Constrained Optimization and Kuhn-Tucker Conditions

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(Calculus 4, 18.4)
Theorem 18.4 (Several Inequality Constraints)

• Suppose $f, g_1, \ldots, g_k$ be $C^1$ functions on $\mathbb{R}^n$

• Let $\bar{x}^* = (x_1^*, \ldots, x_n^*)$ solve max. problem

\[
\max \left\{ f(x_1, \ldots, x_n) \mid g_1(x_1, \ldots, x_n) \leq b_1, \ldots, g_k(x_1, \ldots, x_n) \leq b_k \right\}
\]

• Notation: Constraints $g_1, \ldots, g_{k_0}$ binds

$g_1(x_1^*, \ldots, x_n^*) = b_1, \ldots, g_{k_0}(x_1^*, \ldots, x_n^*) = b_{k_0}$

• Constraints $g_{k_0+1}, \ldots, g_k$ do not bind

$g_{k_0+1}(x_1^*, \ldots, x_n^*) < b_{k_0+1}, \ldots, g_k(x_1^*, \ldots, x_n^*) < b_k$
Theorem 18.4 (Several Inequality Constraints)

- Binding constraints $g_1, \ldots, g_{k_0}$ satisfies NDCQ if its Jacobian matrix has maximum rank $k_0$

\[
\begin{pmatrix}
\nabla g_1 \\
\vdots \\
\nabla g_{k_0}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\partial g_1}{\partial x_1}(\bar{x}^*) & \cdots & \frac{\partial g_1}{\partial x_n}(\bar{x}^*) \\
\vdots & \ddots & \vdots \\
\frac{\partial g_{k_0}}{\partial x_1}(\bar{x}^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(\bar{x}^*)
\end{pmatrix}
\]

- Or, row vectors $Dg_i = \nabla g_i = \left( \frac{\partial g_i}{\partial x_1}(\bar{x}^*), \ldots, \frac{\partial g_i}{\partial x_n}(\bar{x}^*) \right)$ are linearly independent
Theorem 18.4 (Several Inequality Constraints)

- Row vectors

\[ Dg_i = \nabla g_i = \left( \frac{\partial g_i}{\partial x_1}(\vec{x}^*), \ldots, \frac{\partial g_i}{\partial x_n}(\vec{x}^*) \right) \]

are linearly independent if

\[ a_1 \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\vec{x}^*) \\ \vdots \\ \frac{\partial g_1}{\partial x_n}(\vec{x}^*) \end{pmatrix} + \cdots + a_{k_0} \begin{pmatrix} \frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) \\ \vdots \\ \frac{\partial g_{k_0}}{\partial x_n}(\vec{x}^*) \end{pmatrix} = \vec{0} \]

implies \( a_1 = \cdots = a_{k_0} = 0 \)
Theorem 18.4 (Several Inequality Constraints)

For $\mathcal{L} = f(x_1, \cdots, x_n) - \lambda_1 [g_1(x_1, \cdots, x_n) - b_1] - \cdots - \lambda_k [g_k(x_1, \cdots, x_n) - b_k]$

- There exists $\vec{\lambda}^* = (\lambda_1^*, \cdots, \lambda_k^*)$ such that

a) $\frac{\partial \mathcal{L}}{\partial x_1}(\vec{x}^*, \vec{\lambda}^*) = 0, \cdots, \frac{\partial \mathcal{L}}{\partial x_n}(\vec{x}^*, \vec{\lambda}^*) = 0$

b) $\lambda_1^* [g_1(\vec{x}^*) - b_1] = 0, \cdots, \lambda_k^* [g_k(\vec{x}^*) - b_k] = 0$

c) $\lambda_1^* \geq 0, \cdots, \lambda_k^* \geq 0$

d) $g_1(\vec{x}^*) - b_1 \leq 0, \cdots, g_k(\vec{x}^*) - b_k \leq 0$
Theorem 18.7 (Kuhn-Tucker)

- Suppose $f, g_1, \ldots, g_k$ be $C^1$ functions on $\mathbb{R}^n$
- Let $\bar{x}^* = (x_1^*, \ldots, x_n^*)$ solve max. problem

$$\max \left\{ f(x_1, \ldots, x_n) \mid x_1 \geq 0, \ldots, x_n \geq 0, \\
g_1(x_1, \ldots, x_n) \leq b_1, \ldots, g_k(x_1, \ldots, x_n) \leq b_k \right\}$$

