Multi-Stage Games

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(Lecture 4, Micro Theory I-2)

Games Played More than Once

- In each stage, a simultaneous game is played
- History of the game: h_i^t

$$= \text{all information available to player } i \text{ at period } t$$

- The Second Stage Strategy is a function of history h_1^t
- Two/Three stage repeated game strategy:

$$s_i = (s_i^1, s_i^2(h_1^1)) \in S_i \times S_i$$

$$s_i = (s_i^1, s_i^2(h_1^1), s_i^3(h_i^2)) \in S_i \times S_i \times S_i$$

Competition for Market Share Over 2 Periods

Unique stage-game Nash Eq. is (*Low*, *Low*)

Agent 2: Colin

Agent 1: Rowena

		High	Low
	High	100, 100	30, 150
	Low	150, 30	50, 50

Backward Induction: Second Stage

In last stage, unique 2nd stage-game Nash Eq. is (*Low*, *Low*)

Agent 2: Colin

Continuation Payoff $e = 50\delta$

Agent 1: Rowena

	High	Low
High	100, 100	30, 150
Low	150, 30	50, 50

Backward Induction: First Stage

Continuation Payoff makes no difference…

Unique 1st stage-game Nash Eq. is (Low, Low)

Agent 2: Colin

Agent 1: Rowena	High	Low
High	$100+e$, $100+e$	$30+e$, $150+e$
Low	$150+e$, $30+e$	$50+e$, $50+e$

Same for 3 or more stages…

Proposition 9.2-1: Equilibrium of Finitely Repeated Game

- Suppose stage game Nash Equilibrium is
$$\overline{s} = \{\overline{s}_1, \overline{s}_2, \cdots, \overline{s}_n\}$$
- When the stage game is repeated T times
- Playing $\overline{s}$ for T times regardless of history is an equilibrium in the finitely repeated game
- Formally: $\hat{s} = (\hat{s}_1, \cdots, \hat{s}_n) \in \mathbf{R}^{n \times T}$
- where $\hat{s}_i = \left(\hat{s}_i^1, \hat{s}_i^2(h_i^1), \cdot, \hat{s}_i^T(h_i^{T-1})\right) = (\overline{s}_i, \cdots, \overline{s}_i)$
- is an equilibrium of the finitely repeated game

Equilibrium of Finitely Repeated Game

- If the stage game Nash equilibrium is unique,
- This equilibrium also uniquely satisfies Backward Induction.
  - Are there other Nash equilibrium?
- What if there are multiple stage game Nash equilibria?

- Consider the Partnership Game in 9.1…

Nash Equilibrium: Partnership Game

- Two Agents have equal share in a partnership
- Choose Effort: $a_i \in A_i = \{1, 2, 3\}$
- Total revenue: $R = 12a_1a_2$
- Cost to agent i: $C_i(a_i) = a_i^3$
- Payoff: $u_i(s) = R - C_i(a_i) = 12a_1a_2 - a_i^3$

- Game matrix and Nash Equilibrium…

Nash Equilibrium: Partnership Game

Two SGNE: *(1,1), (2,2)*

Combo of SGNE is equilibrium in FRG

Best Payoff = $16+16\delta$

Player 2: Colin

Player 1: Rowena	1	2	3
1	5, 5	11, 4	17, -9
2	4, 11	16, 16	28, 9
3	-9, 17	9, 28	27, 27

Can we do better?

Equilibrium of FRG: Partnership Game

- This is NOT the only two equilibria
- Agents can threat to play the bad equilibrium in stage 2 to induce *(3, 3)* and earn *(27, 27)*…
- EX: Use: $\overline{s}_i^1 = 3, \quad \overline{s}_i^2(h^1) = 2 \text{ if } h^1 = (3,3)$

$$\overline{s}_i^2(h^1) = 1 \text{ if } h^1 \neq (3,3)$$

- If other agent follows this strategy,
- Is it a BR to follow this strategy?
- Yes for Stage 2 (both *(2, 2)* and *(1, 1)* are SGNE)
  - For Stage 1…

