Envelope Theorem

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(Calculus 4, 19.2)

Example: Hunghai (aka Foxconn Tech. Group...)

- Hunghai is a price-taking firm making jpads
  - Sell 3,000 jpads to Pineapple at price $p_i = \$100$
  - Total Cost is $C(q) = \$180,000$
- What is the elasticity of profit w.r.t. price $\epsilon(\Pi, p_i) = \frac{p_i}{\Pi}\frac{\partial \Pi}{\partial p_i}$
  - If output is held fixed?
  - If Hung-Hai responds optimally to price change?
- Hunghai sees a new opportunity and sells 1,500 Vii's to Rentientang at price $p_w = \$200$
  - Production of jpads drop to 2,400, total cost rises to \$300,000. Can you calculate the new $\epsilon(\Pi, p_i)$?

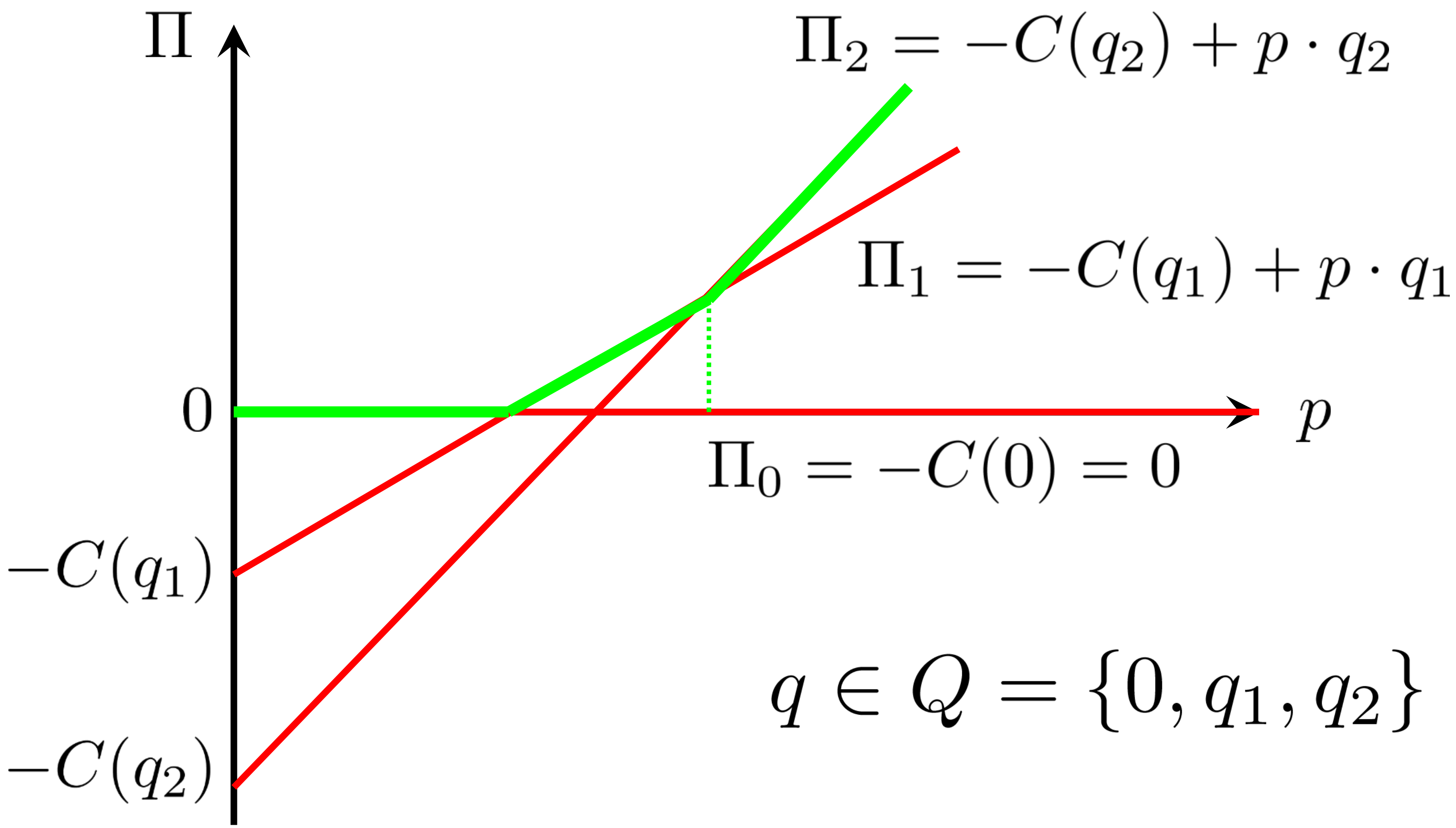
How Firms Adjust to Environment Changes

- A price-taking firm has cost $C(q)$
  - Can sell as much as it wishes at fix price p
- Profit is $\pi = p \cdot q - C(q)$
- Given a change in prices p, how would profit change (as the firm re-optimizes output q)?
  - Direct Effect: $\Delta p \cdot q$
  - Indirect Effect: $\Delta \pi$ due to $q \rightarrow q'$
- First assume only three possible outputs…

