# The Integral Test and the Comparison Tests

Section 11.3-11.4

## Outline

- The Integral Test
  - The Integral Test
  - The *P*-Series
  - Estimate the Sums
- The Comparison Tests
  - The Comparison Test
  - The Limit Comparison Test
  - Estimate the Sums

## The Integral Test

- Example: Are the series convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

## The Integral Test

- The Integral Test:
- Suppose that $f$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

## The Integral Test

- Example:
- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

## Estimate the Sum of a Series

- A partial sum $s_n$ is an approximation to $s$. But how good is such an approximation? To find out, we need to estimate the size of the **remainder** $R_n = s - s_n = a_{n+1} + s_{n+2} + \cdots$
- The remainder $R_n$ is the error made when $s_n$, the sum of the first $n$ terms, is used as an approximation to the total sum.

## Estimate the Sum of a Series

- Under the assumptions of the Integral Test

$$R_n = a_{n+1} + a_{n+2} + \cdots \le \int_n^\infty f(x)dx$$

$$R_n = a_{n+1} + a_{n+2} + \cdots \ge \int_{n+1}^\infty f(x)dx$$

- Hence

$$s_n + \int_{n+1}^\infty f(x)dx \le s \le s_n + \int_n^\infty f(x)dx$$

## Estimate the Sum of a Divergent Series

- Example 1:

$$\ln(n+1) < \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

- Example 2: Stirling's Formula

$$\sqrt{2\pi n}\,(\frac{n}{e})^n < n! < \sqrt{2\pi n}\,(\frac{n}{e})^n(1 + \frac{1}{4n})$$

## The Comparison Test

- The Comparison Test:
- Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
- (1) If $\sum b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum a_n$ is also convergent.
- (2) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all $n$, then $\sum a_n$ is also divergent.

## The Comparison Test

- Remark: Although the condition $a_n \le b_n$ or $a_n \ge b_n$ in the Comparison Test is given for all $n$, we only need to verify that it holds for $n \ge N$, where $N$ is some fixed integer.

## The Comparison Test

- In using the Comparison Test we often compare the unknown series with the following series:
- A $p$-series ( $\sum 1/n^p$ converges if $p > 1$ and diverges if $p \le 1$.)
- A geometric series ( $\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \ge 1$. )

## The Limit Comparison Test

- The Limit Comparison Test:
- Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty}(a_n/b_n) = c$ where $c$ is a finite number and $c > 0$, then either both series converge or both diverge.

## The Limit Comparison Test

- Suppose that $\sum a_n$ is a series with positive terms. Then $\sum \ln(1+a_n)$ converges if and only if $\sum a_n$ converges.
- Remark: We are interested in $\sum \ln(1+a_n)$ because we may need to compute the infinite product $\prod_{n=1}^{\infty}(1+a_n)$.

## The Limit Comparison Test

- The Limit Comparison Test (Part 2):
- Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
  If $\sum b_n$ is convergent and $\lim_{n\to\infty} a_n/b_n = 0$, then $\sum a_n$ is also convergent.
  If $\sum b_n$ is divergent and $\lim_{n\to\infty} a_n/b_n = \infty$, then $\sum a_n$ is also divergent.

## The Limit Comparison Test

- Example: $\sum_{n=1}^{\infty} n^p r^n$ converges for all real numbers $p$ and $0 < r < 1$.
- Example: $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges for all $p > 1$.

## Estimate Sums

- If we have used the Comparison Test to show that a series $\sum a_n$ converges by comparison with a series $\sum b_n$, then we may be able to estimate the sum $\sum a_n$ by comparing remainders.
- Let $R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$
  $T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$
- Then $R_n$ is smaller than $T_n$.

## Review

- State the Integral Test. How do we estimate the remainder term by the Integral Test?
- When will a p-series converge?
- State the Comparison Test and the Limit Comparison Test.