- NDCQ satisfied if $\left( \frac{\partial g_i}{\partial x_j} \right)_{ij}$ has maximum rank

  **Binding constraints**

  **Positive $x_j$**

  where $i \in \{ i \mid g_i(\bar{x}^*) = b_i \}$, $j \in \{ j \mid x_j^* > 0 \}$

- Exists $\bar{\lambda}^* = (\lambda_1^*, \ldots, \lambda_k^*)$, $\lambda_i^* \geq 0$, such that
Theorem 18.7 (Kuhn-Tucker)

For \( \tilde{\mathcal{L}} = f(x_1, \cdots, x_n) - \lambda_1 [g_1(x_1, \cdots, x_n) - b_1] \\
- \cdots - \lambda_k [g_k(x_1, \cdots, x_n) - b_k] \)

A. \[ \frac{\partial \tilde{\mathcal{L}}}{\partial x_1}(\bar{x}^*, \bar{\lambda}^*) \leq 0, \cdots, \frac{\partial \tilde{\mathcal{L}}}{\partial x_n}(\bar{x}^*, \bar{\lambda}^*) \leq 0 \]

\[ x_1^* \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial x_1}(\bar{x}^*, \bar{\lambda}^*) = 0, \cdots, x_n^* \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial x_n}(\bar{x}^*, \bar{\lambda}^*) = 0 \]

B. \[ \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1}(\bar{x}^*, \bar{\lambda}^*) \geq 0, \cdots, \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k}(\bar{x}^*, \bar{\lambda}^*) \geq 0 \]

\[ \lambda_1^* \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1}(\bar{x}^*, \bar{\lambda}^*) = 0, \cdots, \lambda_k^* \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k}(\bar{x}^*, \bar{\lambda}^*) = 0 \]
Theorem 18.7 (Kuhn-Tucker)

- Let $x_1^* > 0, \ldots, x_{n_0}^* > 0, x_{n_0+1}^* = \cdots = x_n^* = 0$
- Binding constraints $g_1, \ldots, g_{k_0}$ satisfies NDCQ if the following matrix has maximum rank $k_0$

\[
\begin{pmatrix}
\widehat{D} g_1 \\
\vdots \\
\widehat{D} g_{k_0}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial g_1}{\partial x_1}(\vec{x}^*) & \cdots & \frac{\partial g_1}{\partial x_{n_0}}(\vec{x}^*) \\
\vdots & \ddots & \vdots \\
\frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_{n_0}}(\vec{x}^*)
\end{pmatrix}
\]

- Or, row vectors $\widehat{D} g_i = \left( \frac{\partial g_i}{\partial x_1}(\vec{x}^*), \cdots, \frac{\partial g_i}{\partial x_{n_0}}(\vec{x}^*) \right)$

($1^{st}$ $n_0$ elements of $\nabla g_i$) are linearly independent
Theorem 18.7 (Kuhn-Tucker)

- Row vectors \((1^{st} n_0 \text{ elements of } \nabla g_i)\)

\[
\hat{D} g_i = \left( \frac{\partial g_i}{\partial x_1}(\vec{x}^*), \ldots, \frac{\partial g_i}{\partial x_{n_0}}(\vec{x}^*) \right)
\]

are linearly independent if

\[
\begin{pmatrix}
\frac{\partial g_1}{\partial x_1}(\vec{x}^*) \\
\vdots \\
\frac{\partial g_1}{\partial x_{n_0}}(\vec{x}^*)
\end{pmatrix}
+ \cdots + a_{k_0}
\begin{pmatrix}
\frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) \\
\vdots \\
\frac{\partial g_{k_0}}{\partial x_{n_0}}(\vec{x}^*)
\end{pmatrix} = \vec{0}
\]

implies \(a_1 = \cdots = a_{k_0} = 0\)
Exercise 18.14 (Generalize Example 18.9)

\[
\max \quad f(x, y, z) = xyz \\
\text{s.t.} \quad P_{xx}x + P_{yy}y + P_{zz}z \leq I \\
\quad x \geq 0, y \geq 0, z \geq 0
\]

