Experimental Economics I

Jury Voting

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Jury Voting Model

- Three jurors $N = \{1, 2, 3\}$ responsible for deciding whether to convict or acquit a defendant.
- Collectively they choose an outcome $x \in \{c, a\}$.
- The jurors simultaneously cast ballots $v_i \in S_i = \{c, a\}$.
- The outcome is chosen by majority rule.
- Each juror is uncertain whether or not the defendant is guilty (G) or innocent (I).
- So the set of state variables is $\Omega = \{G, I\}$.
- Each juror assigns prior prob. $\pi > 1/2$ to state G.
- If the defendant is guilty, the jurors receive 1 unit of utility from convicting and 0 from acquitting; if the defendant is innocent, the jurors receive 1 unit from acquitting and 0 from convicting;

$$\begin{aligned} u(c|G) &= u(a|I) = 1 \\ u(a|G) &= u(c|I) = 0 \end{aligned}$$

Jury Voting Model

- Absent any additional information, each juror receives an expected utility of π from a guilty verdict and $1 - \pi$ from an acquittal.
- Because $\pi > 1/2$, the Nash equ'm that survives the elimination of weakly dominated strategies is the one where each juror votes guilty.
- Now, before voting, each juror receives a private signal about the defendant's guilt $\theta_i \in \{0, 1\}$.
- The signal is informative so that a juror is more likely to receive the signal $\theta_i = 1$ when the defendant is guilty than when the defendant is innocent.
- Assume the prob. of receiving a "guilty" signal ($\theta_i = 1$) when the defendant is guilty is the same as that of receiving an "innocent" signal ($\theta_i = 0$) when the defendant is innocent. Symmetry
- Formally, let $\Pr(\theta_i = 1|\omega = G) = \Pr(\theta_i = 0|\omega = I) = p > 1/2$ so that $\Pr(\theta_i = 0|\omega = G) = \Pr(\theta_i = 1|\omega = I) = 1 - p$.
- Conditional on a state, each signal for an individual is independent with each other (signals are "conditionally independent").

Sincere Voting Strategy

- After receiving her signal, voter i selects her vote $v(\theta_i)$ to maximize the prob. of a correct decision - conviction of the guilty and acquittal of the innocent.
- Suppose that each voter uses the *sincere* strategy $v_i(1) = c$ and $v_i(0) = a$.
- The sincere strategy calls for a vote to convict upon receipt of a guilty signal and a vote to acquit upon an innocent signal.
- Sincere strategies constitute a Bayesian Nash equ'm (BNE) only if voter 1 is willing to use this strategy when she believes that voters 2 and 3 also use it.
- Given these conjectures, the expected utility (EU) of voting to convict is

One other vote c

$$\Pr(\theta_2 = 1, \theta_3 = 0; \omega = G|\theta_1) \quad + \quad \Pr(\theta_2 = 0, \theta_3 = 1; \omega = G|\theta_1)$$
$$+ \quad \Pr(\theta_2 = 1, \theta_3 = 1; \omega = G|\theta_1) \quad + \quad \Pr(\theta_2 = 0, \theta_3 = 0; \omega = I|\theta_1).$$

Both vote c

None vote c

Sincere Voting Strategy

- The EU of voting to acquit is

One other voter vote for a

$$\Pr(\theta_2 = 1, \theta_3 = 0; \omega = I | \theta_1) \quad + \quad \Pr(\theta_2 = 0, \theta_3 = 1; \omega = I | \theta_1)$$
$$+ \quad \Pr(\theta_2 = 0, \theta_3 = 0; \omega = I | \theta_1) \quad + \quad \Pr(\theta_2 = 1, \theta_3 = 1; \omega = G | \theta_1).$$

None vote for a

Both vote for a

- The last two terms of each sum are the same, hence these terms cancel out when comparing utilities.
- Accordingly, voting to convict is a best response if & only if

$$\Pr(\theta_2 = 1, \theta_3 = 0; \omega = G | \theta_1) + \Pr(\theta_2 = 0, \theta_3 = 1; \omega = G | \theta_1)$$
$$\geq \quad \Pr(\theta_2 = 1, \theta_3 = 0; \omega = I | \theta_1) + \Pr(\theta_2 = 0, \theta_3 = 1; \omega = I | \theta_1).$$

- Because these expressions depend on the conditional prob. of observing combinations of the state variable and the signals of the other jurors, juror 1 uses Bayes' rule to evaluate each term.