Nash Equilibrium: Partnership Game

$$u(\text{follow}) = 27 + \delta \cdot 16$$

$$u(\text{defect}) = 28 + \delta \cdot 5$$

$$\textbf{Yes} \text{ if } \delta \geq \frac{1}{11}$$

Player 2: Colin

Player 1: Rowena

	1	2	3
1	5, 5	11, 4	17, -9
2	4, 11	16, 16	28, 9
3	-9, 17	9, 28	27, 27

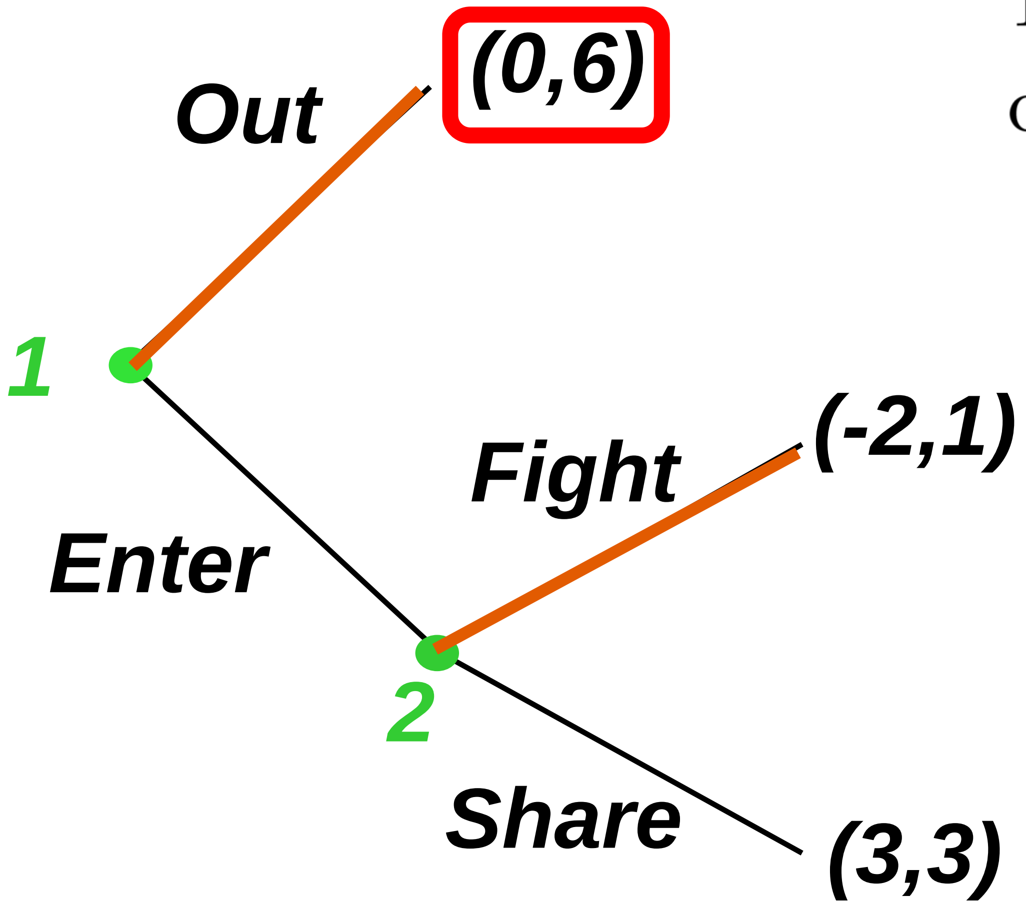
What if MORE rounds?

