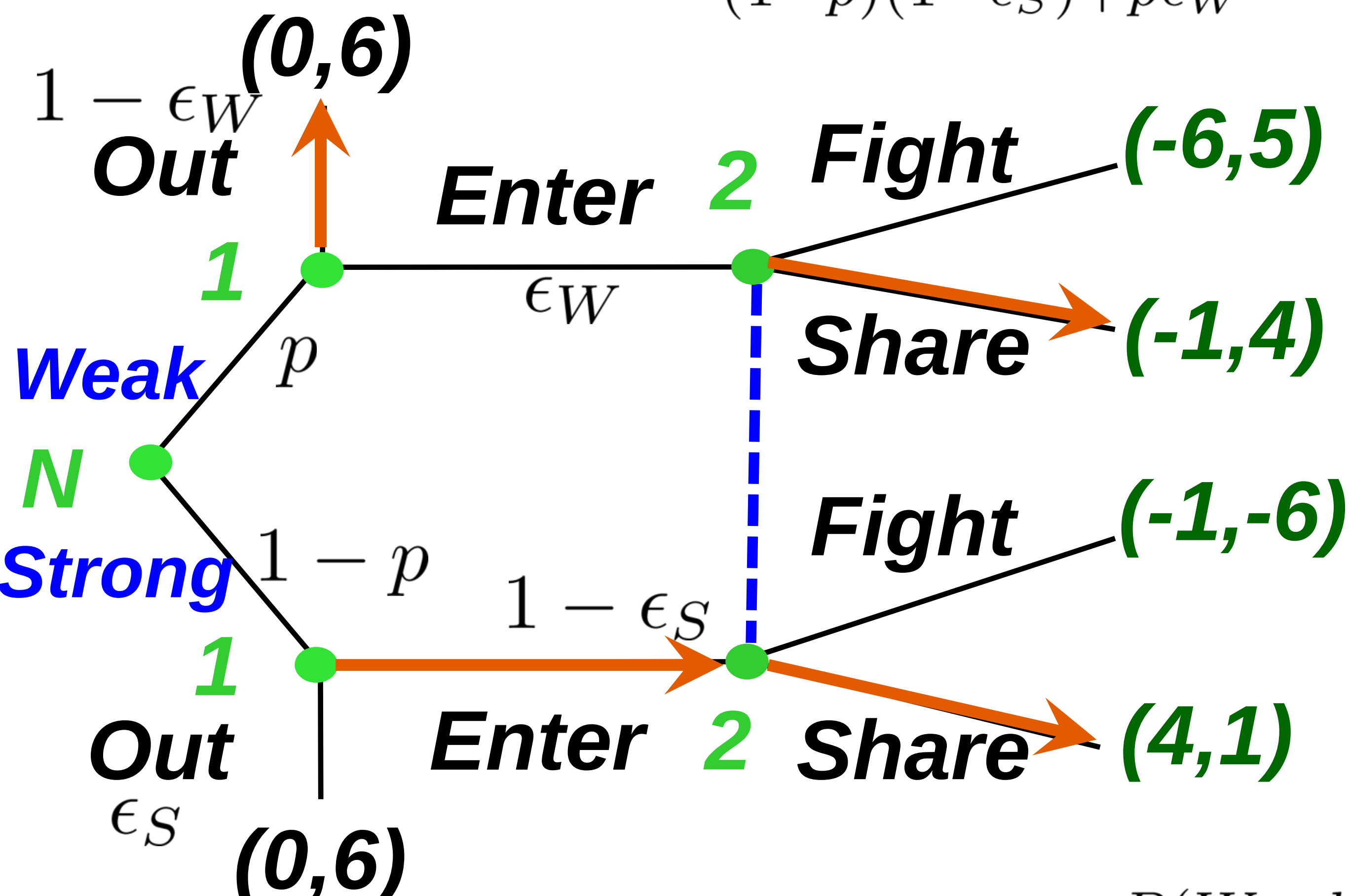
$(\text{action if } Weak, \text{action if } Strong, \text{action if } Enter)$

$$P(Strong|Enter) = \frac{P(Strong,Enter)}{P(Strong,Enter)+P(Weak,Enter)} = 1$$

(Strong-*Enter*; Weak-*Out*) is also a Sequential Equilibrium!