Sincere Voting Strategy

- Suppose that juror 1 receives $\theta_1 = 1$.
- In this case, Bayes' rule yields

$$\Pr(\theta_2 = 1, \theta_3 = 0; \omega = G | \theta_1 = 1)$$
$$= \Pr(\theta_2 = 0, \theta_3 = 1; \omega = G | \theta_1 = 1) = \frac{\pi p^2(1-p)}{\pi p + (1-\pi)(1-p)}$$

event A; event B; $= \frac{\Pr(A \cap B)}{\Pr(B)}$; $\frac{\Pr(\theta_1=1, \theta_2=1, \theta_3=0 | G)\Pr(G)}{\Pr(\theta_1=1|G)\Pr(G) + \Pr(\theta_1=1|I)\Pr(I)}$

and

$$\Pr(\theta_2 = 1, \theta_3 = 0; \omega = I | \theta_1 = 1)$$
$$= \Pr(\theta_2 = 0, \theta_3 = 1; \omega = I | \theta_1 = 1) = \frac{(1-\pi)p(1-p)^2}{\pi p + (1-\pi)(1-p)}.$$

- Thus, $v_i(1) = c$ is optimal for juror 1 if

$$2\frac{\pi p^2(1-p)}{\pi p + (1-\pi)(1-p)} \geq 2\frac{(1-\pi)p(1-p)^2}{\pi p + (1-\pi)(1-p)}.$$

$\Rightarrow 2\pi p^2(1-p) \geq (1-\pi)p(1-p)^2 + \pi p^2(1-p)$

Sincere Voting Strategy

- After simplifying and rearranging, this inequality becomes

(I am pivotal!)

$$\Pr(G \mid \theta_1=1, \theta_j=1, \theta_k=0) = \frac{\pi p^2(1-p)}{\pi p^2(1-p) + (1-\pi)p(1-p)^2} \geq \frac{1}{2}.$$

- LHS is just the conditional prob. of guilt given two signals of $\theta = 1$ and one signal of $\theta = 0$.
- In other words, agent 1 wants to vote to convict if she believes that the defendant is more likely to be guilty than innocent, conditional on her signal and the belief that she is pivotal.
- Similarly, the requirement for a vote of innocence conditional on a signal of 0 is

$$\frac{\pi p(1-p)^2}{\pi p(1-p)^2 + (1-\pi)p^2(1-p)} \leq \frac{1}{2}.$$

- To sum, in any BNE in which voting corresponds to the private signals,

1. Conditional on the supposition that i is pivotal and observes $\theta_i = 1$, the posterior prob. of guilt is greater than $1/2$; and
2. Conditional on the supposition that i is pivotal and observes $\theta_i = 0$, the posterior prob. of guilt is less than $1/2$.

Asymmetric Signal

- Thus, if sincere voting is incentive compatible, then

$$\frac{1-p}{p} \leq \frac{\pi}{1-\pi} \leq \frac{p}{1-p}.$$

- E.g., if $\pi > p$, then sincere voting is not incentive compatible.
- Under majority rule and symmetric signal precision (and equal prior $\pi = 1/2$), sincere voting obtains in equ'm (if $p > 1/2$).
- Alternative way to obtain an *insincere/strategic* voting equ'm is to introduce asymmetric signal:

$$\begin{aligned} p &\equiv \Pr(\theta_i = 1 | \omega = G), & q &\equiv \Pr(\theta_i = 0 | \omega = I), \\ 1-p &= \Pr(\theta_i = 0 | \omega = G), & 1-q &= Pr(\theta_i = 1 | \omega = I), \end{aligned}$$

  and we have here $1 > p > q > 1/2$.
- Then, the posterior probabilities (with equal prior $\pi = 1/2$) are

$$\Pr[\omega = G | \theta_i = 1] = \frac{p}{p + (1-q)}, \; \Pr[\omega = I | \theta_i = 0] = \frac{q}{(1-p) + q}.$$