Sequential Move Games

- T Stages
- Agent $i = i_t \in \mathcal{I}$ moves in stage t
- History prior to stage t observed by i : h_i^{t-1}
- Set of possible pure strategies in stage t is S_t
- Strategy Profile: $s = (s_1, \cdots, s_T)$
- (Expected) Payoffs: $u_i(s)$ depends on s

- Exists other Nash equilibrium not solved by BI…

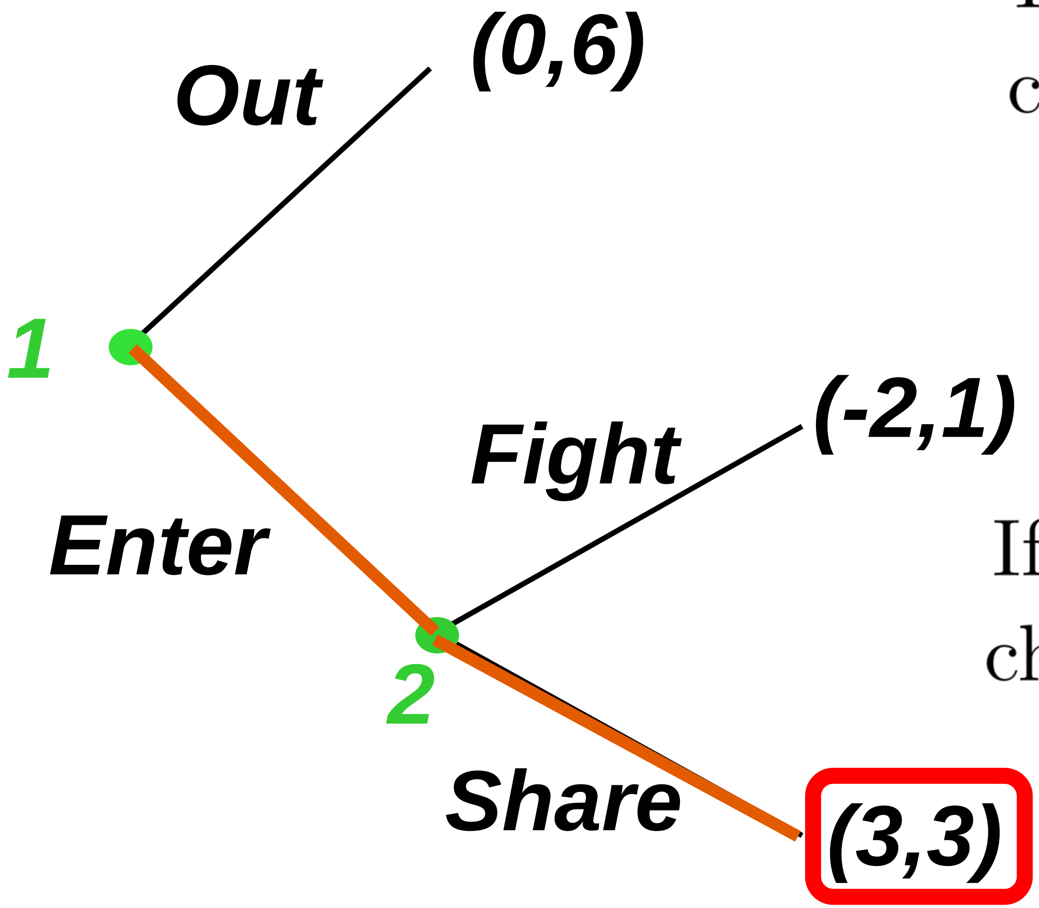
Entry Game with Sub-game (Selten's Chain Store Paradox)



If $s_2 = Fight$

choose $s_1 = Out$

Entry Game with Sub-game (Selten's Chain Store Paradox)