$$P(Weak|Enter) = \frac{p\epsilon_W}{(1-p)(1-\epsilon_S)+p\epsilon_W} \to 0$$

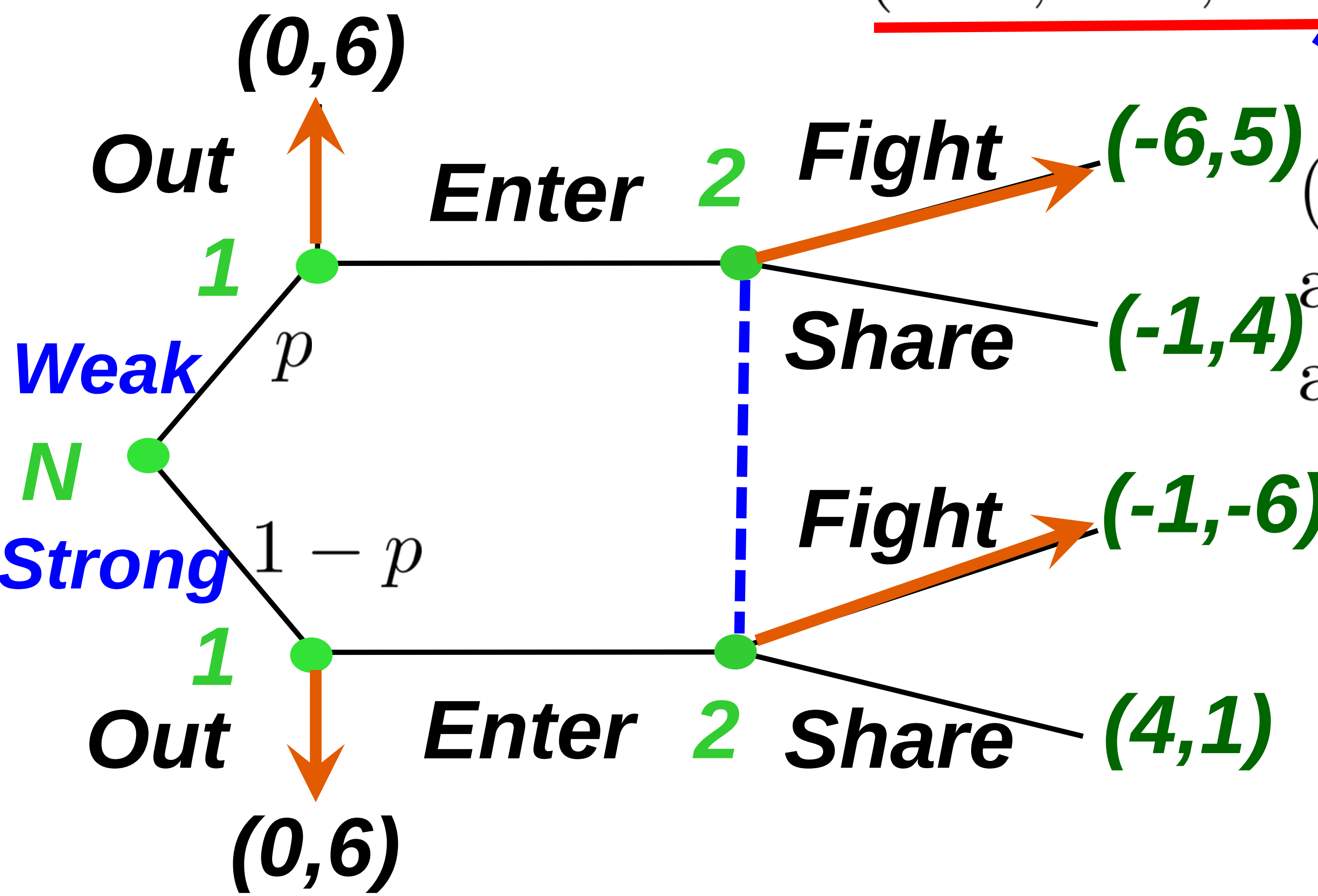
as $\epsilon_W \to 0$



$$P(Weak|Enter) = \frac{P(Weak, Enter)}{P(Strong, Enter) + P(Weak, Enter)}$$

Pooling Equilibrium:
(*Out*, *Out*, *Fight*)

