Games and Strategic Equilibrium

Joseph Tao-yi Wang

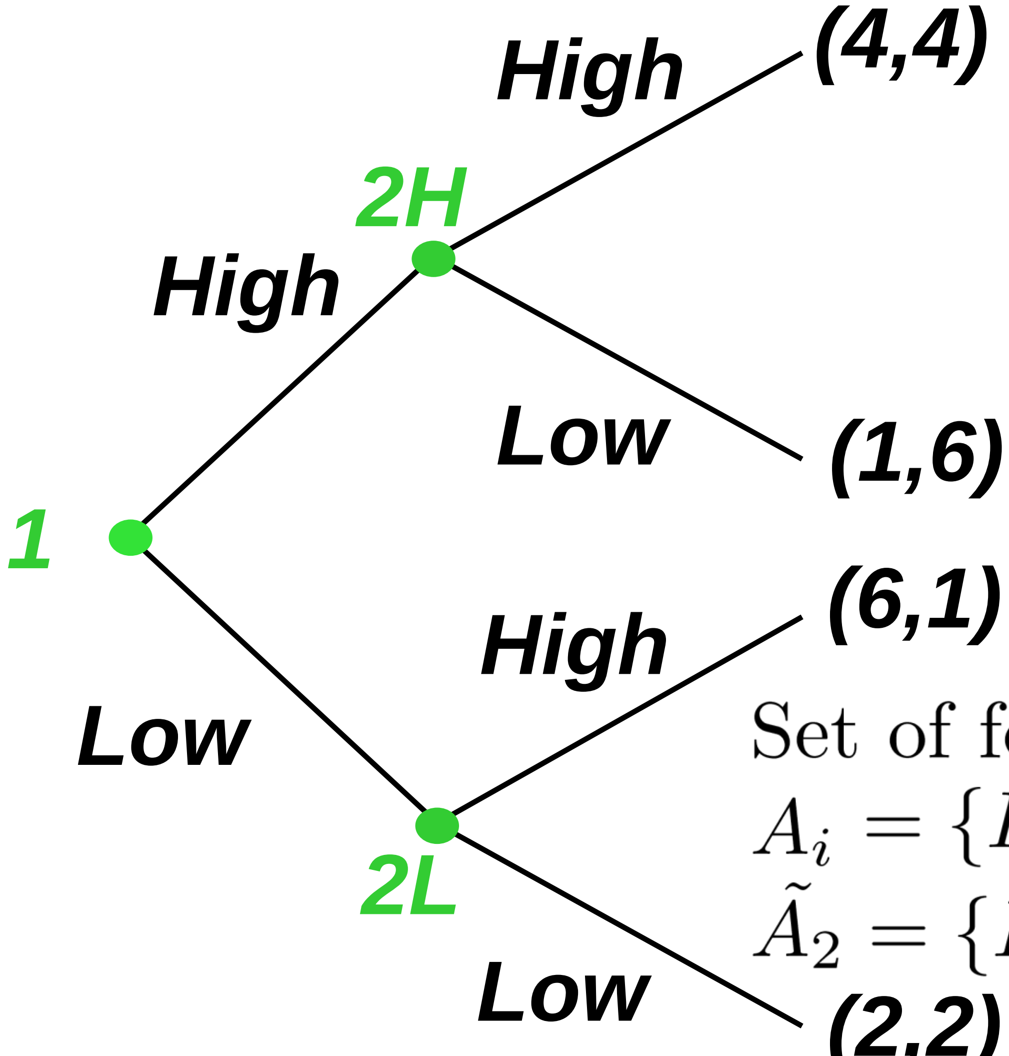
2010/9/17

(Lecture 2, Micro Theory I-2)

What is a Game?

- Example: Two competing firms
- Agents i = manager of firm $i = 1, 2$
- Post next week's price on Sunday Times
  - High price or Low price
- Agent 1 sets price first
  - Sunday Times posts price online instantly; Agent *2* sees opponent's price before setting own price
- Represent game as a game tree

What is a Game?



Set of agents

$\mathcal{I} = \{1, 2\}$

Action a_i

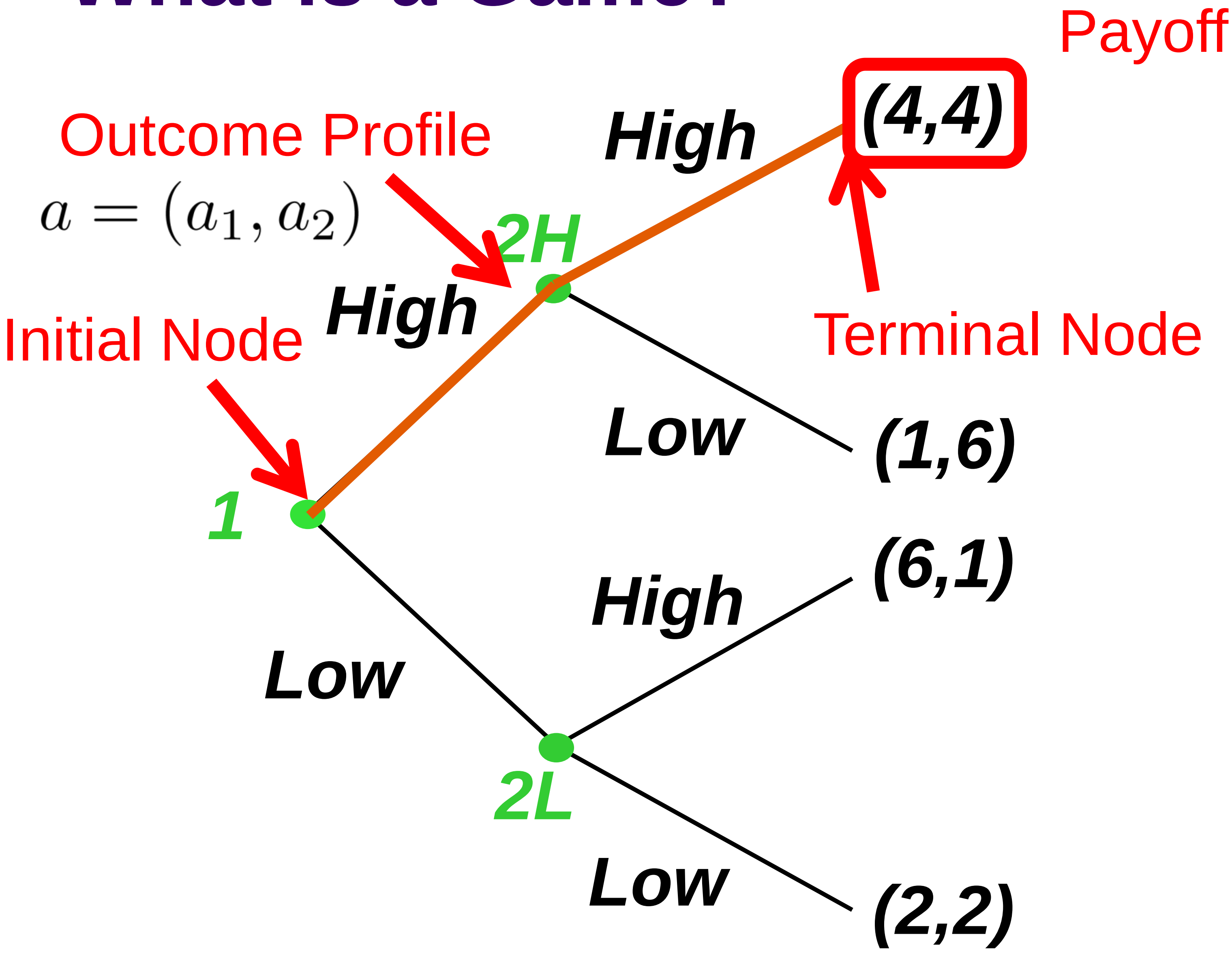
$a_2 = \tilde{a_2}(a_1)$

Set of feasible actions

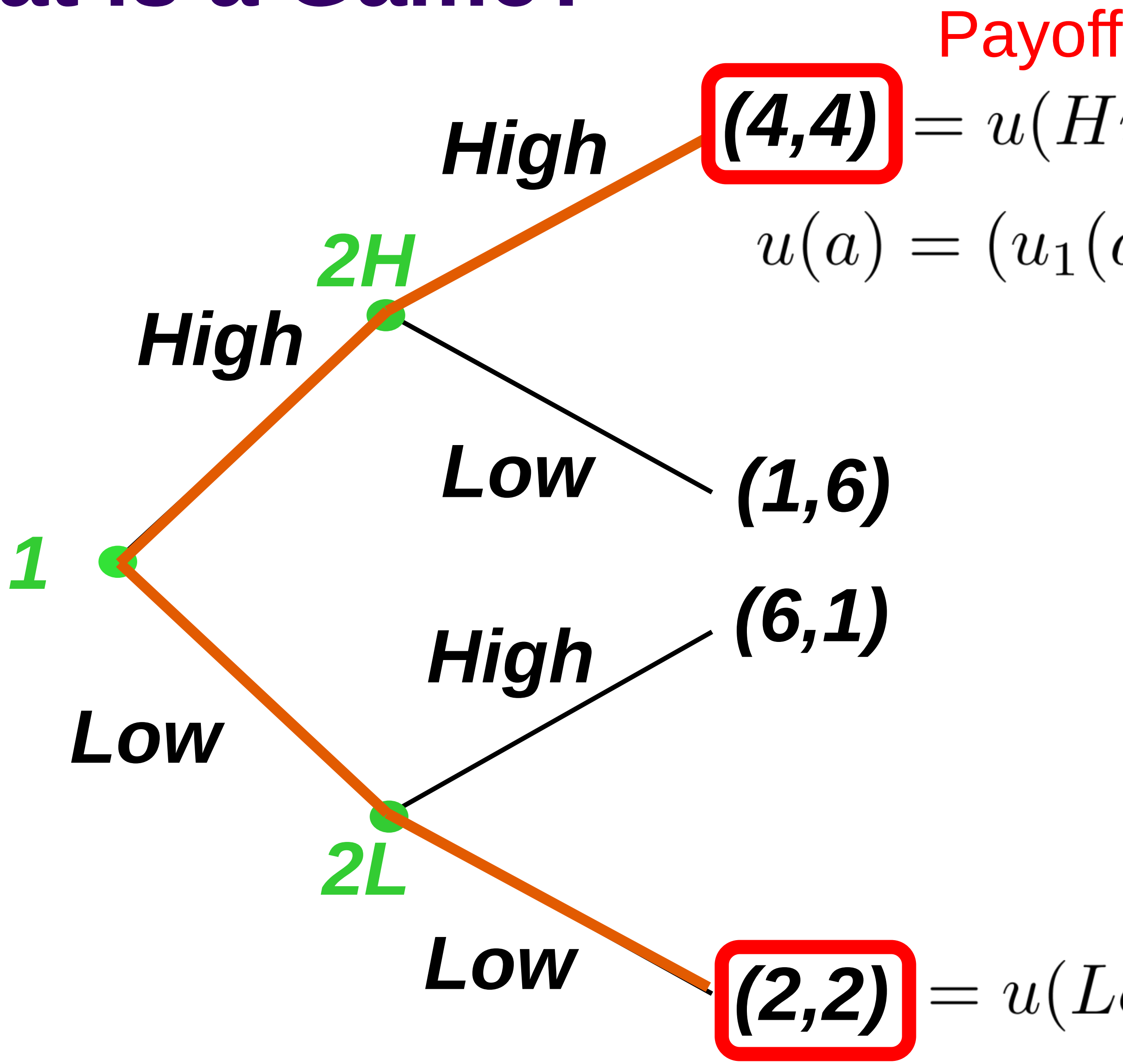
$A_i = \{High, Low\}$

$\tilde{A}_2 = \{HH, HL, LH, LL\}$

What is a Game?