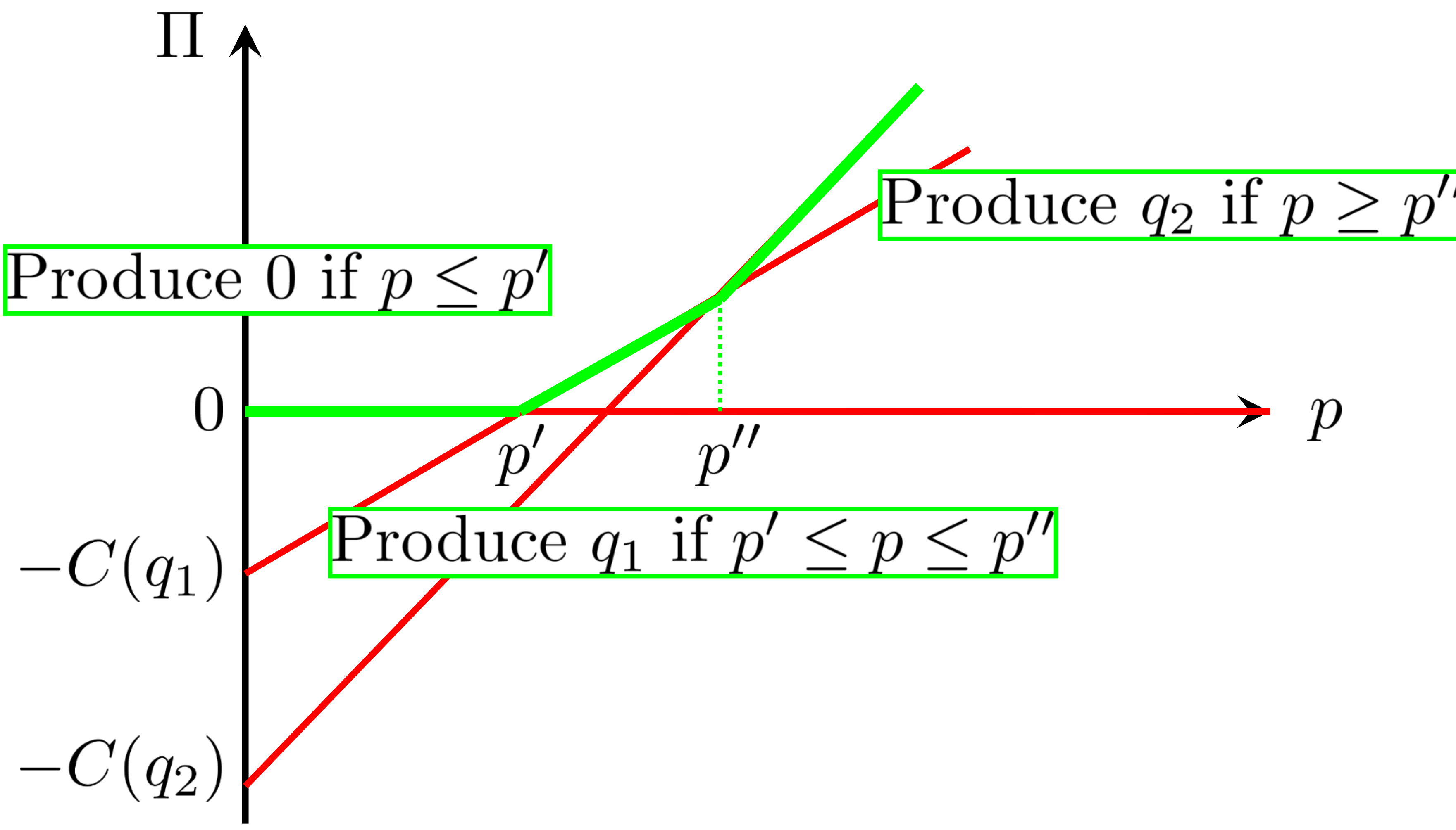
$$q \in Q = \{0, q_1, q_2\}$$

  - Profit is straight line for each possible output