Strategic Voting Equ'm

- Define $\sigma(s) \equiv$ prob. of voting one's signal, $s = 0, 1$.
- Typically, we have in equ'm; $\sigma(1) \in (0, 1)$ and $\sigma(0) = 1$. (semi-pooling equil.)
- Then,

$$
\begin{aligned}
\Pr[c|\omega = G] &= p\sigma(1) + (1-p)(1-\sigma(0)) = p\sigma(1), \\
\Pr[a|\omega = G] &= p(1-\sigma(1)) + (1-p)\sigma(0) = p(1-\sigma(1)) + (1-p), \\
\Pr[c|\omega = I] &= (1-q)\sigma(1) + q(1-\sigma(0)) = (1-q)\sigma(1), \\
\Pr[a|\omega = I] &= (1-q)(1-\sigma(1)) + q\sigma(0) = (1-q)(1-\sigma(1)) + q,
\end{aligned}
$$

- Since the equ'm strategy requires randomization upon signal $s = 1$,

Indifference ⇒ $$\Pr[\omega = G|\theta_i = 1]\Pr[Piv|\omega = G] - \Pr[\omega = I|\theta_i = 1]\Pr[Piv|\omega = I] = 0,$$

where $\Pr[Piv|\omega]$ is the prob. a vote is pivotal at state ω:

(One vote c, the other vote a)

$$
\begin{aligned}
\Pr[Piv|\omega = G] &= \binom{2}{1}\Pr[c|\omega = G]\Pr[a|\omega = G] \\
&= [p\sigma(1)][p(1-\sigma(1)) + (1-p)],
\end{aligned}
$$

Strategic Voting Equ'm

$$\Pr[Piv|\omega = I] = \binom{2}{1}\Pr[c|\omega = I]\Pr[a|\omega = I]$$
$$= [(1-q)\sigma(1)][(1-q)(1-\sigma(1)) + q]$$

- Thus we solve for $\sigma(1)$ from the above equation.
- Since $\sigma(0) = 1$, we finally check whether

$$\Pr[\omega = I|\theta_i = 0]\Pr[Piv|\omega = I] - \Pr[\omega = G|\theta_i = 0]\Pr[Piv|\omega = G] > 0$$

  when $\Pr[Piv|\omega]$ is evaluated at $\sigma(1)$ that solves the indifference condition.
- For example, when $p = 0.9$ and $q = 0.6$, $\sigma(1) = 0.9774$
- Under fixed (p, q), $\sigma(1)$ typically decreases as n gets larger.

Remarks

- Austen-Smith & Banks (1996) show that in many cases the sincere strategy is inconsistent with equilibrium behavior.
- It is easy to find parameters π and p for which one of the necessary conditions does not hold.
- There are alternative strategies jurors might choose.
- Jurors can randomize for some signals, vote the same way regardless of their signal, or use different strategies than other jurors use.
- Feddersen & Pesendorfer (1998) consider the properties of equ'a of this game when one varies the voting rule and number of jurors.

Jury Voting with a Continuum of Signals

- Instead of receiving a binary signal, each juror now receives a signal $\theta_i \in [0,1]$ where θ_i is drawn from a conditional distribution $F(\theta_i|\omega)$.
- This distribution function is associated with a different density function $f(\theta_i|\omega)$ that satisfies the *monotone likelihood ratio* condition.
- A conditional density function satisfies the *strict monotone likelihood ratio condition* (SMLR) if $\frac{f(\theta_i|G)}{f(\theta_i|I)}$ is a strictly monotone function of θ_i on $[0,1]$.
- To see why this assumption is important, note that Bayes' rule implies that

$$\Pr(G|\theta_i) = \frac{f(\theta_i|G)\pi}{f(\theta_i|G)\pi + f(\theta_i|I)(1-\pi)} = \frac{\frac{f(\theta_i|G)}{f(\theta_i|I)}\pi}{\frac{f(\theta_i|G)}{f(\theta_i|I)}\pi + (1-\pi)}.$$

- Accordingly, $\Pr(G|\theta_i)$ is increasing in θ_i if & only if $f(\theta_i|G)/f(\theta_i|I)$ is increasing in θ_i.
- Thus, the SMLR conditioin implies that higher signals correspond to higher posterior probabilities that $\omega = G$.