What is a Game?

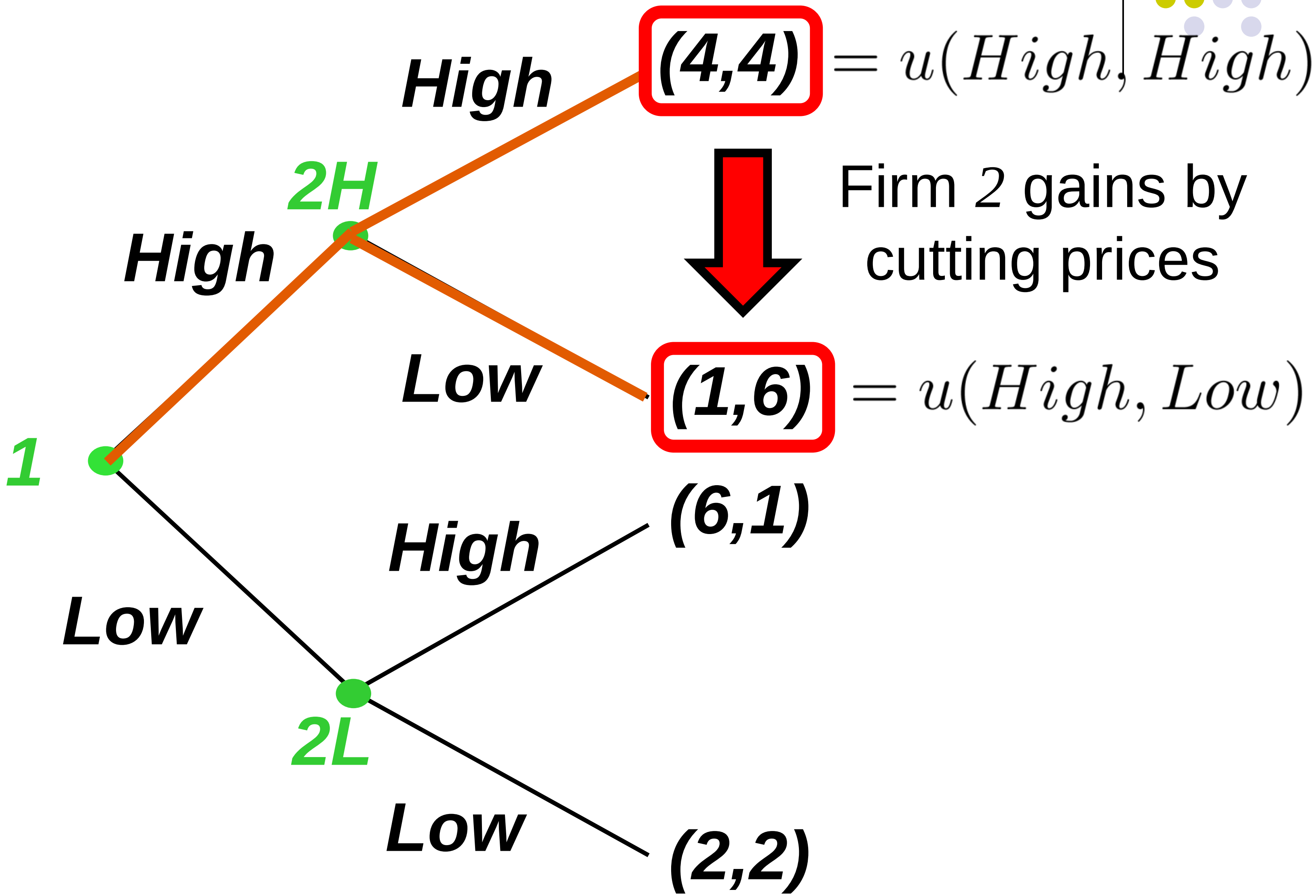


$(4,4) = u(High, High)$

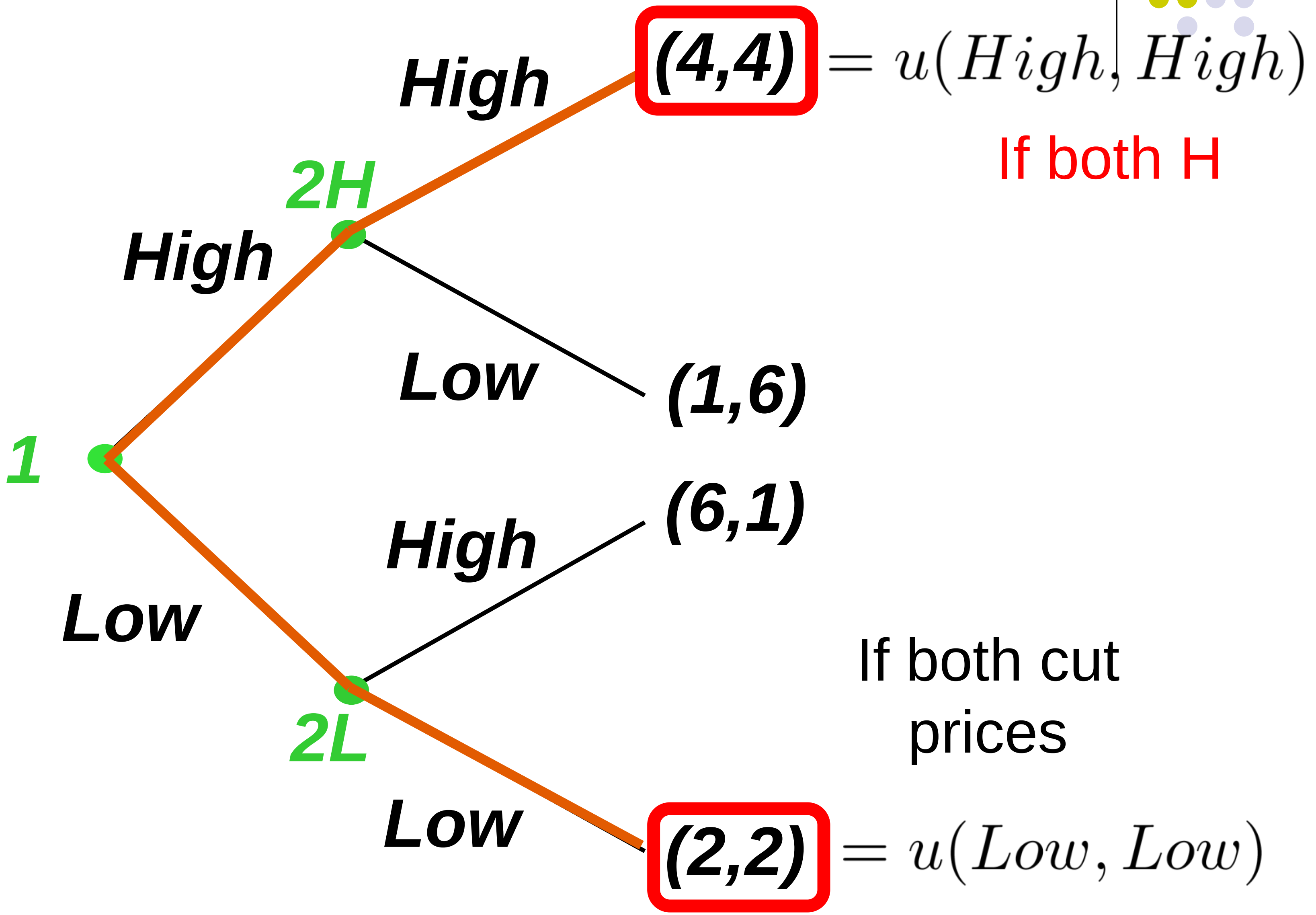
$u(a) = (u_1(a), u_2(a))$

$(2,2) = u(Low, Low)$

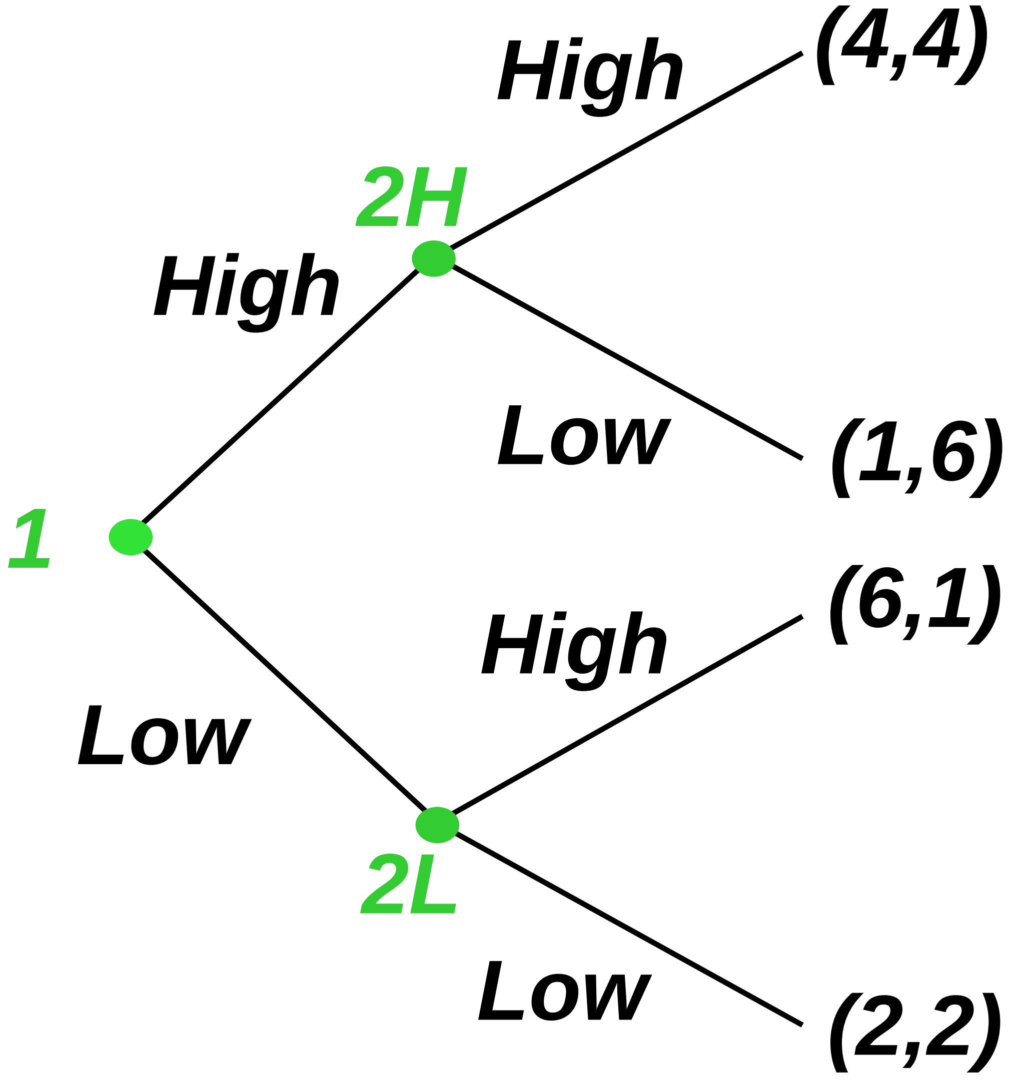
What is a Game?