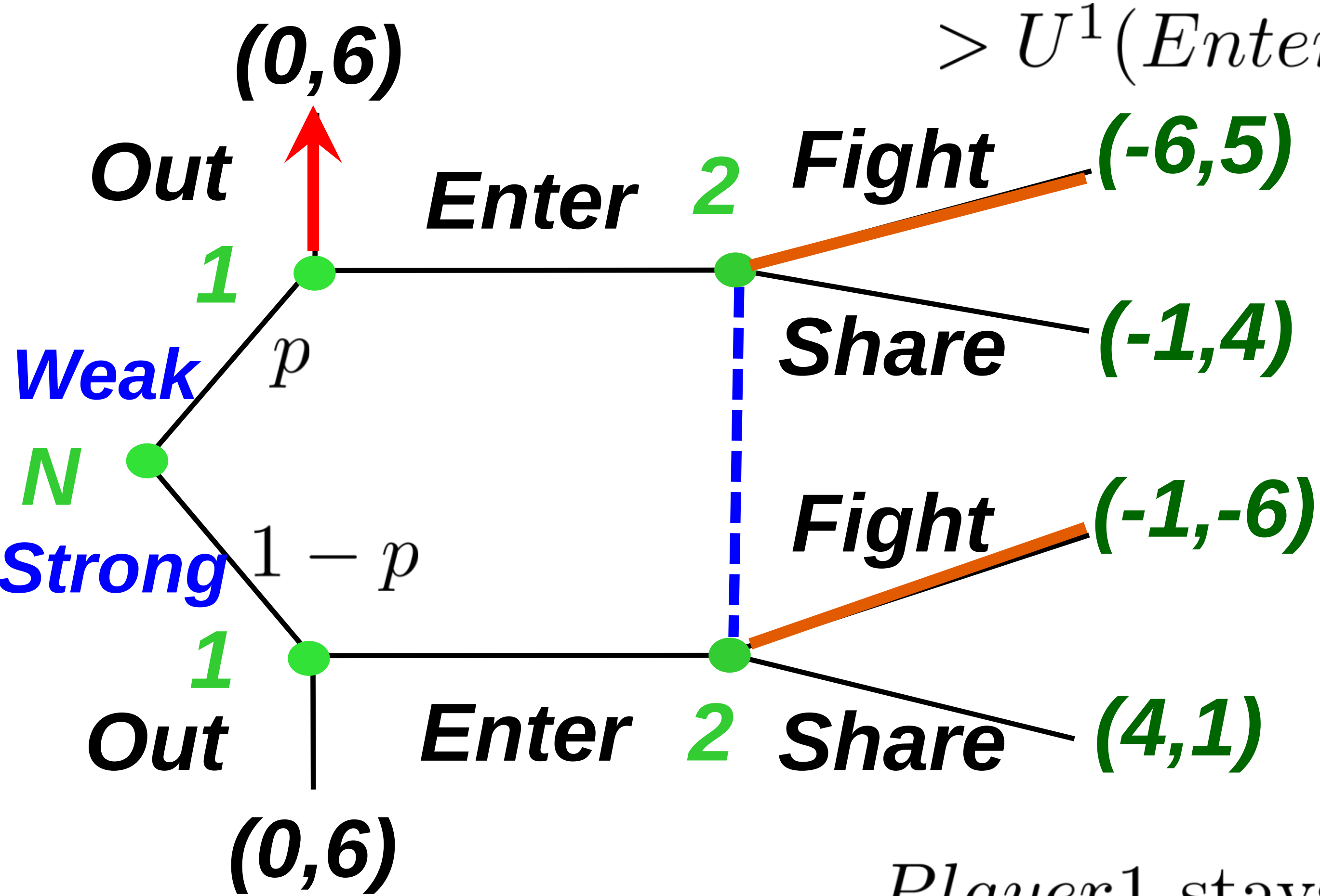
BNE is $(Out, Out, Fight)$



(action if $Weak$,
action if $Strong$,
action if $Enter$)

Pooling Equilibrium: (*Out*, *Out*, *Fight*)