Jury Voting with a Continuum of Signals

- To keep matters simple, we focus exclusively on symmetric strategies where voters who receive the same signal choose the same strategy.
- A symmetric strategy profile is, therefore, a mapping $v_i(\theta_i) : [0, 1] \rightarrow \{c, a\}$.
- As in the binary signal case, BNE strategies are those that are optimal when each agent acts conditionally on her private information and the conjecture that she is pivotal.
- An agent votes to convict if she thinks the prob. of guilt is no less than 1/2 and she votes to acquit if she thinks the prob. of guilt is no more than 1/2.
- Because higher signals are better indicators of guilt, a natural conjecture is that the strategy must be weakly increasing.
- For low values of θ_i an acquittal vote is cast and for high values of θ_i a conviction vote is cast.

Cut Point Strategy

- A monotone strategy of this form can be characterized by a cut point $\hat{\theta} \in [0, 1]$.
- Assume that agents $i \in N \backslash i$ use the monotone strategy

$$v_i(\theta_i) = \begin{cases} c & \text{if } \theta_i \geq \hat{\theta} \\ a & \text{if } \theta_i < \hat{\theta} \end{cases}$$

- If all players other than i use this cut point strategy, the posterior prob. of $\{\omega = G\}$ given signal θ_i and the event that i is pivotal is given by

$$\Pr(G|piv, \theta_i; \hat{\theta}) = \frac{\pi f(\theta_i|G)F(\hat{\theta}|G)^{N-r}[1 - F(\hat{\theta}|G)]^{r-1}}{\pi f(\theta_i|G)F(\hat{\theta}|G)^{N-r}[1 - F(\hat{\theta}|G)]^{r-1} + (1-\pi)f(\theta_i|I)F(\hat{\theta}|I)^{N-r}[1 - F(\hat{\theta}|I)]^{r-1}}$$

- This prob. is a function of the parameter $\hat{\theta}$.

* Here we assume r-rule, so we require r or more votes for conviction (majority rule if $r = (N + 1)/2$ and unanimity rule if $r = N$).

Cut Point Equilibrium

- In this model the existence of a symmetric equ'm in which voters use a cut point hinges on finding a value of $\hat{\theta}$ s.t.

$$\Pr(G|piv, \hat{\theta}; \hat{\theta}) = \frac{1}{2}$$

and demonstrating that $\Pr(G|piv, \theta_i; \hat{\theta}) \leq \frac{1}{2}$ if $\theta_i < \hat{\theta}$ and $\Pr(G|piv, \theta_i; \hat{\theta}) \geq \frac{1}{2}$ if $\theta_i > \hat{\theta}$.

- Although analysis of examples is cumbersome, it is easy to derive conditions on the primitives of the game to ensure that such a $\hat{\theta} \in (0, 1)$ exists.

- First, $\Pr(G|piv, \theta_i; \hat{\theta}) \geq \frac{1}{2}$ if & only if

$$H(\theta_i) = \frac{\pi f(\theta_i|G) F(\hat{\theta}|G)^{N-r}[1 - F(\hat{\theta}|G)]^{r-1}}{(1-\pi) f(\theta_i|I) F(\hat{\theta}|I)^{N-r}[1 - F(\hat{\theta}|I)]^{r-1}}$$

$$= \frac{f(\theta_i|G)}{f(\theta_i|I)} \frac{\pi F(\hat{\theta}|G)^{N-r}[1 - F(\hat{\theta}|G)]^{r-1}}{(1-\pi) F(\hat{\theta}|I)^{N-r}[1 - F(\hat{\theta}|I)]^{r-1}} \geq 1.$$

$H(0) < 1$, $\Rightarrow H(1) > 1$ } $\Rightarrow \exists\, \hat{\theta} \in (0,1)$ s.t. $H(\hat{\theta}) = 1$

Existence of Cut Point Equilibrium

- SMLR then implies that if $\Pr(G|piv, \hat{\theta}; \hat{\theta}) = 1/2$ then $\theta_i < \hat{\theta}$ implies $\Pr(G|piv, \theta_i; \hat{\theta}) \leq 1/2$ and $\theta_i > \hat{\theta}$ implies $\Pr(G|piv, \theta_i; \hat{\theta}) \geq 1/2$.
- If $\Pr(G|piv, 0; 0) \leq 1/2 \leq \Pr(G|piv, 1; 1)$ then the intermediate value theorem implies that such a cut point exists b/c the function $\Pr(G|piv, \cdot; \cdot)$ is continuous.
- For a large class of games these boundary conditions are satisfied.
- In the simple binary signal model, equ'a where everyone uses the same rule and voting is determined by private information may not exist.
- This type of equ'm generally exists in the continuum model, however.
- Using the binary model, Feddersen & Pesendorfer (1998) show that the unanimity rule is a uniquely bad way to aggregate information for large populations b/c in equ'm voters condition on the assumption that everyone else is voting to convict.
- In the continuum model, Meirowitz (2002) shows that the unanimity rule often turns out to be as good as the other voting rules.