• NDCQ?

\[
\tilde{\mathcal{L}} = xyz - \lambda[P_{xx}x + P_{yy}y + P_{zz}z - I]
\]

• FOC?
Exercise 18.14 (Generalize Example 18.9)

\[
\begin{align*}
\max & \quad f(x, y, z) = xyz \\
\text{s.t.} & \quad P_{xx}x + P_{yy}y + P_{zz}z \leq I \\
& \quad x \geq 0, y \geq 0, z \geq 0
\end{align*}
\]

• NDCQ?

\[
\tilde{L} = xyz - \lambda[P_{xx}x + P_{yy}y + P_{zz}z - I]
\]
Exercise 18.14 (Generalize Example 18.9)

\[ \tilde{\mathcal{L}} = xyz - \lambda [P_xx + P_yy + P_zz - 1] \]

FOC:
\[
\begin{align*}
\frac{\partial \tilde{\mathcal{L}}}{\partial x} &= yz - \lambda P_x \leq 0, \quad x \cdot \frac{\partial \mathcal{L}}{\partial x} = 0 \\
\frac{\partial \tilde{\mathcal{L}}}{\partial y} &= xz - \lambda P_y \leq 0, \quad y \cdot \frac{\partial \mathcal{L}}{\partial y} = 0 \\
\frac{\partial \tilde{\mathcal{L}}}{\partial z} &= xy - \lambda P_z \leq 0, \quad z \cdot \frac{\partial \mathcal{L}}{\partial z} = 0 \\
\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} &= I - (P_xx + P_yy + P_zz) \geq 0, \quad \lambda \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} = 0
\end{align*}
\]
Exercise 18.14 (Generalize Example 18.9)

\[ \tilde{L} = xyz - \lambda [P_xx + P_yy + P_zz - I] \]

Solution:

\[ x^* = \frac{I}{3P_x} \]
\[ y^* = \frac{I}{3P_y} \]
\[ z^* = \frac{I}{3P_z} \]

\[ f(x^*, y^*, z^*) = \frac{I^3}{27P_xP_yP_z} \]
Ex: Sales-Maximizing Firm with Advertising

• Suppose \( R(y, a) \), \( C(y) \) are \( C^1 \) functions satisfying \( C'(y) > 0 \), \( R(0, a) = 0 \), \( \frac{\partial R}{\partial a} > 0 \)

• Firms choose \( y, a \) from \( R_+ \) to maximize revenue \( R(y, a) \), without letting profit drop below \( m > 0 \)

\[
\max_{y,a} R(y, a) \\
\text{s.t. } \Pi = R(y, a) - C(y) - a \geq m \\
\quad y \geq 0, a \geq 0
\]
Ex: Sales-Maximizing Firm with Advertising

• Suppose \( C'(y) > 0, \ R(0, a) = 0, \ \frac{\partial R}{\partial a} > 0 \)

\[
\max_{y,a} R(y, a)
\]

\[
s.t. \ \Pi = R(y, a) - C(y) - a \geq m
\]

1. Show that the constraint binds, so the firm will maintain minimum profit

2. Show that output (if positive) is larger than profit-maximizing output
Ex: Sales-Maximizing Firm with Advertising

• Wait, does NDCQ always hold? No!

\[
\max_{y,a} R(y, a)
\]

s.t. \( \Pi = R(y, a) - C(y) - a \geq m \)

\( y \geq 0, a \geq 0 \)

\[ g(a, y) = m - R(y, a) + C(y) + a \]

\[ \nabla g = \left( \frac{\partial g}{\partial y}, \frac{\partial g}{\partial a} \right) = \left( -\frac{\partial R}{\partial y} + C'(y), -\frac{\partial R}{\partial a} + 1 \right) \]

= 0 if MR = MC and advertising MR = 1
The Meaning of the Multiplier

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(Calculus 4, 19.1)
Theorem 19.1 (Single Equality Constraint)

• Consider \( \max_{x,y} \{ f(x, y) | h(x, y) = a \} \)

• Let \( f, h \) be continuously differentiable (\( C^1 \))

• For any fixed value \( a \), let \( (x^*(a), y^*(a), \mu^*(a)) \) be the solution which satisfies NDCQ.
  – (Implicit Function Theorem applies!)