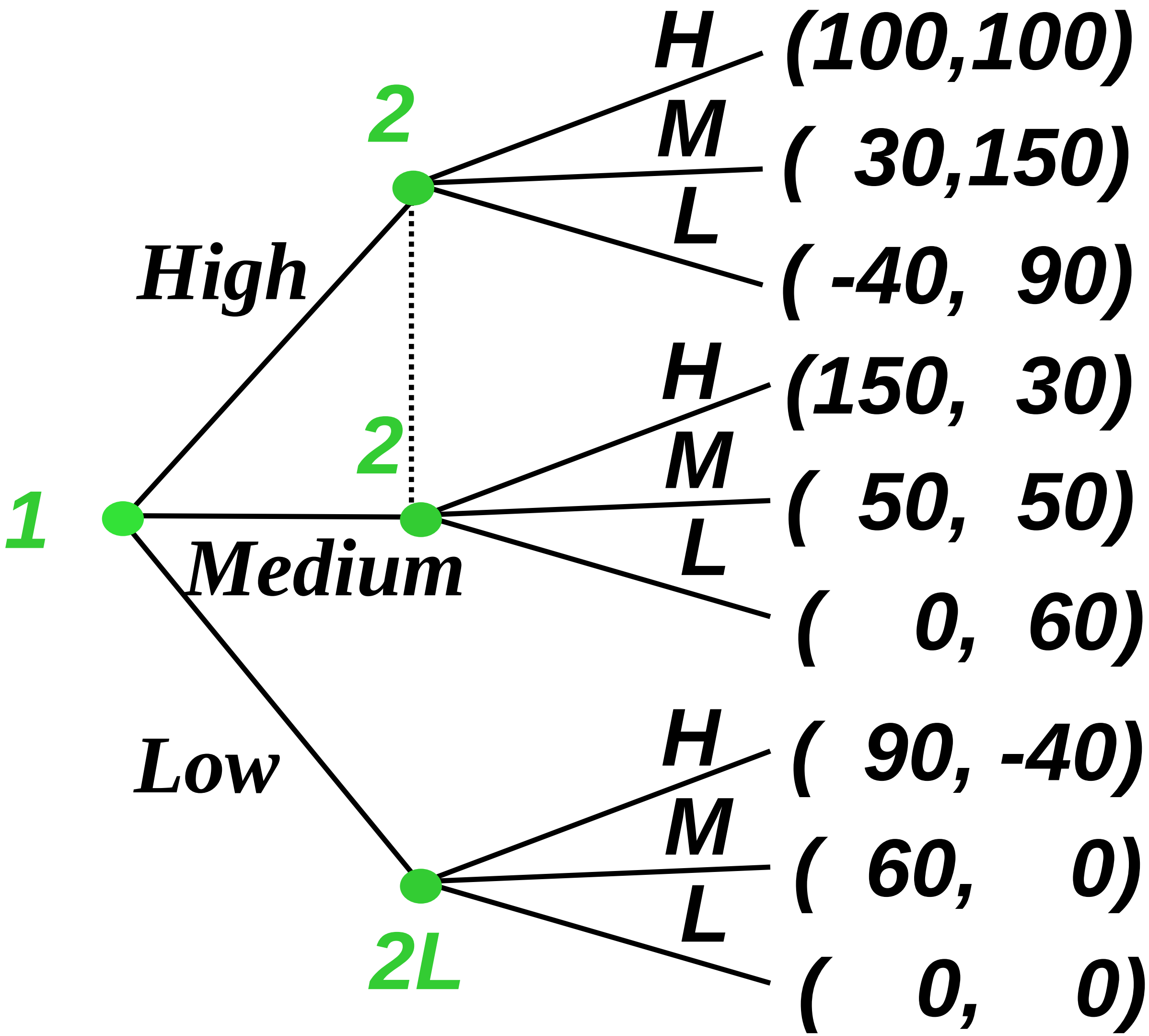
What is a Game?



Extensive Form of the Game

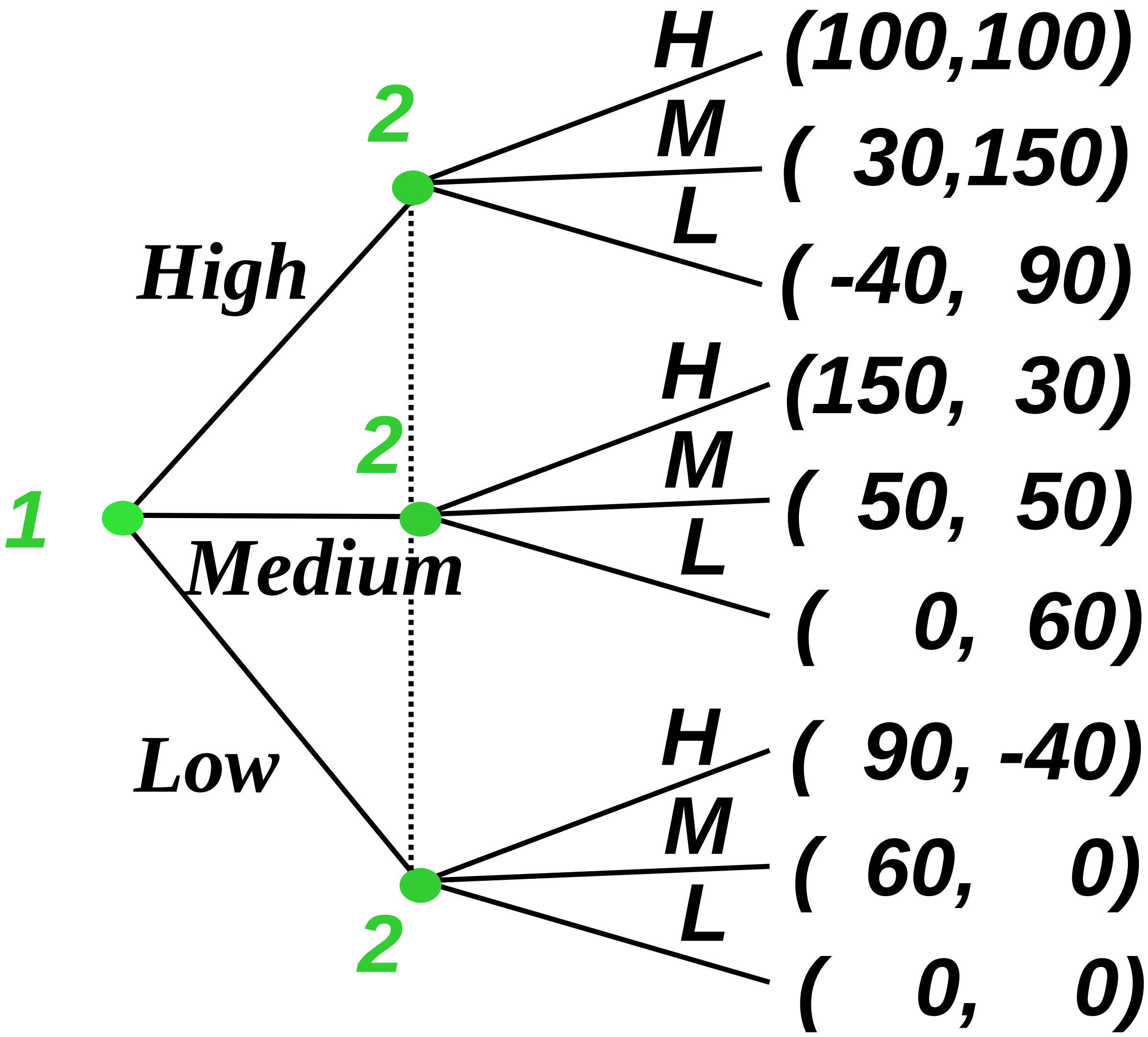


Other Extensive Form Games



- Action: *H, M, L*
- Only posts online if *Low*
- Information Set
- {2H, 2M}, {2L}

Special Case: All Actions Hidden



- Action: *H, M, L*
- Nothing posted online
- Information Set
- {2H, 2M, 2L}

Strict and Weak Dominace

- Set of opponent action space $A_{-i} = \bigotimes_{j \neq i} A_j$
- For agent i,