Voluntary Voting Model

- Two candidates, A and B, in majority voting election.
- Two equally likely states of nature, α and β.
- A is the better choice in state α and B, in state β.
  - In state α, payoff is 1 if A is elected and 0 if B is elected; vice versa in state β.
- The size of the electorate is a random variable, distributed according to a *Poisson* distribution with mean n.

  The Poisson distribution is for analytic convenience.

  - The probability that there are exactly m voters is $e^{-n}n^m/m!$.
- Prior to voting, each voter receives a private signal S_i regarding the true state of nature, either a or b; $\Pr[a|\alpha] = r$ and $\Pr[b|\beta] = s$; the posteriors given by

$$q(\alpha|a) = \frac{r}{r+(1-s)}, \qquad q(\beta|b) = \frac{s}{s+(1-r)}.$$

  - $r \geq s > 1/2$ implies $q(\alpha|a) \leq q(\beta|b)$.

Pivotal Events

- Event (j,k), j votes for A and k votes for B.
- An event is *pivotal* for A if a single additional vote for A changes the outcome, written Piv_A. The additional vote makes or breaks a tie.
- Under majority rule, one additional vote for A makes a difference only if (i) there is a tie; or (ii) A has one vote less than B.

$$T = \{(k,k) : k \geq 0\}, \quad T_{-1} = \{(k-1,k) : k \geq 1\}, \quad Piv_A = T \cup T_{-1}$$

- Similarly, $Piv_B = T \cup T_{+1}$, $T_{+1} = \{(k, k-1) : k \geq 1\}$.
- σ_A, σ_B are the *expected* number of votes for A, B in state α; τ_A, τ_B are the *expected* number of votes for A, B in state β.
- With abstention allowed, $\sigma_A + \sigma_B \leq n$, $\tau_A + \tau_B \leq n$ (equality w/o abstention).

Pivotal Events

- If the realized electorate is of size m with k votes for A and l votes for B ($m-k-l$ abstention),

$$\Pr[(k,l)|\alpha] = e^{-\sigma_A}\frac{\sigma_A^k}{k!}e^{-\sigma_B}\frac{\sigma_B^l}{l!}.$$

* For the probability of the event (k,l) in state β, replace σ by τ.

$$\begin{aligned}
\Pr[T|\alpha] &= e^{-\sigma_A-\sigma_B}\sum_{k=0}^{\infty}\frac{\sigma_A^k}{k!}\frac{\sigma_B^k}{k!},\\
\Pr[T_{-1}|\alpha] &= e^{-\sigma_A-\sigma_B}\sum_{k=1}^{\infty}\frac{\sigma_A^{k-1}}{(k-1)!}\frac{\sigma_B^k}{k!},\\
\Pr[Piv_A|\alpha] &= \frac{1}{2}\Pr[T|\alpha]+\frac{1}{2}\Pr[T_{-1}|\alpha]
\end{aligned}$$

actually from utility ($0 \to \frac{1}{2}$ or $\frac{1}{2} \to 1$)

where $Piv_A = T \cup T_{-1}$ is the set of events where one additional vote for A is decisive, and we have the coefficient $1/2$ because the additional vote for A breaks a tie or leads to a tie.

Pivotal Events

- Similarly,
$$\Pr[Piv_B|\beta] = \frac{1}{2}\Pr[T|\beta] + \frac{1}{2}\Pr[T_{+1}|\beta]$$
where $Piv_B = T \cup T_{+1}$ is the set of events where one additional vote for B is decisive.