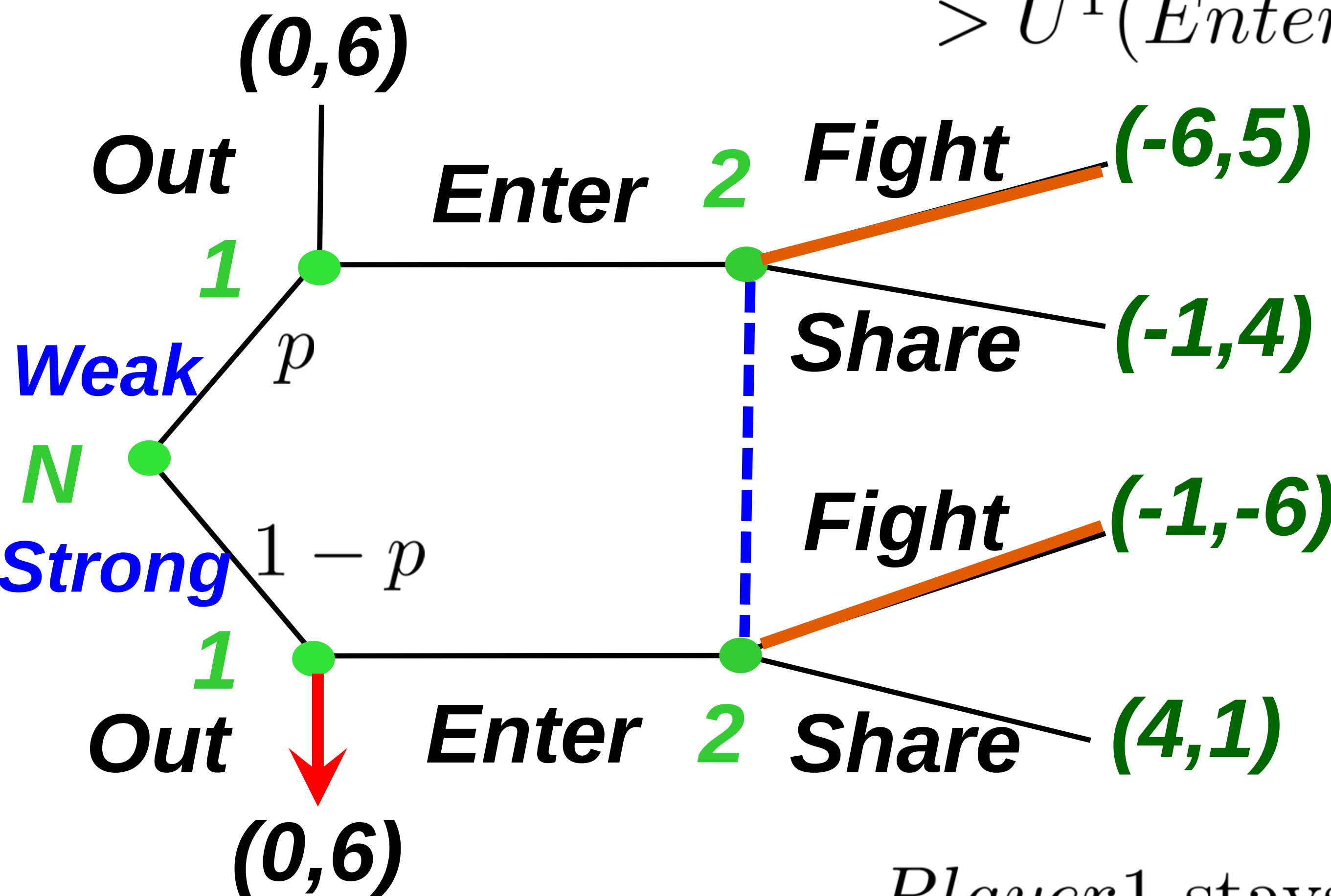
$$U^1(Out; Weak) = 0$$
$$> U^1(Enter; Weak) = -6$$



$Player 1$ stays Out if $Weak$

Pooling Equilibrium:
(*Out*, *Out*, *Fight*)

