Power Series

Section 11.8-11.9

Outline

- Power Series:
  - Definition
  - The Radius of Convergence and the Interval of Convergence
- Representations of Functions as Power Series
  - Geometric Power Series
  - Differentiation and Integration of Power Series
  - Taylor Series of some special functions.

Power Series

- A **power series** is a series of the form
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
where x is a variable and the c_n s are constants called the **coefficients** of the series.
- The sum of the series is a function
$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
whose domain is the set of all x for which the series converges.

Power Series

- More generally, a series of the form
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$
is called a **power series in** $(x-a)$ or a **power series centered at *a*** or a **power series about *a***.
- Notice that when $x = a$ all of the terms are 0 for $n \geq 1$ and so the power series always converges when $x = a$.

The Radius of Convergence

- Theorem: For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ only one of the following is true.
- 1. It converges at only one point $x = a$.
- 2. It converges for all x .
- 3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The Radius of Convergence

- The number R in case 3 is called the **radius of convergence** of the power series. By convention, $R = 0$ in case 1 and $R = \infty$ in case 2 .

The Radius of Convergence

- Theorem: If $\sum_{n=0}^{\infty} c_n(x_0-a)^n$ converges, then for any x such that $|x-a|<|x_0-a|$, $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely.
- Corollary: If $\sum_{n=0}^{\infty} c_n(x_0-a)^n$ diverges, then for any x such that $|x-a|>|x_0-a|$, $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges.

The Radius of Convergence

- How to find the radius of convergence?
- Theorem: Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has nonzero coefficients. If $\lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = L$ (or $\lim_{n\to\infty} \sqrt[n]{|c_n|} = L$), then the radius of convergence of the power series is $R=1/L$ when $L \neq 0$, and $R=\infty$ when $L=0$.

The Interval of Convergence

- The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.
- It may be the single point $\{a\}$, or the whole real line or a finite interval centered at a.
- When x is an *endpoint* of the interval, that is, $x=a\pm R$, anything can happen.

Examples of Power Series

Series	Radius of convergence	Interval of convergence
$\sum_{n=0}^{\infty} x^n$	$R=1$	$(-1,1)$
$\sum_{n=0}^{\infty} n!\, x^n$	$R=0$	$\{0\}$
$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$	$R=1$	$[1,3)$
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$R=\infty$	$(-\infty,\infty)$

Representations of Functions as Power Series

- Geometric Series

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n$$

- This equation expresses the function $f(x)=1/(1-x)$ as a power series.

Representations of Functions as Power Series

- Differentiation and Integration of Power Series:
- We would like to be able to differentiate and integrate the sum of power series, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as for a polynomial.
- This is called **term-by-term differentiation** and **integration**.

Representations of Functions as Power Series

- Theorem: If the power series $\sum c_n(x-a)^n$ has radius of converges $R>0$, then the function f defined by $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R,a+R)$ and $f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^n$

  $\int f(x)\,dx = C+\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

  Moreover, the radius of convergence of the above power series are both R.

Representations of Functions as Power Seri

- Examples: For $|x|<1$,

$$\frac{1}{(1-x)^2}=\frac{d}{dx}\frac{1}{1-x}=\sum_{n=1}^{\infty} nx^{n-1}$$

$$\ln(1+x)=\int \frac{1}{1+x}dx=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

$$\tan^{-1}x=\int \frac{1}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

Taylor Series

Function	Taylor Series	Radius of Convergence
e^x	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$R=\infty$
$\sin x$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$R=\infty$
$\cos x$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	$R=\infty$
$(1+x)^k$	$\sum_{n=0}^{\infty}\binom{k}{n}x^n$	$R=1$

Taylor Series

- Let $T_n(x)=\sum_{i=0}^{n}\frac{f^{(i)}}{i!}(x-a)^i$ be the nth degree Taylor polynomial, and $R_n(x)=f(x)-T_n(x)$ be the nth remainder term.
- If $|f^{(n+1)}(x)|\le M$ for $|x-a|\le d$, then

$$|R_n(x)|\le \frac{M}{(n+1)!}|x-a|^{n+1}$$

  for $|x-a|\le d$.

Review

- What is a power series?
- What are the radius of convergence and the interval of convergence of a power series?
- How do we differentiate or integrate a power series function?
- Review some representations of functions as power series.
- Review Taylor series of some special functions.