Partial Derivatives in Economics

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Just as derivatives describe "marginal" cost for single variable cost functions, partial derivatives can be used to describe marginal product of different inputs for production functions! In particular, a manufacturer produces its product with several inputs, and the **output quantity**, $Q^* = F(K^*, L^*)$, depends on the inputs, say, capital K^* and labor L^*. The **marginal product of capital (MPK)** is the change in output due to an increase in capital ΔK, or

$$\Delta Q = F(K^* + \Delta K, L^*) - F(K^*, L^*) = \frac{F(K^*+1, L^*) - F(K^*, L^*)}{1} \text{ when } \Delta K = 1.$$

If the input is divisible, we can let it be as small as we want and the marginal product becomes

$$\lim_{\Delta K \to 0} \Delta Q = \frac{F(K^* + \Delta K, L^*) - F(K^*, L^*)}{\Delta K} = \frac{\partial F}{\partial K}(K^*, L^*).$$

In the same way, we define the **marginal product of labor (MPL)** as the partial derivative of the production function with respect to L^*, etc.

1. Compute the partial derivatives of the Cobb-Douglas production function

$$q_1 = f_1(K, L) = AK^{\alpha_K} L^{\alpha_L}, \quad (A, \alpha_K, \alpha_L > 0)$$

2. Consider the Cobb-Douglas production $q_1 = f_1(K, L)$ when $A = 3$, $\alpha_K = \frac{2}{3}$, and $\alpha_L = \frac{1}{3}$.

   (a) What is the output q_1 when $K = 1000$ and $L = 125$?

   (b) Use linear approximation to estimate output q_1 when $K = 998$ and $L = 128$.

   (c) Use a calculator to compute output q_1 and verify your estimates.

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Consider the Constant Elasticity of Substitution (CES) production function

$$q = f(K,L) = \left(\frac{3}{4}K^{-\frac{1}{4}} + \frac{1}{4}L^{-\frac{1}{4}}\right)^{-4}$$

Suppose that due to worker recruiting pace and investment conditions, the inputs K and L vary with time t and interest rate r, via the following expressions:

$$K(t,r) = \frac{10t^2}{r} \text{ and } L(t,r) = 6t^2 + 250r.$$

1. Calculate the rate of change of output q with respect to t when $t = 10$ and $r = 10\%$. What is the meaning of this rate?

2. Calculate the rate of change of output q with respect to r when $t = 10$ and $r = 10\%$. What is the meaning of this rate?

3. In what proportions should we add K and L to $(10000, 625)$ to increase production most rapidly?

Marginal Rate of Substitution (MRS) and Special Production Functions

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1. For a production function, the **Marginal Rate of Substitution (MRS)** of its inputs is the ratio of marginal products (MP). Specifically, for $F(K, L)$ the MRS between capital K and labor L is $MRS_{KL} = \dfrac{MPK}{MPL}$ where $MPK = \dfrac{\partial F}{\partial K}$ and $MPL = \dfrac{\partial F}{\partial L}$.

   (a) Compute the slope of the tangent line, $\frac{dL}{dK}$, for the level curve $F(K, L) = C$ in terms of MRS_{KL}. What is the economic meaning of this slope?

   (b) Given the production function $F(K, L) = 3K^{\frac{2}{3}}L^{\frac{1}{3}}$, compute MRS_{KL} when $K = 1000$ and $L = 125$. Find the tangent line of $F(K, L) = 1500$ at $K = 1000$ and $L = 125$.

2. Marginal and Average Products (optional):

   (a) Consider a production function $q = f(K, L)$ such that its marginal product of labor (MPL), $\dfrac{\partial f}{\partial L}$, equals to the average product of labor, $\dfrac{q}{L}$. Which function is this?

   (b) What if MPL is proportion to the average product of labor such that $\dfrac{\partial f}{\partial L} = \alpha_L \dfrac{q}{L}$?

   (c) Which production function $q = f(K, L)$ would let marginal product of capital (MPK) be proportion to the average product of capital, or $\dfrac{\partial f}{\partial K} = \alpha_K \dfrac{q}{K}$?

(d) Show that the Cobb-Douglas production function $q = f(K, L) = AK^{\alpha_K} L^{\alpha_L}$ satisfies these three properties:

   i. Zero output when one lacks any of the inputs: $q = 0$ if $K = 0$ or $L = 0$.

   ii. MPL is proportion to the average product of labor: $\dfrac{\partial f}{\partial L} = \alpha_L \dfrac{q}{L}$.

   iii. MPK is proportion to the average product of capital: $\dfrac{\partial f}{\partial K} = \alpha_K \dfrac{q}{K}$.

(e) Show that the Cobb-Douglas production function is the *only* production function that satisfies the above three properties.

3. Special Cases of the CES production function (optional):

The Constant Elasticity of Substitution (CES) production function is

$$q = f(K, L) = A\left(\alpha K^{-\gamma} + (1-\alpha)L^{-\gamma}\right)^{-\frac{1}{\gamma}}$$

where all the parameters are positive constants.

(a) Show that as $\gamma \to 0$, the CES production function $q = f(K, L)$ converges to the Cobb-Douglas production function $q = f(K, L) = AK^{\alpha_K} L^{\alpha_L}$.

(b) Show that as $\gamma \to \infty$, the CES production function converges to the Leontief production function $q = f(K, L) = A \cdot \min\{K, L\}$.