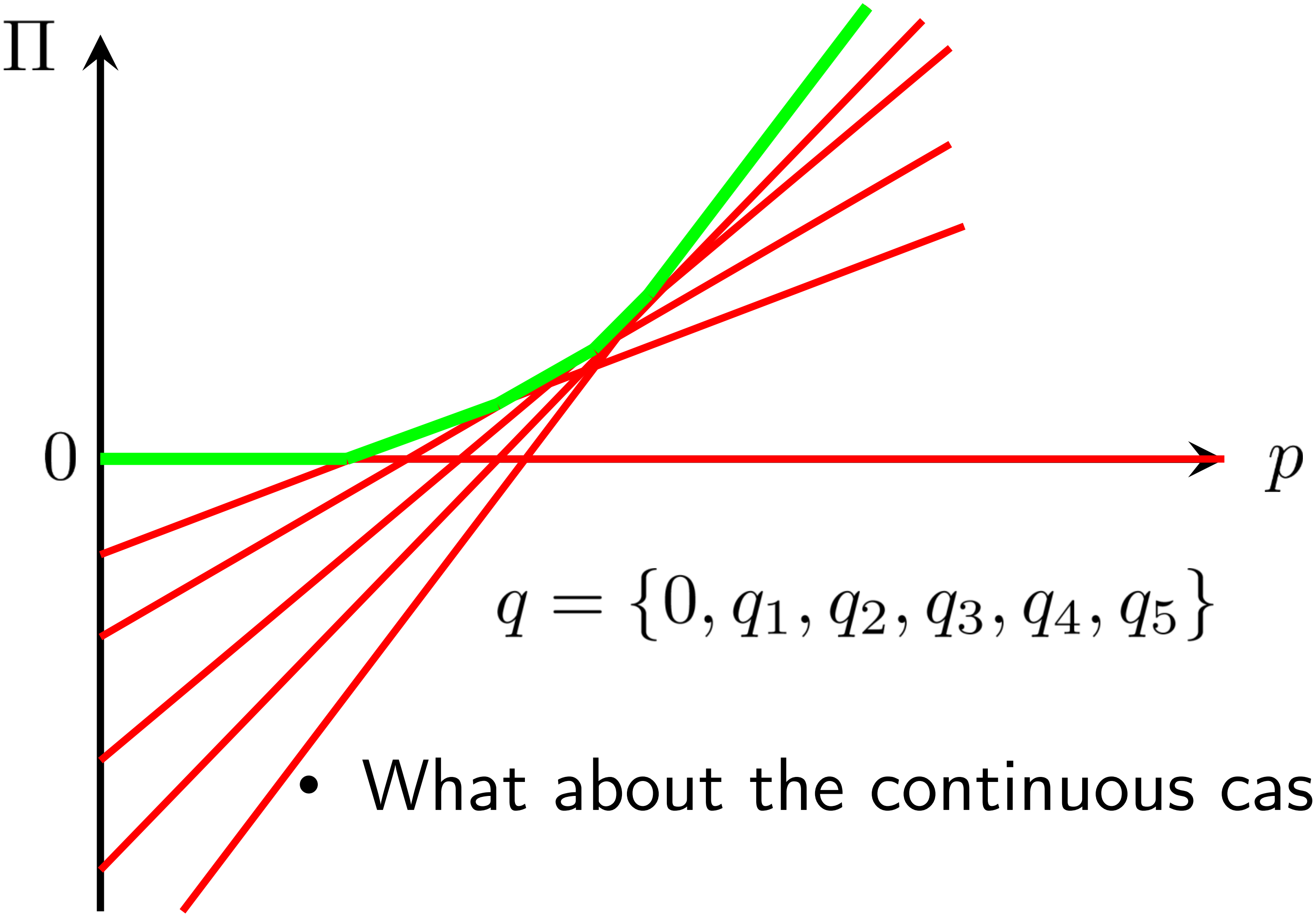
Three Output States



Upper Envelope for Three Output States