$$U^1(Out; Strong) = 0$$
$$> U^1(Enter; Strong) = -1$$

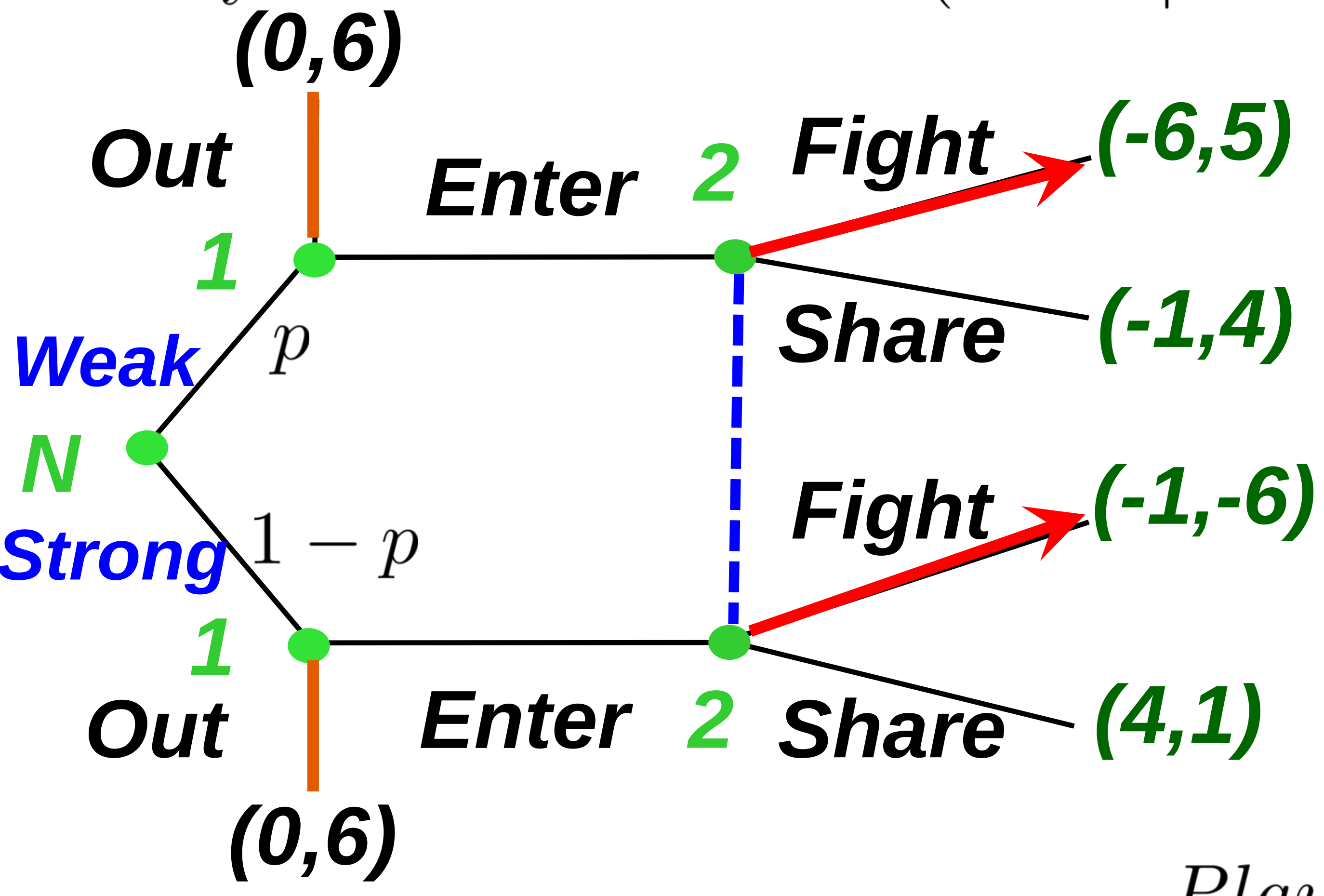


$Player 1$ stays Out if $Strong$

Pooling Equilibrium: (*Out*, *Out*, *Fight*)

