Alternating Series and Absolute Convergence

Section 11.5-11.6

Outline

Alternating Series

- Definition:
- An **alternating series** is a series whose terms are *alternately positive and negative*.
- Example:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

- Try to plot partial sums of the above "alternating harmonic series".

Alternating Series Test

- Alternating Series Test:
- If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+\cdots,\quad b_n>0$$

satisfies (i) $b_{n+1}\le b_n$ for all n (ii) $\lim_{n\to\infty} b_n=0$ then the series converges.

Estimating Sums

- Alternating Series Estimation Theorem:
- If $s=\sum(-1)^{n-1}b_n$ where $b_n>0$ and the series satisfies (i) $b_{n+1}\le b_n$ for all n (ii) $\lim_{n\to\infty} b_n=0$, then we can estimate the remainder term

$$|R_n|=|s-s_n|\le b_{n+1}\,.$$

Absolute Convergence

- Definition: A series $\sum a_n$ is called **absolutely convergent** if $\sum|a_n|$, the series of absolute values, is convergent.
- Definition: A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.
- Theorem: If a series is absolutely convergent, then it is convergent.

Absolute Convergence

- What is the difference between absolutely convergent series and conditionally convergent series?
- Theorem:
- Given a series , let $a_n^+ = \frac{a_n + |a_n|}{2}$, $a_n^- = \frac{a_n - |a_n|}{2}$. If is absolutely convergent, then both $\sum a_n^+$ and $\sum a_n^-$ are convergent. If is conditionally convergent, then both $\sum a_n^+$ and $\sum a_n^-$ are divergent.

The Ratio Test

- The ratio test:
- Case 1: If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then $\sum a_n$ is absolutely convergent (and hence convergent).
- Case 2: If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$, or $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty$ then $\sum a_n$ is divergent.
- Case 3: If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, the ratio test is inconclusive.

The Root Test

- The root test:
- Case 1: If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent (and hence convergent).
- Case 2: If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then $\sum a_n$ is divergent.
- Case 3: If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

The Root Test

- Note:
- If $L = 1$ in the Ratio Test, don't try the Root Test because the limit will again be 1.
- If $L = 1$ in the Root Test, don't try the Ratio Test because the limit either won't exist or the limit will be 1.

Rearrangements of Series

- By a **rearrangement** of an infinite series $\sum a_n$ we mean a series obtained by simply changing the order of the terms.
- If we rearrange the order of the terms in a finite sum, then the value of the sum remains unchanged. But this is not always the case for an infinite series. (This is the main difference between conditional convergence series and absolute convergence series.)

Rearrangements of Series

- Theorem:
- If $\sum a_n$ is absolutely convergent with sum S, then any rearrangement of $\sum a_n$ has the same sum S.
- If $\sum a_n$ is conditionally convergent and r is any real number, then there is a rearrangement of $\sum a_n$ that has the sum equal to r. We can even rearrange $\sum a_n$ so that the new series diverges to positive infinity or negative infinity.

Rearrangements of Series

- Examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\cdots=\ln 2$$

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3}{2}\ln 2$$

- You can find out some other interesting rearrangements of the alternating harmonic series !!

Review

- State the Alternating Series Test.
- How do we estimate the remainder term of an alternating series?
- What are the definitions of “absolute convergence” and “conditional convergence” of a series?
- State the Ratio Test and the Root Test.
- When will the rearrangements of a series always lead to the same sum?