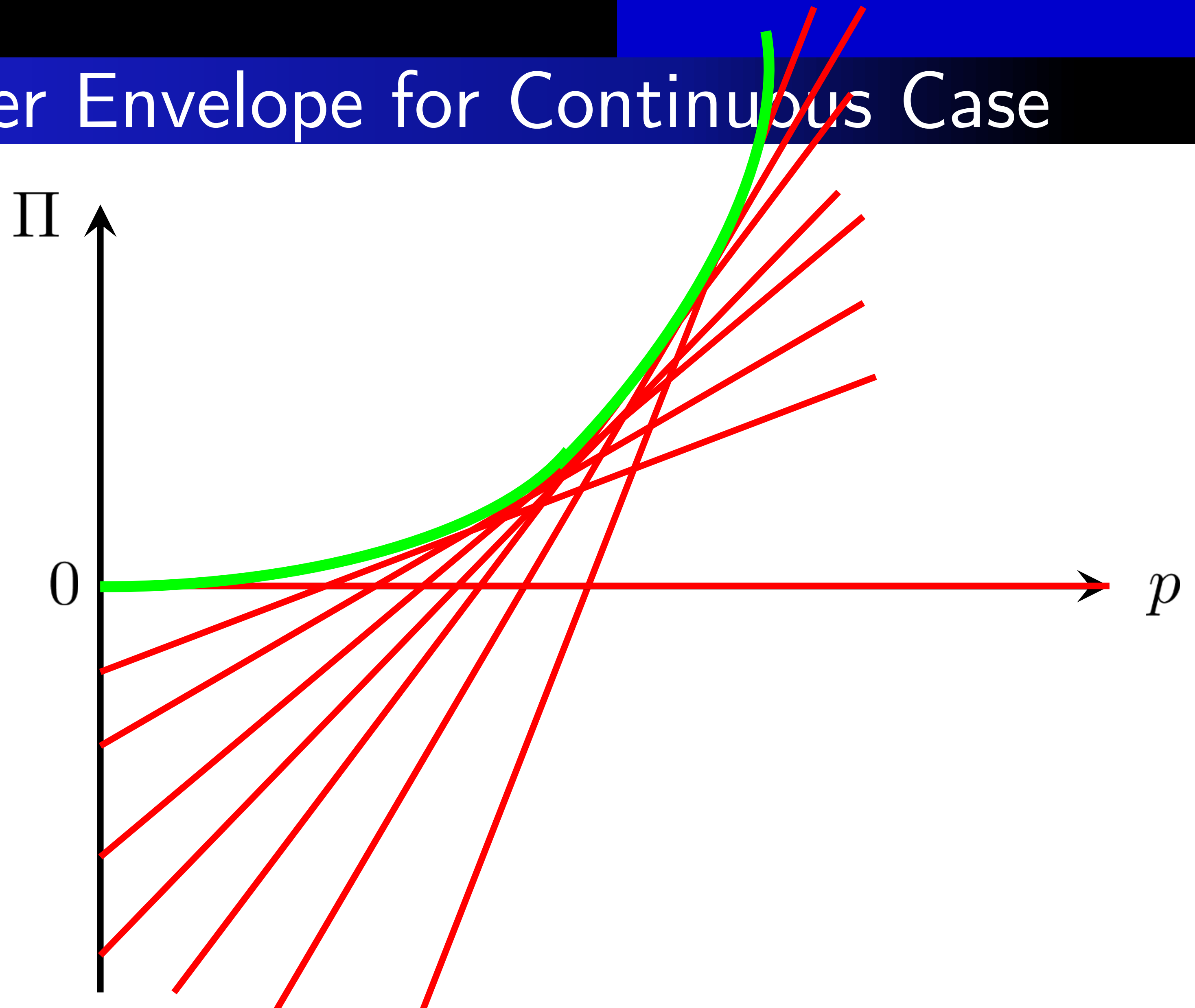


Upper Envelope for Six Output States



- What about the continuous case?