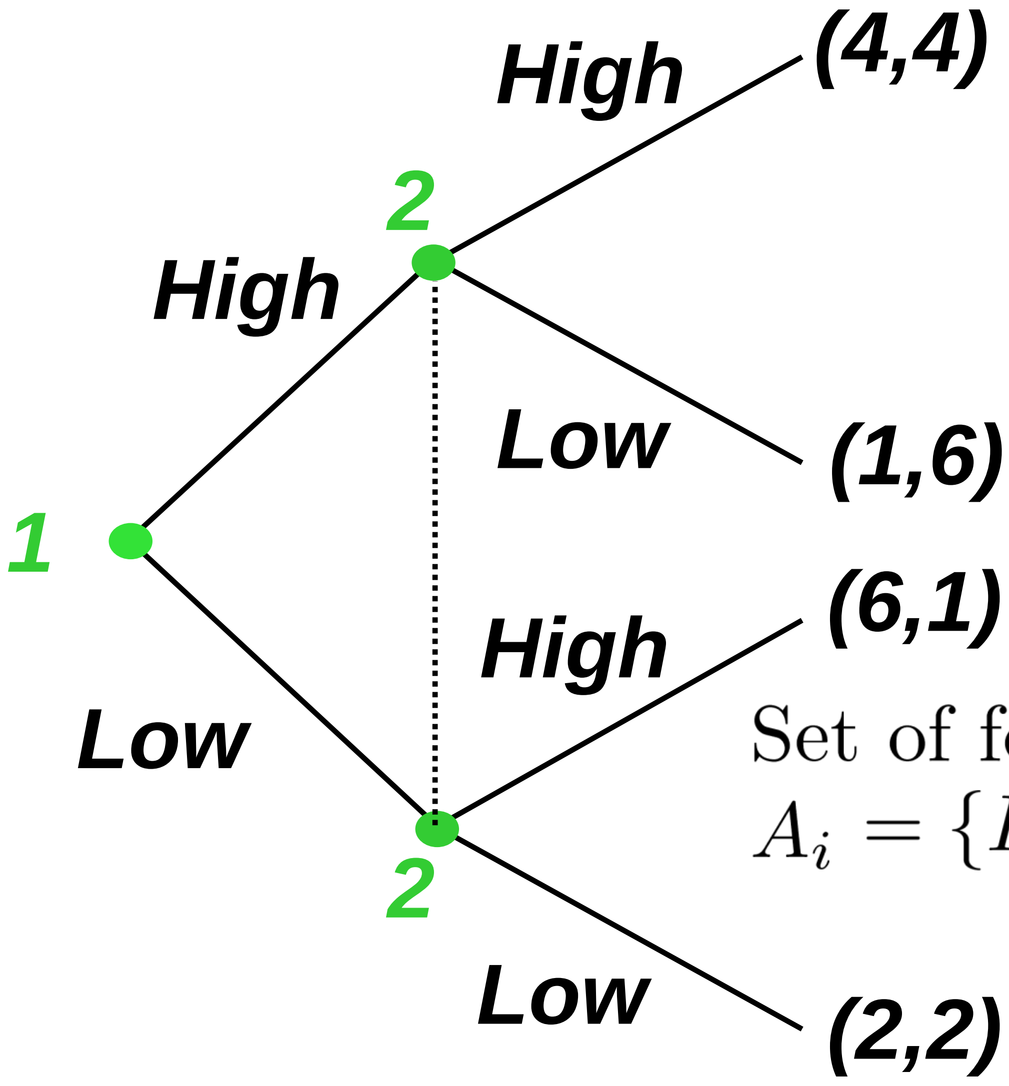
a_i is **strictly dominated** by $\overline{a_i}$ if

$$u_i(\overline{a_i}, a_{-i}) > u_i(a_i, a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

a_i is **weakly dominated** by $\overline{a_i}$ if

$$u_i(\overline{a_i}, a_{-i}) \geq u_i(a_i, a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$
$$u_i(\overline{a_i}, a_{-i}) > u_i(a_i, a_{-i}) \text{ for some } a_{-i} \in A_{-i}$$