If $s_2 = Fight$

choose $s_1 = Out$

If $s_2 = Share$

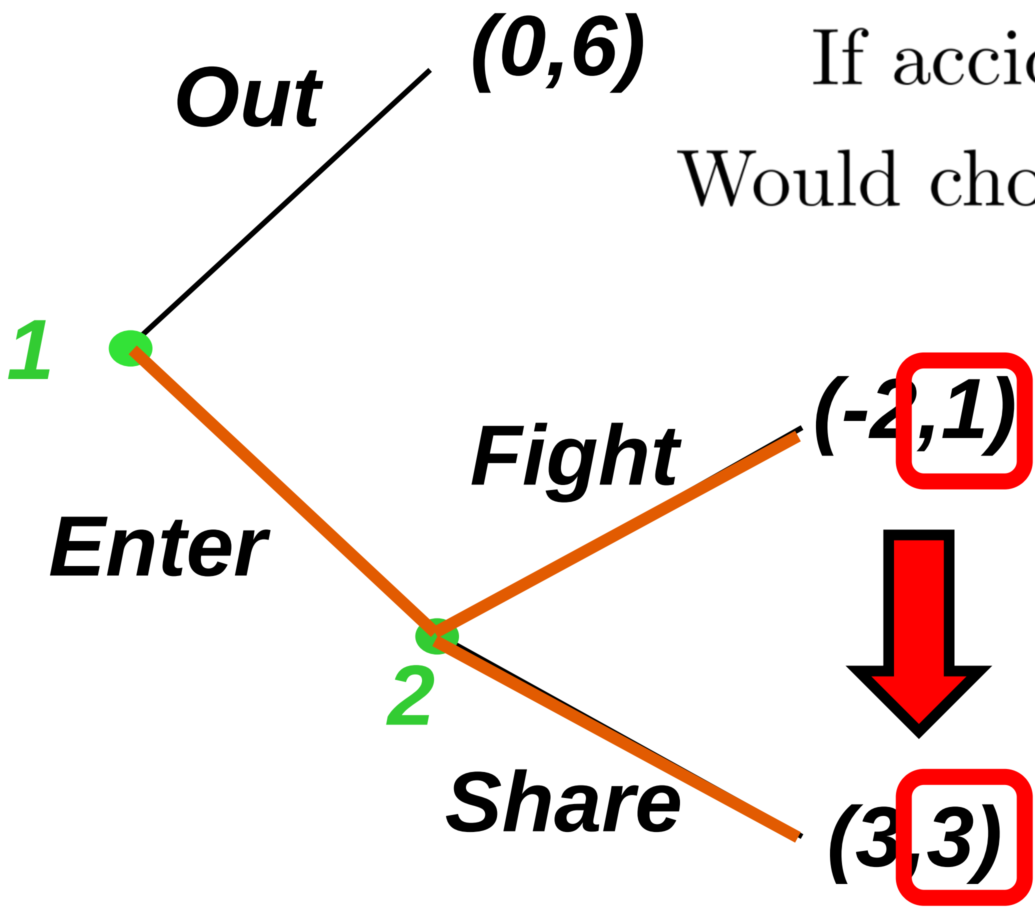
choose $s_1 = Enter$

Entry Game with Sub-game (Selten's Chain Store Paradox)