- Define *modified Bessel functions*
$$I_0(z) = \sum_{k=0}^{\infty} \frac{(z/2)^k}{k!}\frac{(z/2)^k}{k!}, \qquad I_1(z) = \sum_{k=1}^{\infty} \frac{(z/2)^{k-1}}{(k-1)!}\frac{(z/2)^k}{k!}$$
and rewrite the probabilities of close elections in terms of these functions
$$\Pr[T|\alpha] = e^{-\sigma_A-\sigma_B} I_0(2\sqrt{\sigma_A\sigma_B})$$
$$\Pr[T_{\pm1}|\alpha] = e^{-\sigma_A-\sigma_B}\left(\frac{\sigma_A}{\sigma_B}\right)^{\pm1/2} I_1(2\sqrt{\sigma_A\sigma_B}).$$

- For z large, we also have
$$I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}} \approx I_1(z).$$

Compulsory Voting

- By compulsory voting each voter must cast a vote for either A or B.
- Vote sincerely in compulsory voting equilibrium?
- Given sincere and compulsory voting, $\sigma_A = nr$, $\sigma_B = n(1-r)$, $\tau_A = n(1-s)$, $\tau_B = ns$.
- As n increases, both $\sigma \to \infty$, $\tau \to \infty$, and so the previous approximations for $I_0(z)$, $I_1(z)$ imply

$$\frac{\Pr[Piv_A|\alpha] + \Pr[Piv_B|\alpha]}{\Pr[Piv_A|\beta] + \Pr[Piv_B|\beta]} \approx \frac{e^{2n\sqrt{r(1-r)}}}{e^{2n\sqrt{s(1-s)}}} \times K(r,s) \longrightarrow 0$$

(as $n \to \infty$)

where $K(r,s)$ is positive and independent of n.

- $r > s > 1/2$ also implies $s(1-s) > r(1-r)$ and so RHS goes to zero as n increases.

Compulsory Voting

- This implies that, when n is large and a voter is pivotal, state β is infinitely more likely than state α.
- Thus, voters with *a* signals will not wish to vote sincerely.

Proposition 1: *Suppose $r > s$. If voting is compulsory, sincere voting is not an equilibrium in large elections.*

- This result also holds for a fixed number of voters (Feddersen & Pesendorfer APSR 1998).

Voluntary Voting

- Costly voting: one's cost of voting is private info and an independent draw from a continuous distribution F with support $[0, 1]$ - F admits a density $f > 0$ on $[0, 1)$.
- Voting costs are independent of the signals.
- There exists an equilibrium of this voluntary (and costly) voting game with the following features;

(i) There exists a pair of positive *threshold costs* c_a, c_b s.t. a voter with cost c and signal $i = a, b$ votes (does not abstain) if & only if $c \leq c_i$. The threshold costs determine differential *participation rates* $F(c_a) = p_a$, $F(c_b) = p_b$.

(ii) All those who vote do so sincerely - i.e., all those with signal a vote for A and those with signal b vote for B.

Equ'm Participation Rates

- We show that when all those who vote do so sincerely, there is an equ'm in cutoff strategies.
- There exists a threshold cost $c_a > 0$ ($c_b > 0$) s.t. all voters with signal i and cost $c \leq c_a$ ($c \leq c_b$) go to the polls and vote for A (B).
  strictly positive!
- These then determine participation probabilities $p_a = F(c_a)$, $p_b = F(c_b)$ for voters with signal a, b, respectively.
- Now the expected numbers of votes for A, B in state α are $\sigma_A = nrp_a$, $\sigma_B = n(1-r)p_b$; and those in state β are $\tau_A = n(1-s)p_a$, $\tau_B = nsp_b$, respectively.
- We look for participation rates p_a, p_b s.t. a voter with signal a and cost $c_a = F^{-1}(p_a)$ is indifferent b/w going to the polls and staying home;
  meaning of threshold cost

$$(IRa) \quad U_a(p_a, p_b) \equiv q(\alpha|a)\Pr[Piv_A|\alpha] - q(\beta|a)\Pr[Piv_A|\beta] = F^{-1}(p_a)$$

benefit of voting = cost of voting

where the pivot probabilities are determined using the expected vote totals σ, τ.