Strict and Weak Dominance

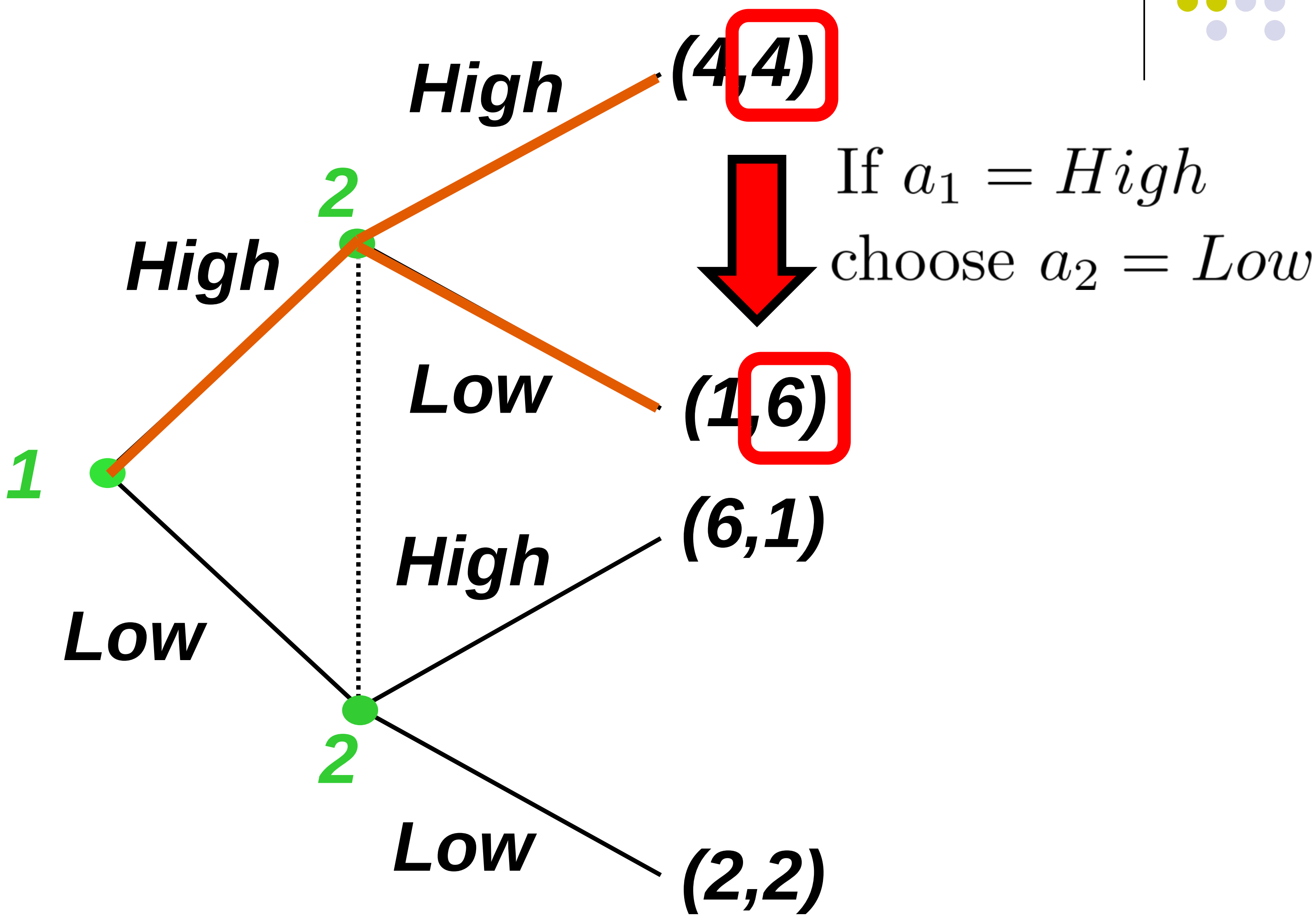


Set of agents
$\mathcal{I} = \{1, 2\}$

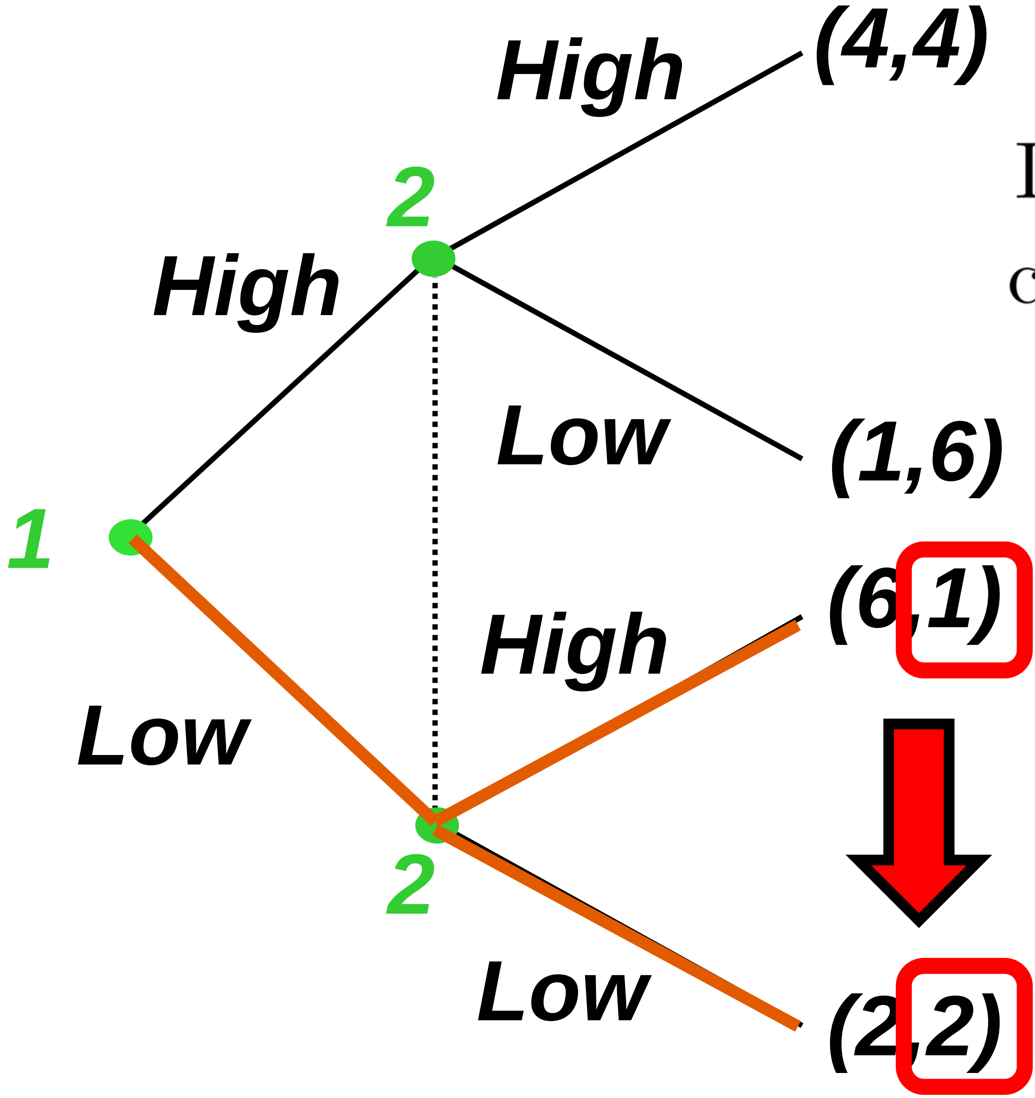
Action a_i

Set of feasible actions
$A_i = \{High, Low\}$

Strict and Weak Dominance



Strict and Weak Dominance

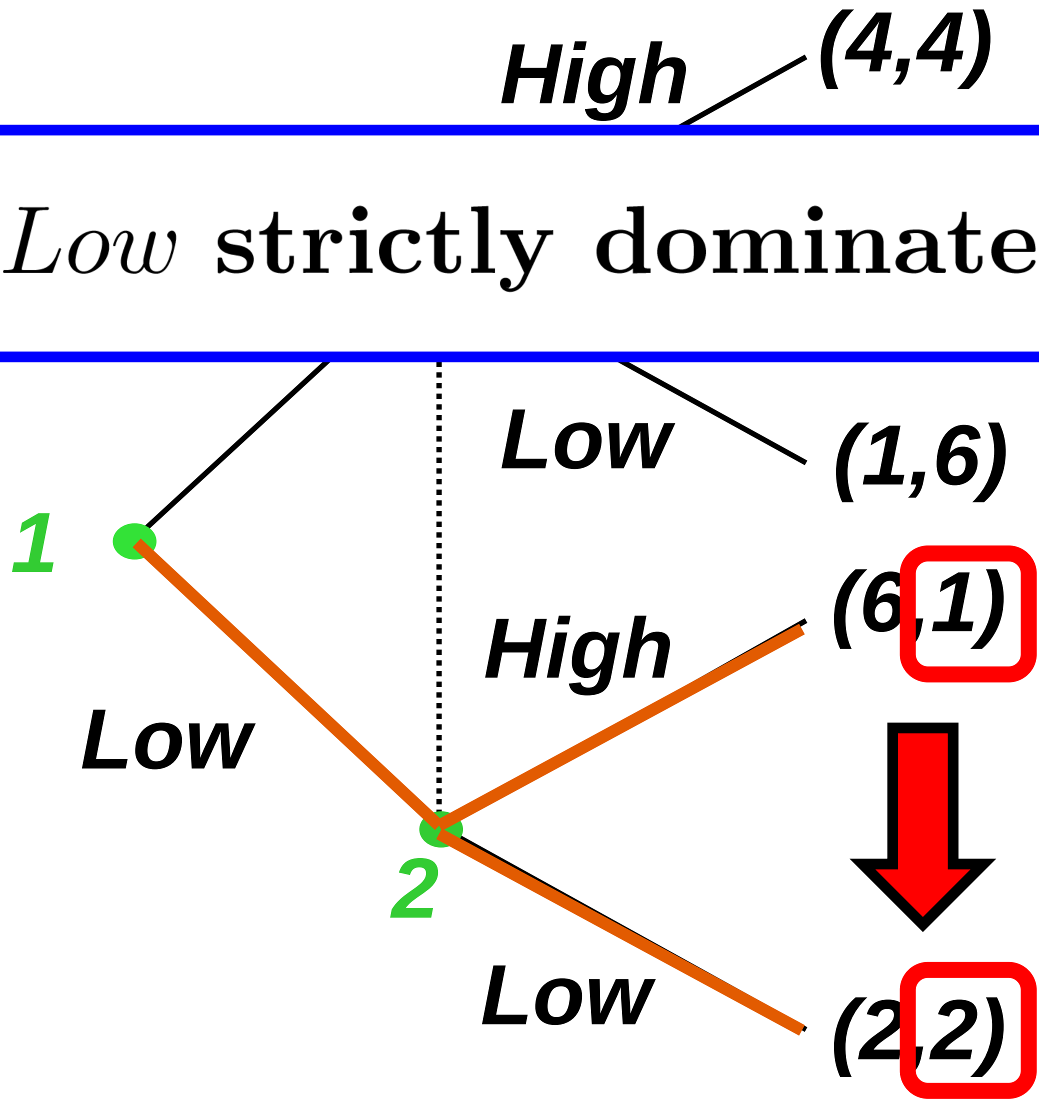


If $a_1 = High$ choose $a_2 = Low$

If $a_1 = Low$ choose $a_2 = Low$

Strict and Weak Dominance

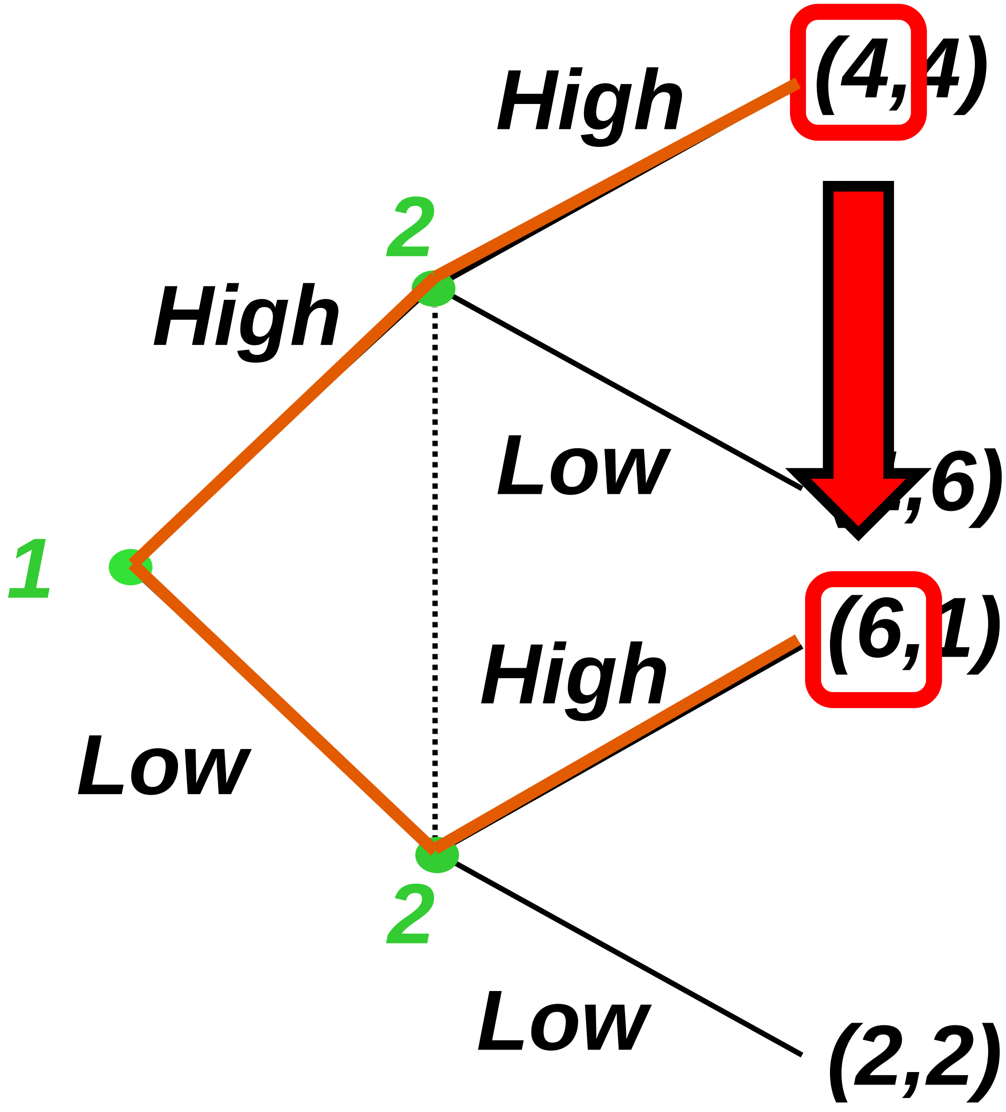
$a_2 = Low$ **strictly dominates** $a_2 = High$