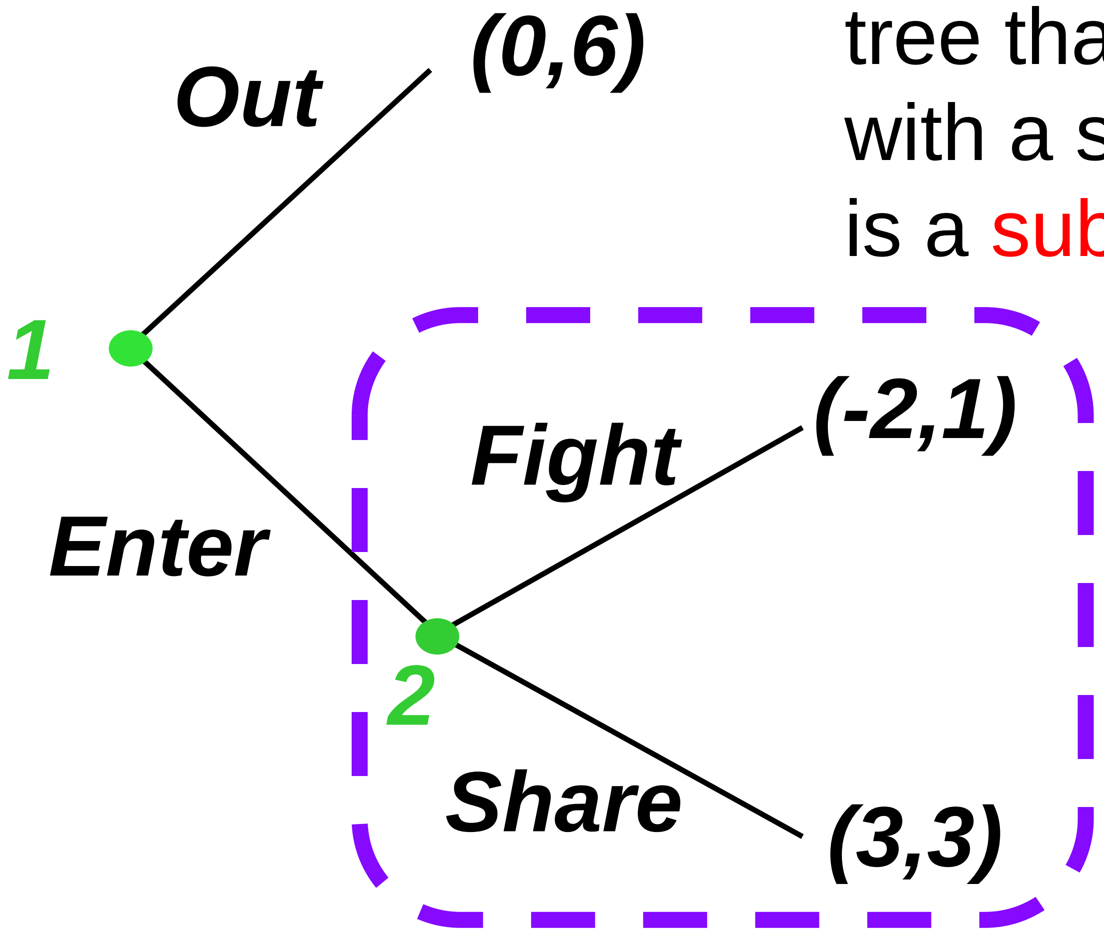
But $(Out, Fight)$ is **not credible**:

If accidentally $Enter$,

Would choose $s_2 = Share$!!