• Suppose \( x^*, y^*, \mu^* \) are \( C^1 \) functions of \( a \)

• Then, \( \mu^*(a) = \frac{d}{da} f(x^*(a), y^*(a)) \)
Theorem 19.2 (Several Equality Constraints)

For \( \max \{ f(x_1, \ldots, x_n) \mid h_1(x_1, \ldots, x_n) = a_1, \ldots, h_m(x_1, \ldots, x_n) = a_m \} \)

- Let \( f, h_1, \ldots, h_m \) be \( C^1 \) functions on \( \mathbb{R}^n \)
- For \( \vec{a} = (a_1, \ldots, a_m) \), \( x_1^*(\vec{a}), \ldots, x_n^*(\vec{a}) \) is the solution with Lagrange Multipliers \( \mu_1^*(\vec{a}), \ldots, \mu_m^*(\vec{a}) \) which satisfies NDCQ
- Suppose \( x_i^*, \mu_j^* \) are \( C^1 \) functions of \( \vec{a} \), then

\[
\frac{\partial f}{\partial a_j}(\vec{a}) = \frac{\partial}{\partial a_j} f (x_1^*(\vec{a}), \ldots, x_n^*(\vec{a})) = \mu_j^*(\vec{a}) \quad (j=1, \ldots, m)
\]
Theorem 19.3 (Several Inequality Constraints)

For \( \max \{ f(x_1, \cdots, x_n) \mid g_1(x_1, \cdots, x_n) \leq a_1^*, \cdots, g_k(x_1, \cdots, x_n) \leq a_k^* \} \)

- Let \( f, g_1, \ldots, g_k \) be \( C^1 \) functions on \( \mathbb{R}^n \)
- For \( \bar{a}^* = (a_1^*, \cdots, a_k^*) \), \( x_1^*(\bar{a}^*), \cdots, x_n^*(\bar{a}^*) \) is the solution with Lagrange Multipliers \( \lambda_1^*(\bar{a}^*), \cdots, \lambda_m^*(\bar{a}^*) \) which satisfies NDCQ
- Suppose \( x_i^*, \lambda_j^* \) are \( C^1 \) functions near \( \bar{a}^* \), then

\[
\frac{\partial f}{\partial a_j}(\bar{a}^*) = \frac{\partial}{\partial a_j} f(x_1^*(\bar{a}^*), \cdots, x_n^*(\bar{a}^*)) = \lambda_j^*(\bar{a}^*) \quad (j=1, \ldots, k)
\]
Ex: Limited Resources, Profit-maximizing Firm

For \( \max \left\{ f(x_1, \cdots, x_n) \mid g_1(x_1, \cdots, x_n) \leq a_1^*, \right. \)
\[ \cdots , g_k(x_1, \cdots, x_n) \leq a_k^* \left. \right\} \]

• Firm provide services \( 1, \ldots, n \) at levels \( x_1, \ldots, x_n \)

• To maximize profit \( f(x_1, \ldots, x_n) \) by allocating inputs \( 1, \ldots, k \) at levels \( g_1, \ldots, g_n \)
  – Inputs \( 1, \ldots, k \) constrained by \( \vec{a}^* = (a_1^*, \cdots, a_k^*) \)
  – Addition profit for adding 1 more unit of input \( j \)
    \( = \text{firm's WTP for adding 1 more unit of input} \ j \)
    \( = \lambda_j^*(\vec{a}^*) \)
Exercise 18.14 (Generalize Example 18.9)

\[
\max f(x, y, z) = xyz \\
\text{s.t. } P_x x + P_y y + P_z z \leq I \\
x \geq 0, y \geq 0, z \geq 0 \\
\tilde{L} = xyz - \lambda [P_x x + P_y y + P_z z - I] \\
\Rightarrow f(x^*, y^*, z^*) = \frac{I^3}{27P_x P_y P_z} \\
\frac{\partial \tilde{L}}{\partial I}(x^*, y^*, z^*; I) = \lambda^* = \frac{y^* z^*}{P_x} = \frac{I^2}{9P_x P_y P_z}
\]