If $a_1 = Low$

choose $a_2 = Low$

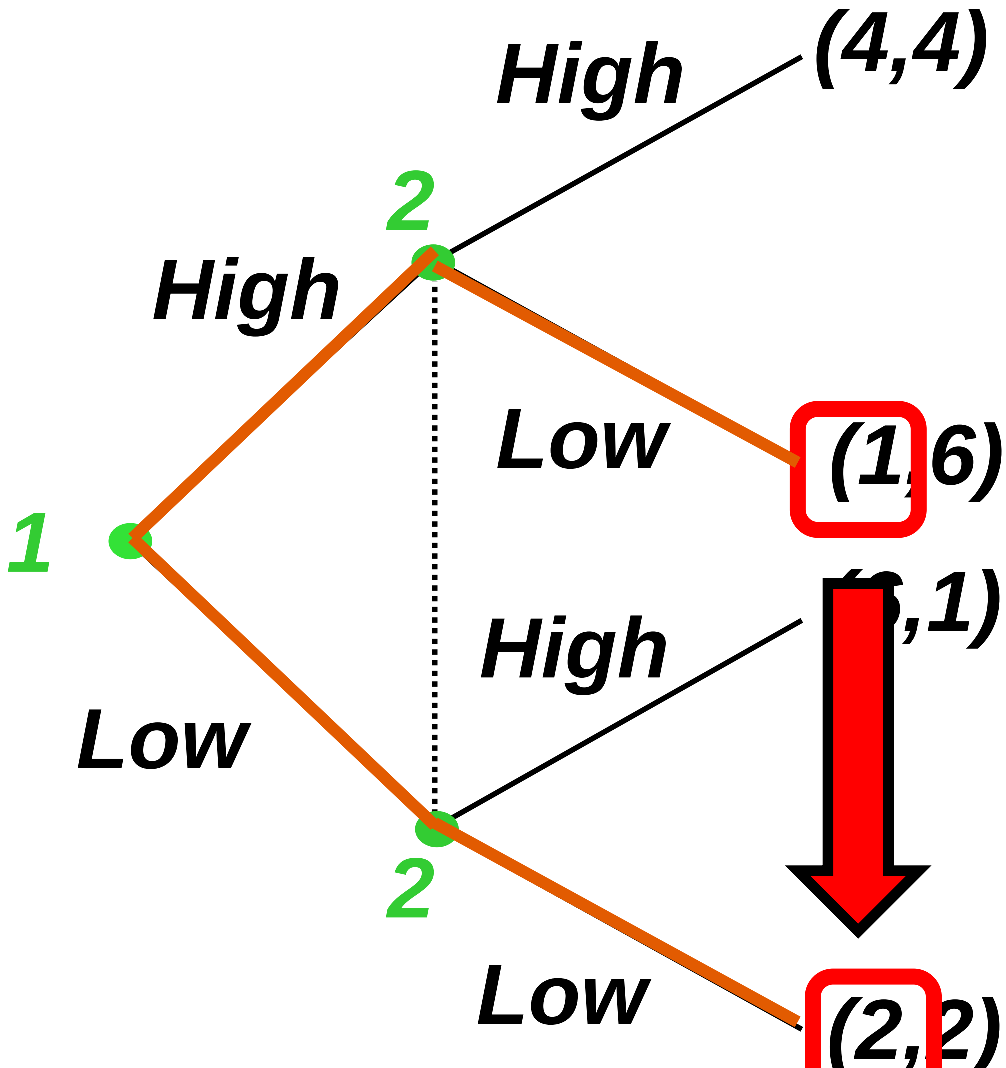
Strict and Weak Dominance



If $a_2 = High$

choose $a_1 = Low$

Strict and Weak Dominance

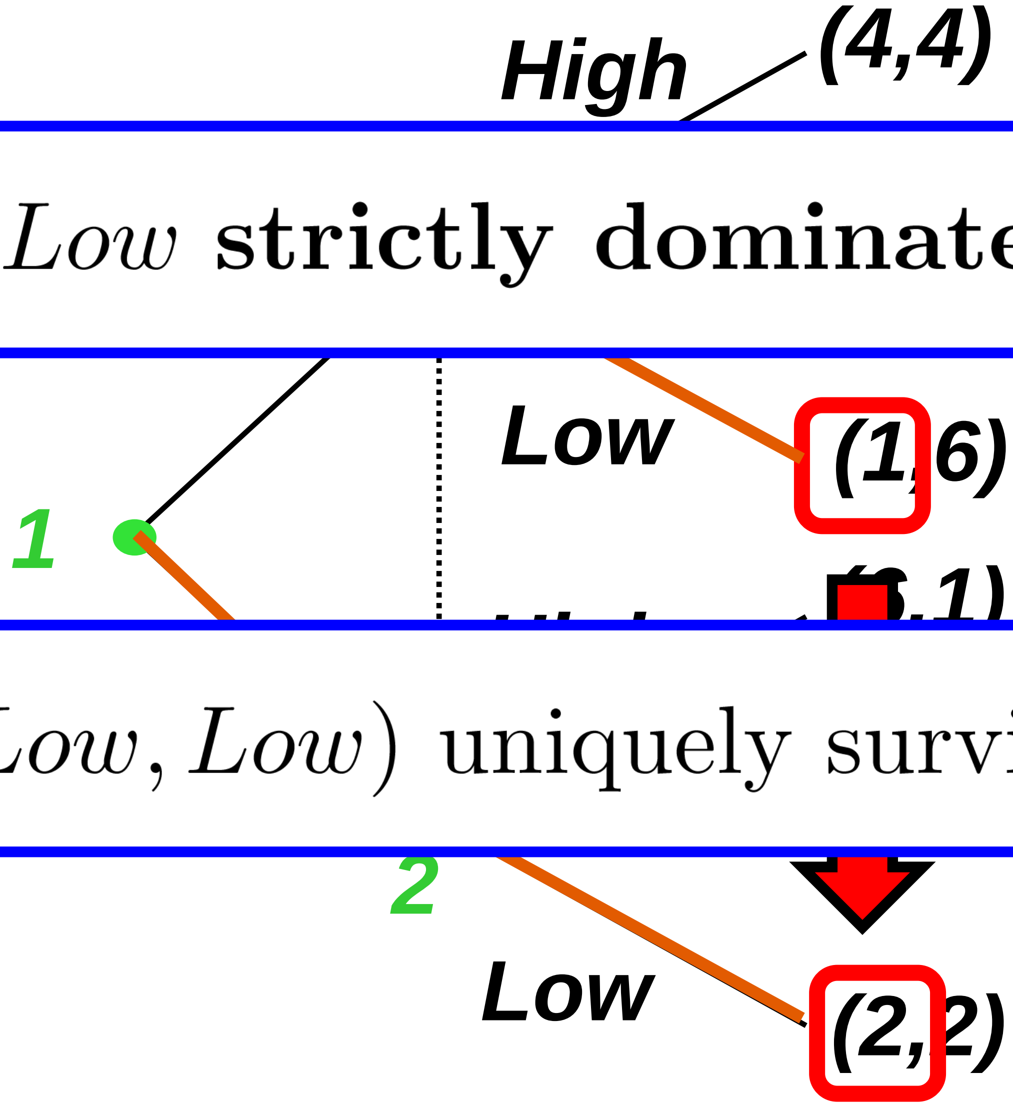


If $a_2 = Low$

choose $a_1 = Low$

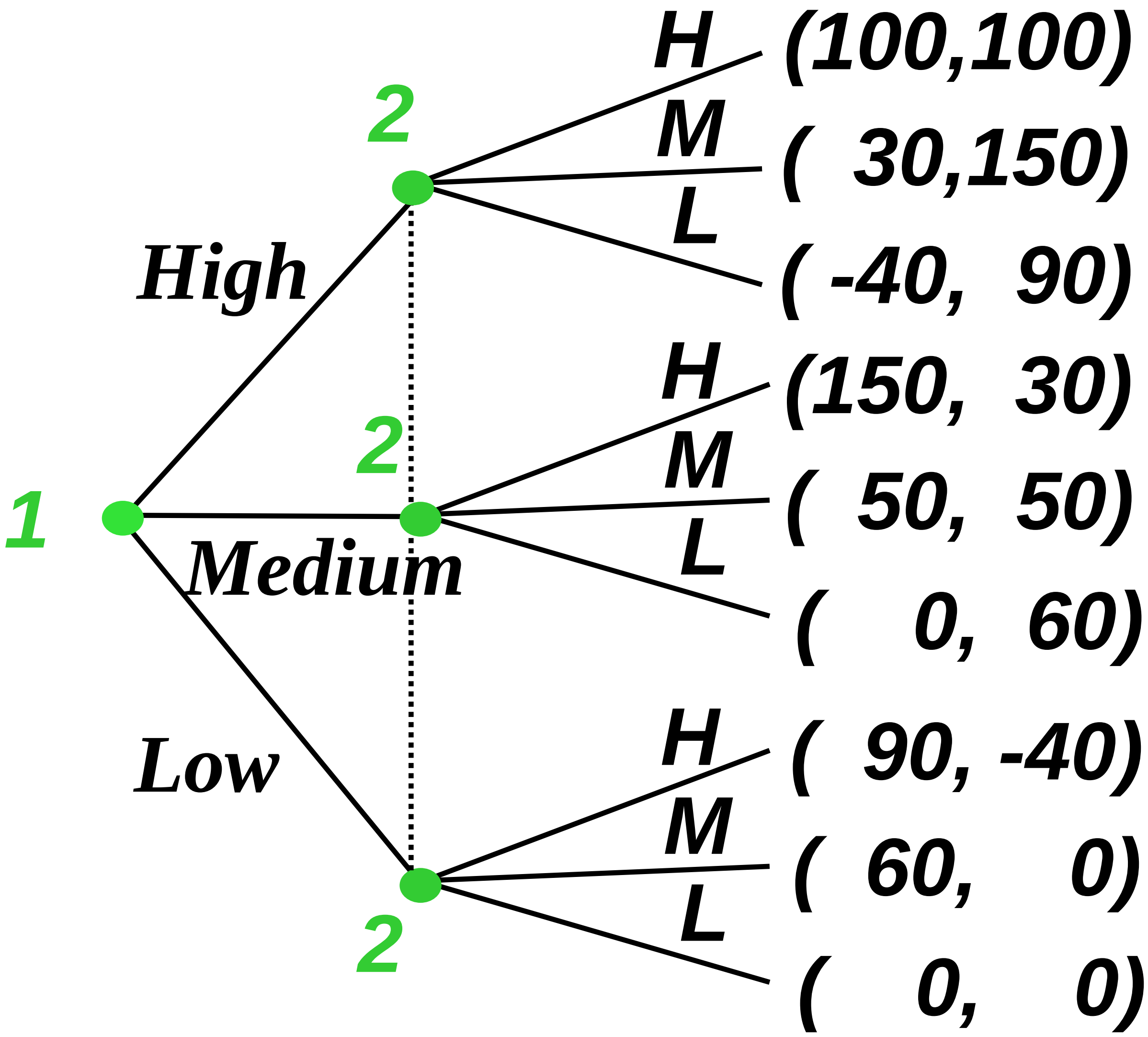
Strict and Weak Dominance

$$a_1 = Low \textbf{ strictly dominates } a_1 = High$$



$$(Low, Low) \text{ uniquely survives EDS}$$

Simultaneous Game: Extensive Form



- Action: *H, M, L*
- Nothing posted online
- Information Set
- {2H, 2M, 2L}

Simultaneous Game: Strategic Form (Normal Form)

High strictly dominated by *Medium*

Player 2: Colin

Player 1: Rowena

	High	Medium	Low
High	100, 100	30,150	-40, 90
	^	^	^
Medium	150, 30	50, 50	0, 60
Low	90, -40	60, 0	0, 0

Elimination of Dominated Strategies (EDS)

Medium weakly dominated by *Low*

Player 2: Colin

Player 1: Rowena

	Medium	Low
Medium	50, 50	0, 60
Low	60, 0	0, 0

Iterative Elimination of Dominated Strategies

Player 2: Colin

$$(Low, Low) \text{ uniquely survives IEDS}$$

		Low
Player 1: Rowena	Low	0, 0

Mixed Strategy and Dominance

(2/3,1/3)-mixture of *(Middle, Down)* weakly dominates *Up*

Player 2: Colin

Player 1: Rowena			Left	Right
		~~Up~~	~~0, 6~~	~~-1, 8~~
	2/3	Middle	-2, 1	4, 0
	1/3	Down	4, 2	-8, 1

Mixed Strategy and IEDS

Left strictly dominates *Right*

Down strictly dominates *Middle*

Player 2: Colin

Player 1: Rowena

	Left	Right
Middle	-2, 1	4, 0
Down	4, 2	-8, 1

Equilibrium of "One-Shot" Simultaneous Game

- Each Agent $i \in \mathcal{I}$
- Has finite Action Set $A_i = \{a_{i1}, a_{i2}, \cdots, a_{im}\}$
- Agent i's Strategy Set

$$S_i = \Delta(A_i) = \left\{ \pi \middle| \pi \geq 0, \sum_{j=1}^{m_i} \pi_j = 1 \right\}$$

- Mixed Strategy: $\pi_i(a_i)$
- Strategy Profile:

$$s = (s_1, \cdots, s_I) \in S = S_1 \times \cdots \times S_I$$

Equilibrium of "One-Shot" Simultaneous Game

- Consequence of the game (for agent *i*): $\pi_i(a)$
- Outcome of the game (for agent *i*): $x_i(a)$
- Agent *i*'s Expected Utility

$$u_i = \sum_{a \in A} \pi_i(a) v_i(x_i(a)) = u_i(a) \cdot \pi_(a)$$

- Mixing in Continuous Action Space: $\mu_i \in \Delta(A_i)$
- Expected Utility in Continuous Action Space:

$$u_i(s) = \int_{a \in A} u_i(a) d\mu(a)$$

Nash Equilibrium

- Strategy Profile: $s \in S = \Delta_1(A_1) \times \cdots \times \Delta_I(A_I)$
- Best Response: $BR_i(s_{-i})$
- Best Response Mapping:

$$BR(s) = (BR_1(s_{-1}), \cdots, BR_I(s_{-I}))$$

- Nash Equilibrium: s such that $BR(s) = s$
  - Fixed Point in the BR mapping
- Consider a strategy profile $\overline{s} = (\overline{s}_1, \cdots, \overline{s}_I)$
- Is there any other strategy strictly better for agent i (if others play according to $\overline{s}_{-i}$)

Nash Equilibrium

- For simultaneous game played by agents *1~I*
- The strategy profile $\overline{s} = (\overline{s}_1, \cdots, \overline{s}_I)$ is a Nash Equilibrium if the strategies are mutual BR.
- In other words,
- For each $i \in \mathcal{I}$ and all $a_i \in A_i$

$$u_i(\overline{s}_i, \overline{s}_{-i}) \geq u_i(a_i, \overline{s}_{-i})$$

  - Note that you only need to check pure strategies since mixed strategies yield a weighted average of payoffs among pure strategies

Nash Equilibrium: Partnership Game

- Two Agents have equal share in a partnership
- Choose Effort: $a_i \in A_i = \{1, 2, 3\}$
- Total revenue: $R = 12a_1a_2$
- Cost to agent i: $C_i(a_i) = a_i^3$
- Payoff: $u_i(s) = R - C_i(a_i) = 12a_1a_2 - a_i^3$