If Player 2 believes that $P(Weak|Enter) = \frac{1}{1+\theta}$

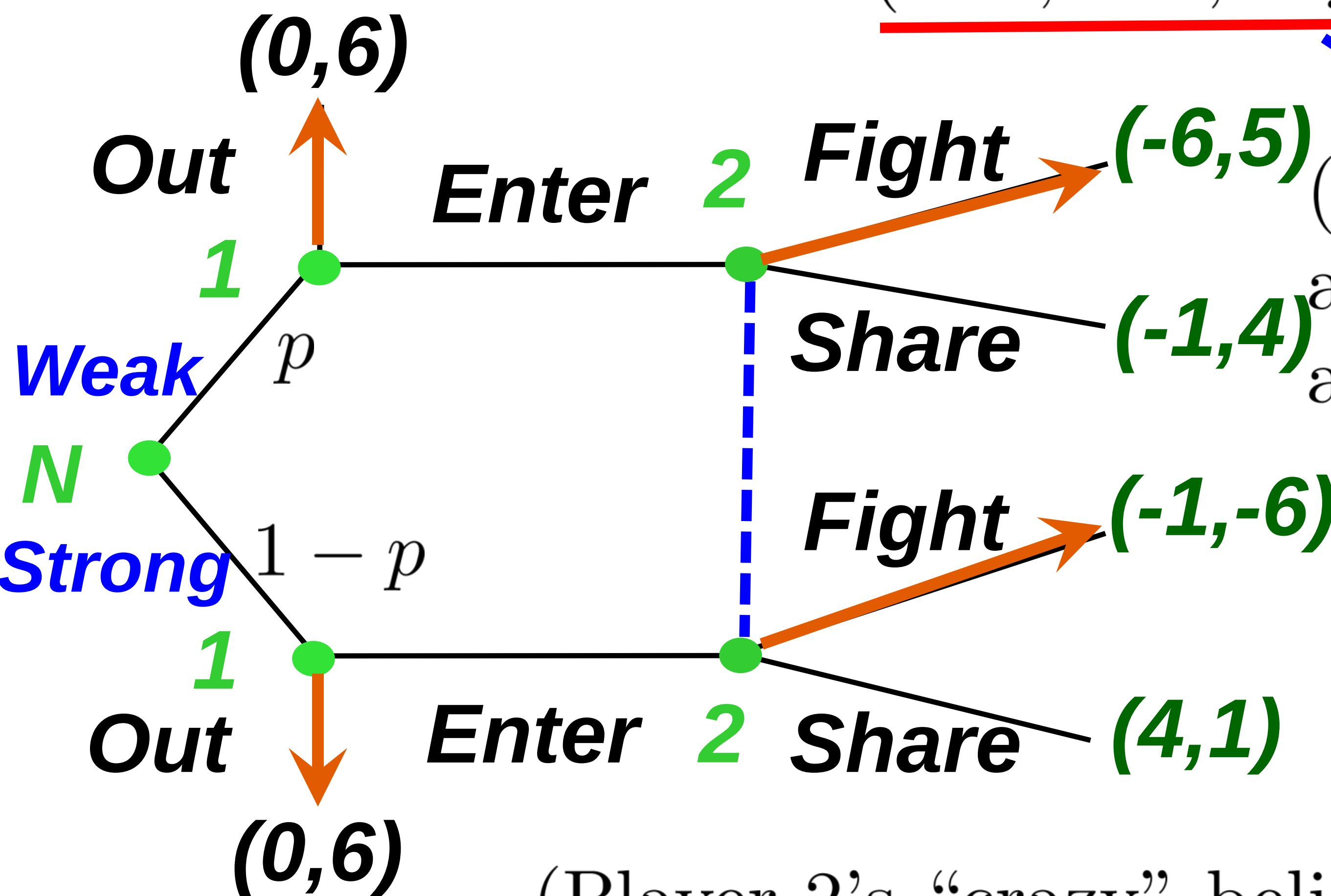
$> \frac{6}{11}$



$Player2$ will $Fight$

Pooling Equilibrium: (*Out*, *Out*, *Fight*)