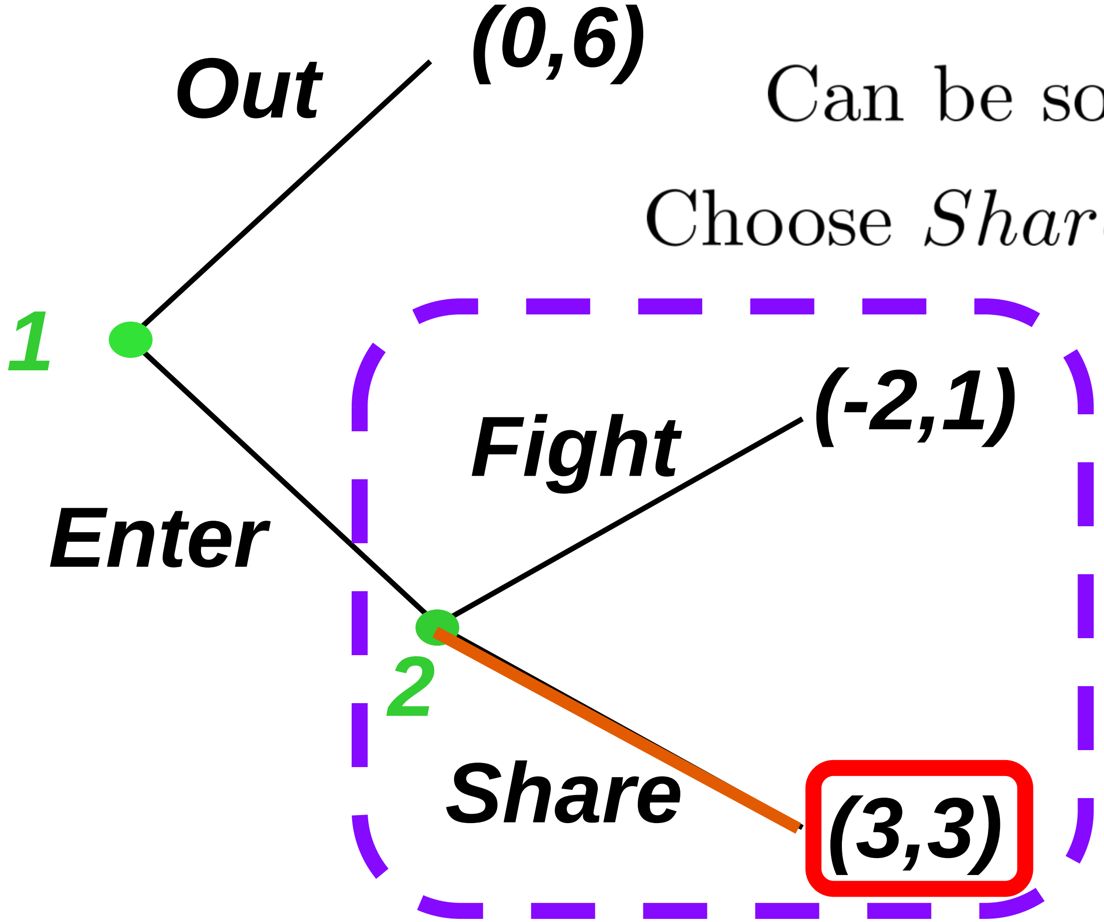
Definition of a Sub-game



- "Branch of game tree that begins with a single node is a sub-game

Definition of Sub-game Perfect Equilibrium

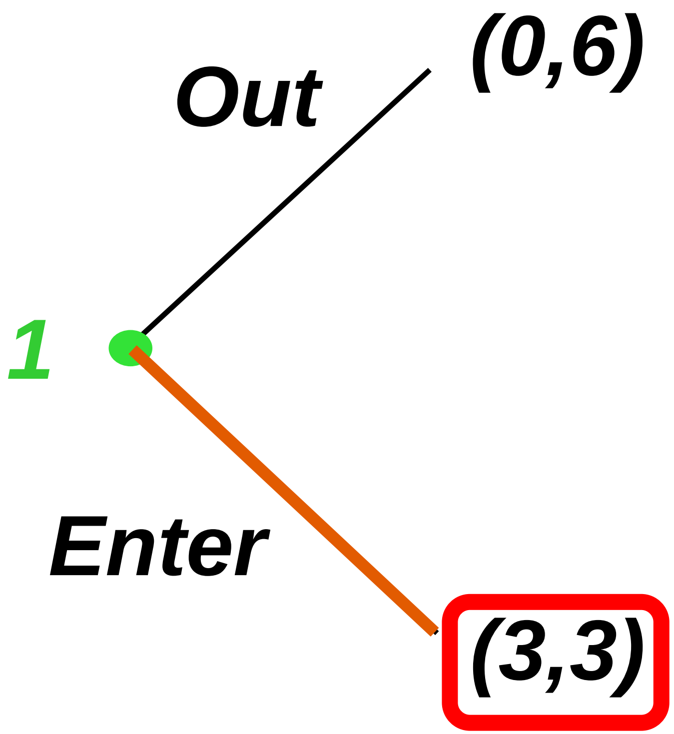
- SPE: Strategy must be Nash in all sub-game!



Can be solved by **BI**...

Choose *Share* is sub-game

Sub-game Perfect Equilibrium of the (reduced) Entry Game



- Reduced entry game (with payoffs from the sub-game)

$$\text{choose } s_1 = Enter$$

- Unique SPE is *(Enter, Share)*

Summary of 9.2

- Finitely Repeated Games
  - Equilibrium Threat and Efficiency
- Sequential Move Game
- Sub-game Perfect Equilibrium
  - Solved by Backward Induction
- HW 9.2: Riley – 9.2-1~3