- Game matrix and Nash Equilibrium…

Nash Equilibrium: Partnership Game

1 is a BR if other picks *1*

2 is a BR if other picks *2* or *3*

Player 2: Colin

Player 1: Rowena		1	2	3
	1	5, 5	11, 4	17, -9
	2	4, 11	16, 16	28, 9
	3	-9, 17	9, 28	27, 27

Nash Equilibrium: Partnership Game

1 is a BR if other picks *1*

2 is a BR if other picks *2* or *3*

Player 2: Colin

Player 1: Rowena		1	2	3
	1	5, 5	11, 4	17, -9
	2	4, 11	16, 16	28, 9
	3	-9, 17	9, 28	27, 27

Nash Equilibrium: Partnership Game

$(1,1)=BR(1,1)$

$(2,2)=BR(2,2)$

Player 2: Colin

Player 1: Rowena		1	2	3
	1	5, 5	11, 4	17, -9
	2	4, 11	16, 16	28, 9
	3	-9, 17	9, 28	27, 27

Nash Equilibrium: Partnership Game

- This is NOT the only two NE
- Solve for MSE:
- For $s_2 = (p, 1-p, 0) \in \Delta(A_2)$

$$u_1(1, s_2) = 5p + 11(1-p) = 11 - 6p$$

- $= u_1(2, s_2) = 4p + 16(1-p) = 16 - 12p$
- Hence,

$$p = \frac{5}{6}$$

- By symmetry, MSE is $s_1 = s_2 = \left(\frac{5}{6}, \frac{1}{6}, 0\right)$

Common Knowledge

- Common Knowledge of the Game
- Common Knowledge of Rationality
- Common Knowledge of Equilibrium

- Exercise: Is “九二共識” truly a consensus in terms of common knowledge?

Existence of Equilibrium

- Use: Kakutani's Fixed Point Theorem (FPT)

  If $S \subseteq \mathbf{R}^n$ is **closed, bounded & convex** and if ϕ is an **upper hemi-continuous** correspondence from S to S, such that

  $\phi(s)$ is non-empty and convex, then

  $\phi(s)$ has a **fixed point.**

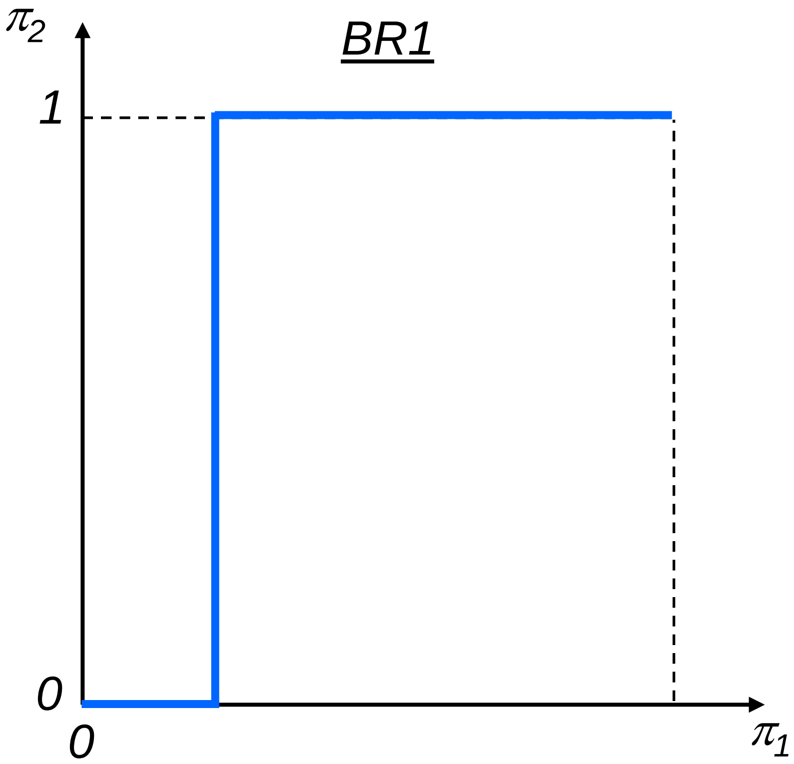
- Proposition 9.1-1: Existence of NE (Nash, 1950)
- In a game with finite action sets, if players can choose either pure or mixed strategies, there exists a Nash Equilibrium.

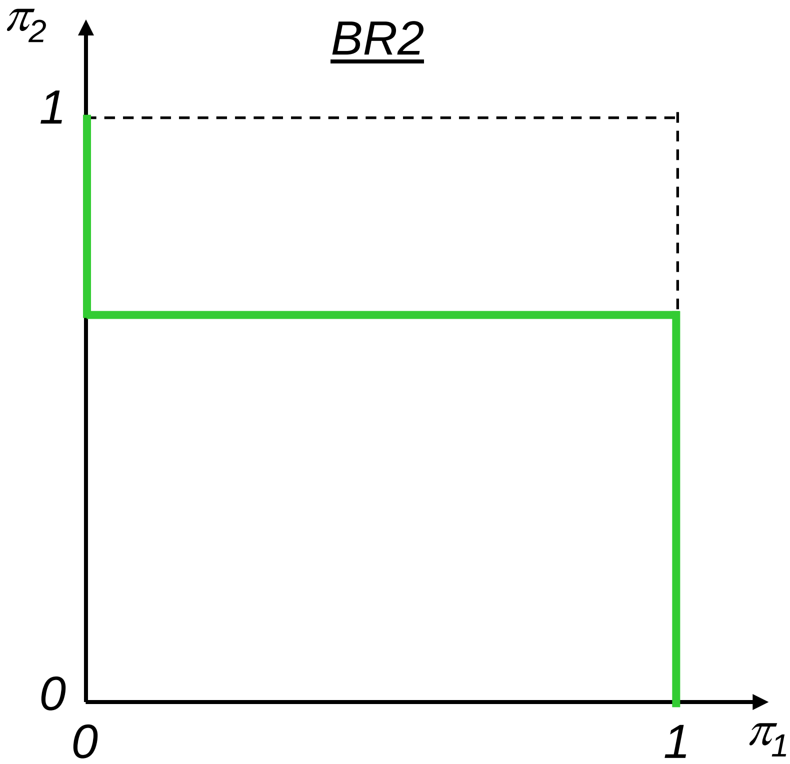
Existence of Equilibrium

- Consider the following simpler FPT:
  If $S_1, S_2 \subseteq \mathbf{R}$ is **closed, bounded and convex** and $\phi_1(s_2), \phi_2(s_1)$ are **continuous** functions from S_{-i} to S_i, then $\phi = (\phi_1, \phi_2)$ has a **fixed point.**

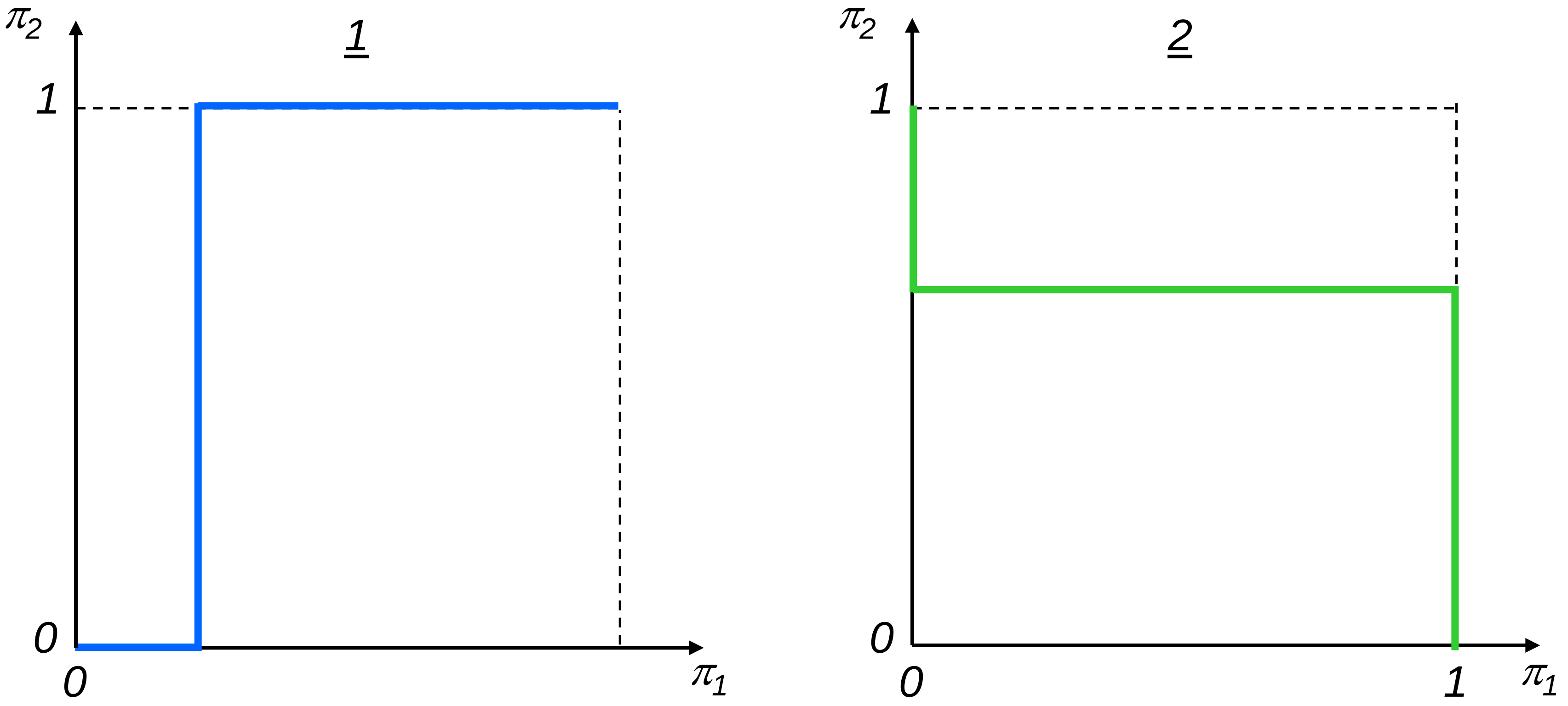
- Existence of Nash Equilibrium requires:
- Strategy sets are closed, bounded and convex,
- BR functions are indeed continuous…