BNE is $(Out, Out, Fight)$

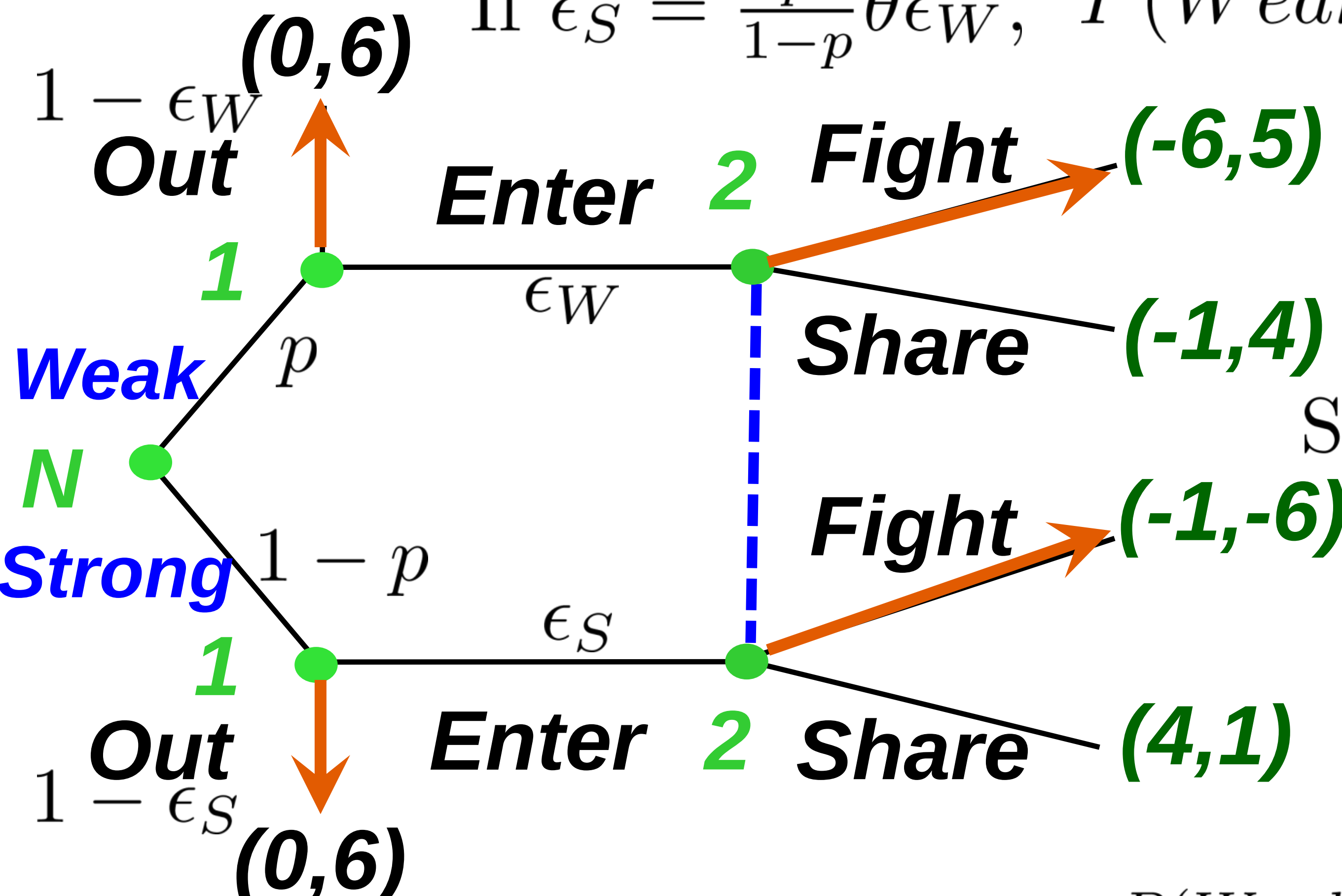


$(\text{action if } Weak,$
$\text{action if } Strong,$
$\text{action if } Enter)$

(Player 2's "crazy" belief never tested)

(*Out*, *Out*, *Fight*) is also a Sequential Equilibrium!

If $\epsilon_S = \frac{p}{1-p}\theta\epsilon_W$, $P(Weak|Enter) = \frac{p\epsilon_W}{(1-p)\epsilon_S + p\epsilon_W} = \frac{1}{1+\theta}$



Still Sequential!!

$$P(Weak|Enter) = \frac{P(Weak, Enter)}{P(Strong, Enter) + P(Weak, Enter)}$$

(*Out*, *Out*, *Fight*) is also a Sequential Equilibrium!

- (*Out*, *Out*, *Fight*) is not ruled out by THP, and hence, is also a Sequential Equilibrium…
- But why can't the *Strong* type say,
- "If I enter, I will be credibly signaling that I am *Strong*, since if I were weak and chose to *Enter*, my possible payoffs would be -1 or -5, smaller than 0 (equilibrium payoff if weak)."
- Seeing this, player 2's BR is *Share*
  - It is profitable for player 1 to *Enter* (& signal)…

Definition: Intuitive Criterion (Cho and Kreps)

- Consider $\hat{a}_i$, a strategy of player i that is not chosen in the Bayesian Nash equilibrium,
- Let $u_i(\hat{a}_i, t_i)$ be the payoff of player *i*'s if he chooses $\hat{a}_i$ and is believed to be type $t_i \in T_i$
- Let $u_i^N(t_i)$ be this types' equilibrium payoff
- The BNE fails the Intuitive Criterion if, for some player i of type $\hat{t}_i \in T_i$, $u_i(\hat{a}_i, \hat{t}_i) > u_i^N(\hat{t}_i)$
- And for all other types in $t_i \in T_i$,

$$u_i(\hat{a}_i, t_i) < u_i^N(t_i)$$

Intuitive Criterion (Cho and Kreps)