Upper Envelope for Continuous Case



How Firms Adjust to Environment Changes

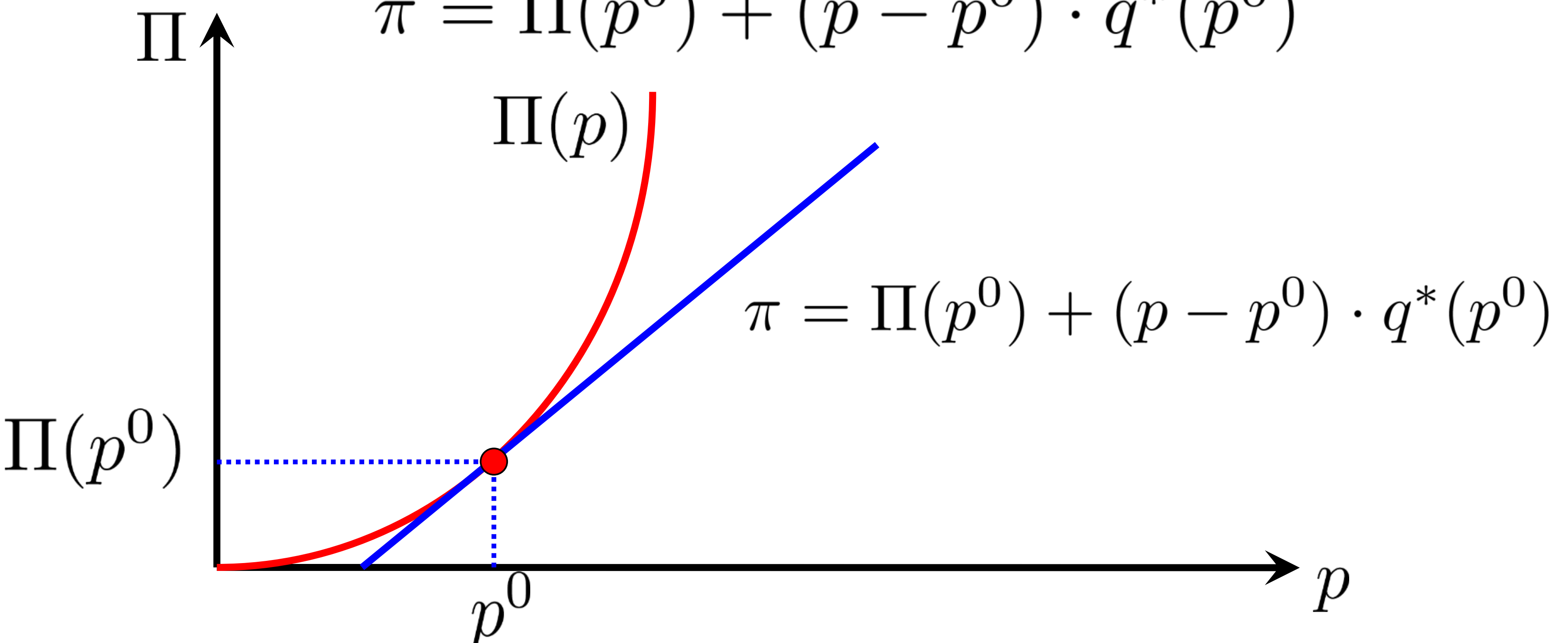
- Output can be any real number
- Firm solves $q^*(p)$ to $\max\{\pi = p \cdot q - C(q)\}$
- Maximized profit is $\Pi(p) = p \cdot q^*(p) - C\left(q^*(p)\right)$
- Initial output price p^0 (fixed)
  - Initial output $q^*(p^0)$
  - Initial profit $\Pi(p^0)$
- Profit (with fixed output) is