Existence of Equilibrium



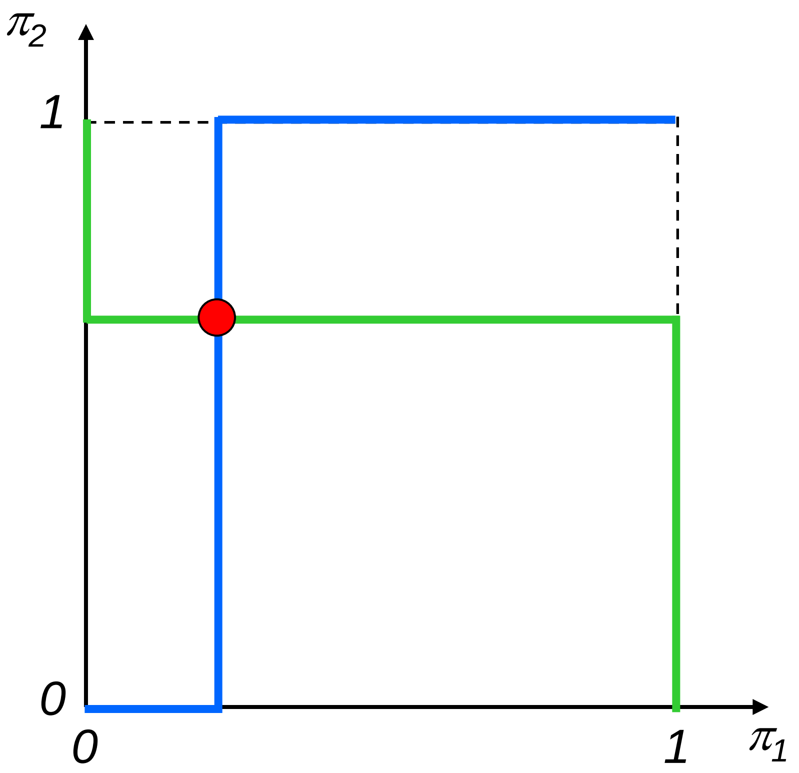


Existence of Equilibrium



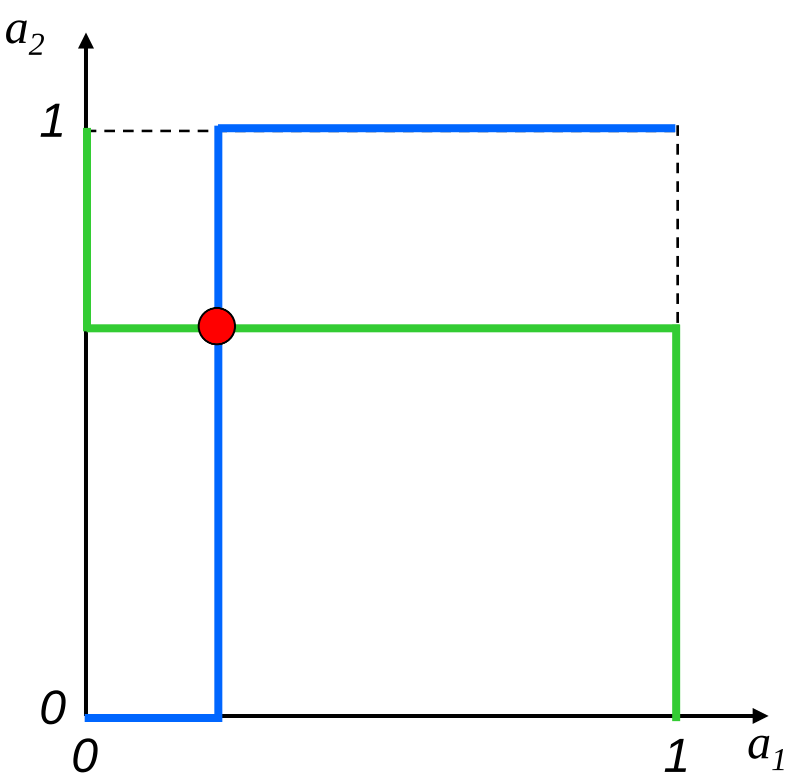
Existence of Equilibrium

Mixed-strategy NE in which player <u>1</u> plays Up with probability π_1 and player <u>2</u> plays Left with probability π_2.



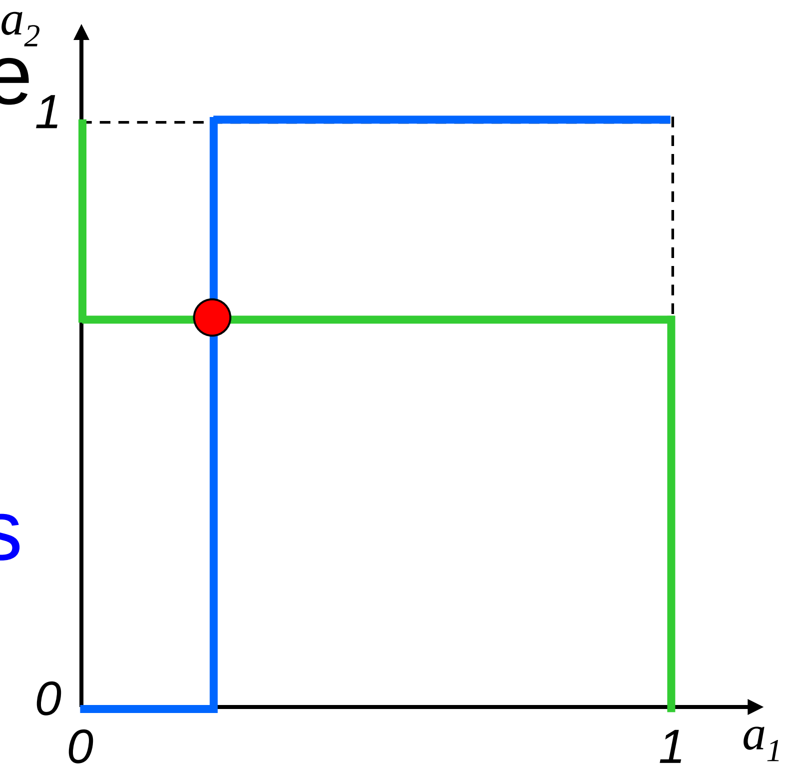
Existence of Equilibrium: For Continuous Action Space

For continuous action space (where each player chooses a pure strategy a_i), there exists a pure strategy NE in which player 1 plays a_1 and player 2 plays a_2.



Existence of Equilibrium: For Non-unique BR

- Why do we need Kakutani's FPT?
- Because best response may not be unique!!!
- BR correspondences,
  - Not only BR "functions"
- Upper hemi-continuous
  - Not "Continuous"



Existence of Equilibrium

- Use: Kakutani's Fixed Point Theorem (FPT)

If $S \subseteq \mathbf{R}^n$ is **closed, bounded & convex** and if ϕ is an **upper hemi-continuous** correspondence from S to S, such that

$\phi(s)$ is non-empty and convex, then

$\phi(s)$ has a **fixed point.**

- Closed and Bounded
- Convex
- Upper hemi-continuous

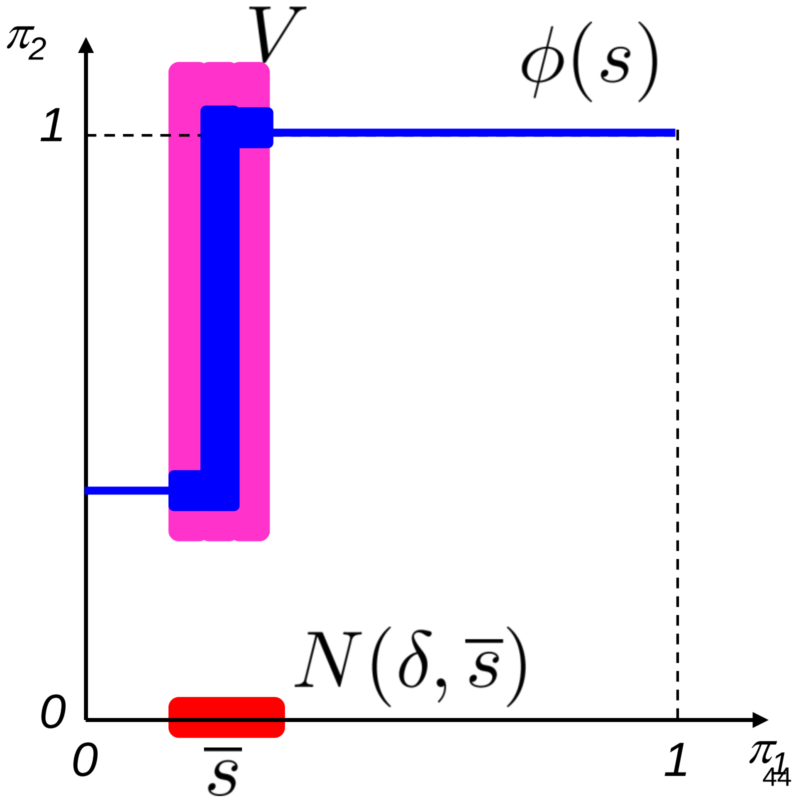
Existence of Equilibrium

- Closed
  $$\text{If } \{s^n,\} \in S, \ \lim_{n\to\infty} s^n = \overline{s} \in S.$$
- Bounded
  $$S \subseteq B(s,r), r < \infty$$
  - Contained in a ball of radius r (centered at s)
- Convex
  $$\text{If } s^0, s^1 \in C, \text{ for } 0 < \lambda < 1,$$
  $$s^\lambda = (1-\lambda)s^0 + \lambda s^1 \in C.$$

Existence of Equilibrium

- $\phi(s)$ is upper hemi-continuous at $\overline{s}$ if
- For any open neighborhood

  V of $\phi(\overline{s})$
- There exists $N(\delta, \overline{s})$ a δ-neighborhood of $\overline{s}$
- such that $\phi(s) \subseteq V$ for all $s \in N(\delta, \overline{s})$



Existence of Equilibrium

- Using Kakutani's Fixed Point Theorem (FPT)
- Proposition 9.1-1: Existence of NE (Nash, 1950)
- In a game with finite action sets, if players can choose either pure or mixed strategies,
  - Mixed strategy profile $(\pi_1, \pi_2, \ldots, \pi_n), 0 \leqq \pi_i \leqq 1$
  - Closed, bounded and convex
- there exists a Nash Equilibrium.
  - BR correspondence is non-empty, convex (mixing among BR is also BR), and upper hemi-continuous

Existence of Equilibrium

- Proposition 9.1-2: Existence of pure NE
- In a game with action sets $A_i \subseteq \mathbf{R}^n$ is closed, bounded and convex, and utility u is continuous,
- If BR sets $BR_i(a_{-i}) \subseteq A_i$ are convex,
- there exists a pure strategy Nash Equilibrium.
- Corollary 9.1-3: Existence of pure NE
- If BR sets $BR_i(a_{-i}) \subseteq A_i$ are single-valued, or

  $\text{If } u_i(a_i, a_{-i}) \text{ are } \textbf{quasi-concave} \text{ over } a_i$
- there exists a pure strategy Nash Equilibrium.

Summary of 9.1

- Game Tree
  - Extensive Form and Information Sets
- Simultaneous Game
  - Strategic Form (Normal Form)
- Nash Equilibrium
  - Existence of Nash Equilibrium (by Kakutani's FPT)
- HW 9.1: Riley – 9.1-1~4