- In the previous Example,
- (*Out*, *Out*, *Fight*) fails the Intuitive Criterion
  - "If I enter, I will be credibly signaling that I am *Strong*, since if I were weak and chose to *Enter*, my possible payoffs would be -1 or -5, smaller than 0 (equilibrium payoff if weak)."
- (*Out*, *Enter*, *Share*) satisfies the Intuitive Criterion
  - Such argument is not credible…

Continuous Types: An Auction Game

- One single item for sale
- n risk neutral bidders
- Valuation is continuously distributed on the unit interval with cdf $F(.) \sim [0,1]$
  - All this is common knowledge
- Bidder's type = Valuation (private information)
- Pure Strategy = Bid function $b = b_j(v_j)$

Sealed High-Bid Auction (aka First Price Auction)

- Each buyer submits one sealed bid

$$b_j \geq 0, j = 1, \cdots, n$$

- Buyer who makes highest bid is the winner
  - If there is a tie, the winner is chosen randomly from the tying high bidders
- The winning bidder pays his bid and receivers the item

Sealed High-Bid Auction (aka First Price Auction)

- Bidder j, $j=1, \ldots, n$, knows own valuation v_j
  - Risk neutral, pay b, wins with probability p
- Payoff is $u_j(b, p, v_j) = p(v_j - b)$

- Solve for Equilibrium Bidding Strategy $b_j(v_j)$
- For the special case of 2 bidders of Independent Private Value (IPV)
- Assume valuation is uniform [0,1], cdf $F(x) = x$

BR to a Linear Strategy

- If buyer 2's bidding strategy is $b_2(v_2) = \alpha_2 v_2$
- Then the distribution of bids is uniform
  - Since valuation is uniform
- If buyer bids b, he wins with probability

$$p_2(b) = Pr[b_2 \leq b] = \frac{b}{\alpha_2}, b \in [0, \alpha_2]$$

- Buyer 1: $U_1(b) = p_2(b)(v_1 - b)$

$$= \frac{b}{\alpha_2}(v_1 - b) = \frac{1}{\alpha_2}(bv_1 - b^2)$$

Equilibrium of the Sealed High-Bid (aka First Price) Auction

- Solve the maximization problem:

$$\max_b U_1(b) = p_2(b)(v_1 - b) = \frac{1}{\alpha_2}(bv_1 - b^2)$$

- FOC: $v_1 - 2b = 0$
- Maximum at $b_1(v_1) = \frac{1}{2}v_1$
  - I.e. The BR to a linear strategy is a linear strategy
- By symmetry, the BNE is $b_j(v_j) = \frac{1}{2}v_j$

3 Bidder Case

- What if there are 3 bidders?
- Intuition is you would bid higher (competition)
- Assume bidder 3 bids $b_3(v_3) = \alpha_3 v_3$
- If buyer bids b, he wins with probability

$$p_3(b) = Pr[b_3 \leq b] = \frac{b}{\alpha_3}, b \in [0, \alpha_3]$$

- Buyer 1: $U_1(b) = p_2(b)p_3(b)(v_1 - b)$

$$= \frac{b}{\alpha_2}\frac{b}{\alpha_3}(v_1 - b) = \frac{1}{\alpha_2\alpha_3}(b^2 v_1 - b^3)$$

Equilibrium of the Sealed High-Bid (aka First Price) Auction

- Solve the maximization problem:

$$\max_b U_1(b) = p_2(b)p_3(b)(v_1 - b) = \frac{1}{\alpha_2\alpha_3}(b^2 v_1 - b^3)$$

- FOC: $2bv_1 - 3b^2 = 0$
- Maximum at $b_1(v_1) = \frac{2}{3}v_1$
  - I.e. The BR to a linear strategy is a linear strategy
- By symmetry, the BNE is $b_j(v_j) = \frac{2}{3}v_j$

Summary of 9.7

- Pooling Equilibrium vs. Separating Equilibrium
- Semi-Pooling Equilibrium (MSE)

- Intuitive Criteria
- Continuous Type Models: Auction Games

- HW 9.7: Riley – 9.7-1~3