$$\pi = \Pi(p^0) + (p - p^0) \cdot q^*(p^0)$$

How Firms Adjust to Environment Changes

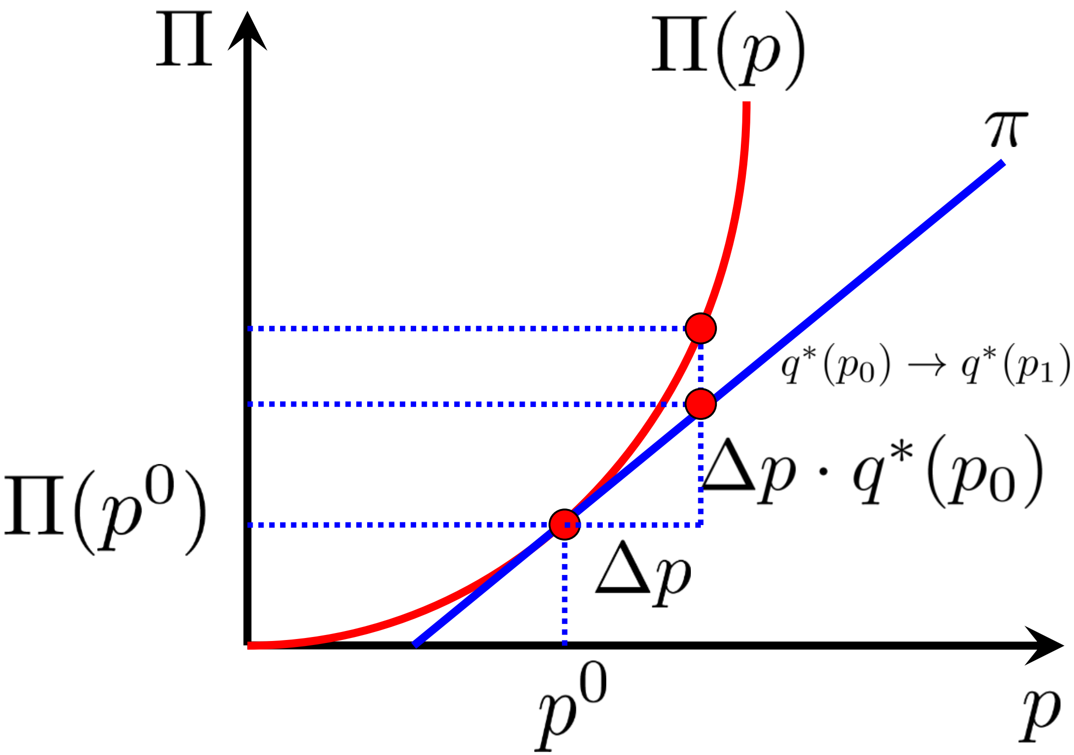
- Fixing output, increase in price changes profit by $q^*(p)$ per dollar, so (fixed output) profit is