- Similarly, a voter with signal b and cost $c_b = F^{-1}(p_b)$ must also be indifferent;

$$(IRb) \quad U_b(p_a, p_b) \equiv q(\beta|b)\Pr[Piv_B|\beta] - q(\alpha|b)\Pr[Piv_B|\alpha] = F^{-1}(p_b).$$

Proposition 2: *There exist participation rates $p_a^* \in (0,1)$ and $p_b^* \in (0,1)$ that simultaneously satisfy (IRa) and (IRb).*

- Intuition for positive participation rates: assume $p_a = 0$.
- Then the only pivotal events are $(0,0)$ and $(0,1)$.

Equ'm Participation Rates

- Hence conditional on being pivotal

$$\frac{\Pr[Piv_A|\alpha]}{\Pr[Piv_A|\beta]} = \frac{e^{-n(1-r)p_b}}{e^{-nsp_b}} \times \frac{1+n(1-r)p_b}{1+nsp_b}.$$

- The ratio of the exponential terms favors state α while the ratio of the linear terms favors state β; and the exponential terms always dominate.
- Since state α is perceived more likely than β by a voter with signal a who is pivotal, the payoff from voting is positive. ⇒ Contradicting $p_a = 0$ $(-)\times(-)$
- We also have

Lemma 1: *If $r > s$, then any solution to (IRa) and (IRb) satisfies $p_a^* < p_b^*$, with equality if $r = s$.*

Sincere Voting

- Given the (equ'm) participation rates, we can show that it is a best-response for every voter to vote sincerely.
- We begin with a lemma;

  Lemma 2: *If voting behavior is s.t.* $\sigma_A > \tau_A$ *and* $\sigma_B < \tau_B$, *then*

$$\frac{\Pr[Piv_B|\alpha]}{\Pr[Piv_B|\beta]} > \frac{\Pr[Piv_A|\alpha]}{\Pr[Piv_A|\beta]}.$$

(Need Poisson distribution ?!)

- On the set of "marginal" events where the vote totals are close (i.e., a voter is pivotal), A is more likely to be leading in state α and more likely to be trailing in state β.
- Let (p_a^*, p_b^*) be equ'm participation rates.
- A voter with signal a and cost $c_a^* = F^{-1}(p_a^*)$ is just indifferent b/w voting and staying home;

$$(IRa) \quad q(\alpha|a)\Pr[Piv_A|\alpha] - q(\beta|a)\Pr[Piv_A|\beta] = F^{-1}(p_a^*).$$ > 0

Sincere Voting

- To show: sincere voting is optimal for a voter with signal a if others are voting sincerely;

$$
\begin{aligned}
(ICa) \qquad & q(\alpha|a)\Pr[Piv_A|\alpha] - q(\beta|a)\Pr[Piv_A|\beta] \\
& \geq \quad q(\beta|a)\Pr[Piv_B|\beta] - q(\alpha|a)\Pr[Piv_B|\alpha].
\end{aligned}
$$

- LHS is the payoff from voting for A whereas RHS is the payoff to voting for B.
- $p_a^* > 0$ combined with (IRa) implies

$$
\frac{\Pr[Piv_A|\alpha]}{\Pr[Piv_A|\beta]} > \frac{q(\beta|a)}{q(\alpha|a)}.
$$

- Then by Lemma 2,

$$
\frac{\Pr[Piv_B|\alpha]}{\Pr[Piv_B|\beta]} > \frac{q(\beta|a)}{q(\alpha|a)}.
$$

- But then, the last inequality is equivalent to

$$
q(\beta|a)\Pr[Piv_B|\beta] - q(\alpha|a)\Pr[Piv_B|\alpha] < 0. \quad (*)
$$

Sincere Voting

- Similarly, we combine $p_b^* > 0$, Lemma 2, and

$$(IRb) \quad q(\beta|b)\Pr[Piv_B|\beta] - q(\alpha|b)\Pr[Piv_B|\alpha] = F^{-1}(p_b^*)$$

to show

$$\begin{aligned}(ICb) \qquad & q(\beta|b)\Pr[Piv_B|\beta] - q(\alpha|b)\Pr[Piv_B|\alpha] \\ \geq \quad & q(\alpha|b)\Pr[Piv_A|\alpha] - q(\beta|b)\Pr[Piv_A|\beta].\end{aligned}$$

Proposition 3: *Under voluntary participation, sincere voting is incentive compatible.*

- We can also show that all equ'a involve sincere voting (Krishna & Morgan JET 2012).