$$\pi = \Pi(p^0) + (p - p^0) \cdot q^*(p^0)$$



How Firms Adjust to Environment Changes

$$\frac{\partial \Pi}{\partial p}(p^0) = q^*(p^0)$$

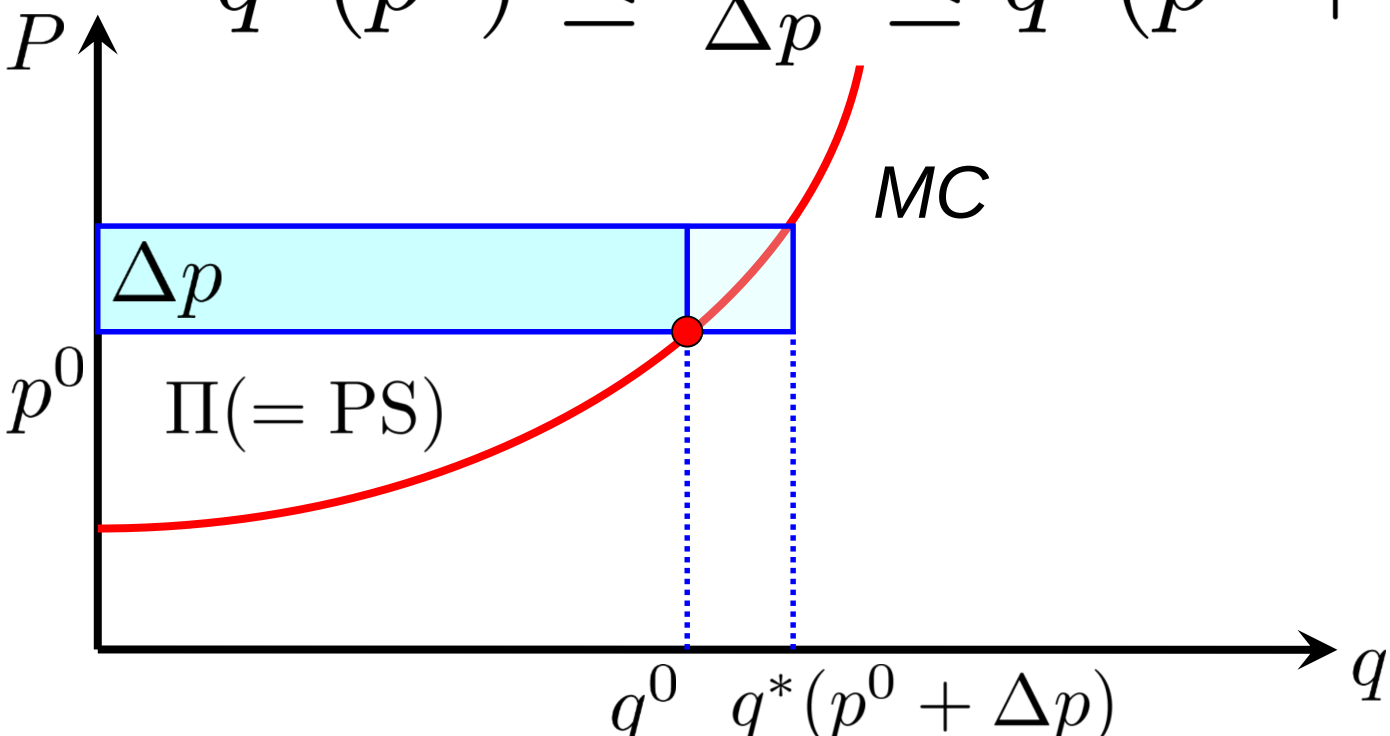


- Firm cannot be worse off if it can change quantity
- $\Pi(p)$ is above π
  - Tangent to π if $\Pi(p)$ smooth
- Total effect = Direct effect only
  - Ignore indirect eff.

Another Graphic Presentation (P-q)

$$q^*(p^0)\Delta p \le \Delta\Pi \le q^*(p^0+\Delta p)\Delta p$$

$$q^*(p^0) \le \frac{\Delta\Pi}{\Delta p} \le q^*(p^0+\Delta p)$$



In fact, we have Thm 19.4: Envelope Theorem

- Assume: (Feasible output)
  - $X \in \mathbf{R}^n$ is closed and bounded, a is a scalar $\boxed{a = p}$
  - $f(\vec{x}, a)$ is C^1 (continuously differentiable) (Profit)

$\boxed{q^*(p)}$ $\vec{x}^*(a) = \arg\max_{\vec{x} \in X} \{f(\vec{x}, a)\}$ is C^1,

$\boxed{\Pi(p)}$

- Then, $F(a) = \max_{\vec{x} \in X} \{f(\vec{x}, a)\} = f(\vec{x}^*(a), a)$, the value function is differentiable and has

$$F'(a) = \frac{df}{da}(\vec{x}^*(a), a) = \frac{\partial f}{\partial a}(\vec{x}^*(a), a).$$

(Only Direct Effect)

Thm 19.4: Envelope Theorem (Unconstrained)

– $X \in \mathbf{R}^n$ is closed and bounded, a is a scalar

– $f(\vec{x}, a)$ is C^1 (continuously differentiable)

$\vec{x}^*(a) = \arg\max_{\vec{x} \in X} \{f(\vec{x}, a)\}$ is C^1,

For, $F(a) = \max_{\vec{x} \in X} \{f(\vec{x}, a)\} = f(\vec{x}^*(a), a)$,

$$F'(a) = \sum_i \underline{\frac{\partial f}{\partial x_i}(\vec{x}^*(a), a)} \cdot \frac{dx_i^*}{da}(a) + \frac{\partial f}{\partial a}(\vec{x}^*(a), a)$$

$$= \frac{\partial f}{\partial a}(\vec{x}^*(a), a) \text{ since } \underline{\frac{\partial f}{\partial x_i}(\vec{x}^*(a), a)} = 0$$

Thm 19.4: Envelope Theorem (Unconstrained)

- Direct Effect = Total Effect (at the margin)
- This only allows the maximand to be affected by the parameter change...
- To allow for both the maximand and the constraints to be affected by the parameter change, need slightly stronger assumptions...

Thm 19.5: Envelope Theorem (Constrained)

$$F(a) = \max_{\vec{x}} \{ f(\vec{x}, a) | h_1(\vec{x}, a) = 0, \cdots, h_k(\vec{x}, a) = 0 \}$$

Suppose: $\mathcal{L} = f(\vec{x}, a) + \sum_i \mu_i h_i(\vec{x}, a)$

- $f(\vec{x}, a)$ and $h_i(\vec{x}, a)$ are C^1
- $\vec{x}^*(a), \vec{\mu}^*(a)$ unique solutions; NDCQ hold.
- $\vec{x}^*(a), \vec{\mu}^*(a)$ are C^1 continuously differentiable at a^0 (implicit function theorem applies)

Then,

$$F'(a^0) = \frac{d}{da} f(\vec{x}^*(a^0), a^0) = \frac{\partial \mathcal{L}}{\partial a} (\vec{x}^*(a^0), \vec{\mu}^*(a^0), a^0)$$

Example: Hunghai (aka Foxconn Tech. Group...)

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$$\begin{aligned}\Pi &= p \times q - C(q) \\ &= \$100 \times 3,000 - \$180,000 = \$120,000\end{aligned}$$

$$\frac{\partial \Pi}{\partial p_i} = q_i = 3,000 \Rightarrow \epsilon = \frac{p_i}{\Pi}\frac{\partial \Pi}{\partial p_i} = \frac{\$100 \cdot 3,000}{\$120,000} = \frac{5}{2}$$

- Hunghai's elasticity of profit wrt. jpad price is 2.5 for both fixed and variable output (by ET!)

Example: Hunghai

- Hunghai sees a new opportunity and sells 1,500 Vii's to Rentientang at price $p_w = \$200$
  - Production of jpads drop to 2,400, price $p_i = \$100$
  - total cost rises to \$300,000. Calculate new $\epsilon(\Pi, p_i)$

$$\begin{aligned}\Pi &= \$100 \times 2,400 + \$200 \times 1,500 - \$300,000 \\ &= \$240,000\end{aligned}$$

$$\frac{\partial \Pi}{\partial p_i} = q_i' = 2,400 \Rightarrow \epsilon = \frac{p_i}{\Pi}\frac{\partial \Pi}{\partial p_i} = \frac{\$100 \cdot 2,400}{\$240,000} = 1$$

What does this all mean?

- Hunghai used to only produce jpads
- Since it is a price-taker, if Pineapple Corp. decides to lower prices by 10%, Hunghai's profit would decrease by 25%
- Even if Hunghai tries to re-optimize! (ET)
- After diversifying to producing also Vii's, it's profit is now less prone to Pineapple's price cuts (lowers by 10% if prices are cut by 10%)
- Isn't this what firms in Hsinchu Science Park do?

Summary of 19.2

- Re-maximize under environmental change
  - Direct Effect: Change in profit (objective function)
  - Indirect Effect: Change due to re-optimization
- Envelope Theorem(s):
  - Only have Direct Effect at the margin
- Homework:
  - Exercise 19.11, 19.13
  - Find a Hunghai example in the news