# The egalitarian solution versus the nucleolus: a strategic comparison

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# Research agenda

- Each solution (rule) represents a core value system.
- Logical relations between core value and "fairness" or other criteria.
- A core value can be represented or equivalent to the implications of various combinations of fairness criteria.
- What makes one solution (core value) different from others.

# Research approaches

1. Definition: simplicity and intuition.
2. Axiomatic approach: The departure point of the approach is the fairness properties. These properties are formally used to compare solutions. The ultimate object of the axiomatic study is to understand the implications of various combinations of different fairness properties. It is a centralized system.

# Research approaches

3. Strategic approach: How to implement a core value (or a solution)? We can adopt the axiomatic approach to convince individuals that the socially desirable outcome recommended by the solution is "fair" and "justice". Alternatively, the desirable outcome (or implementing the solution) recommended by the solution can be achieved through designing a non-cooperative game in which agents behave based on their own interests. It is a decentralized system. The axiomatic and strategic approaches are complement to each other. The departure point of the approach is to bridge the gap between the two counterparts (namely, cooperative and non-cooperative) of game theory. It is now called Nash program.

# Motivation

The nucleolus (Schmeidler, 1969) and the egalitarian solution (Dutta and Ray, 1989) are central solution concepts in the theory of TU games. They have been applied to a number of resource allocation problems. An allocation chosen by the nucleolus (follow John Rawls maximin principle) maximizes the worth of the worst-off coalitions in the lexicographic order. An allocation chosen by the egalitarian solution (follow egalitarianism principle) is the Lorenz-maximal element from the set of allocations satisfying core-like participation constraints.

# Motivation

The differences between two solutions have been investigated from axiomatic viewpoint. For instances, in TU games,

- Theorem (Dutta, 1990): $\phi$ satisfies Davis-Maschler consistency + for two-agent case, $\phi$ coincides with the constrained egalitarianism solution $\Leftrightarrow \phi =$ the egalitarian solution.
- Theorem (Sobolev, 1975): $\phi$ satisfies Davis-Maschler consistency + for two-agent case, $\phi$ coincides with the standard solution $\Leftrightarrow \phi =$ the nucleolus.

# Motivation

However, the differences between the two solutions have not been explored (or satisfactorily answered) from strategic viewpoint. Our aim of this paper is to fill the gap. To be precise about strategic difference (or strategic comparison) between two solutions, suppose that a game $\Gamma$ implements (or strategically justifies) a solution $\alpha$. Suppose that another game $\Omega$ obtained by revising $\Gamma$ based on certain criteria implements another solution $\beta$. We say the criteria are the strategic differences between $\alpha$ and $\beta$.

# Motivation

Serrano (1993) offers a three-agent strategic justification of the nucleolus in TU games and moreover, points out that this three-agent result is impossibly extended to the case with more than three agents. Since then, attention has been drawn to smaller domains. Here, we consider a class of cost allocation problems, which exemplifies the following real-life applications.

## Taxi-Fare Sharing Problem

- Several agents are jointly riding a taxi, different agents having different destinations and different uses for it.
- The further the destination an agent has, the longer the distance the agent needs.
- The taxi that accommodates a given agent with a certain distance accommodates any nearer distance that any agent has.

How should the taxi-fare be shared among them?

## Airport Problem

- Several airlines are jointly using an airstrip, different airlines having different needs for it.
- The larger the planes an airline flies, the longer the airstrip it needs.
- An airstrip that accommodates a given airplane accommodates any smaller airplane any airline flies.

How should the maintenance cost of the airstrip be shared among the airlines?

# Applications

**Other applications**

- Irrigation ditch problem
- Highway toll-fee problem
- Public transportation ticketing problem
- Elevator user-fee problem
- *etc.*

# Applications

## General applications

- Users in a group are linearly ordered by their needs for a facility.
- Accommodating a given user implies accommodating all users whose needs are smaller than his at no extra cost.
- The facility should satisfy a user with the largest need.

How should the cost of building up or maintaining such facility be shared among the users?

# The model: formal definition

- This class of cost sharing problems has been studying for many years. One famous example is the study of 25 irrigation ditches located in south-central Montana, USA.
- However, there was no formal discussion and rigorous analysis about this class of cost sharing problems until Littlechild and Owen (1973).
- How do they formulate the class of the problems?

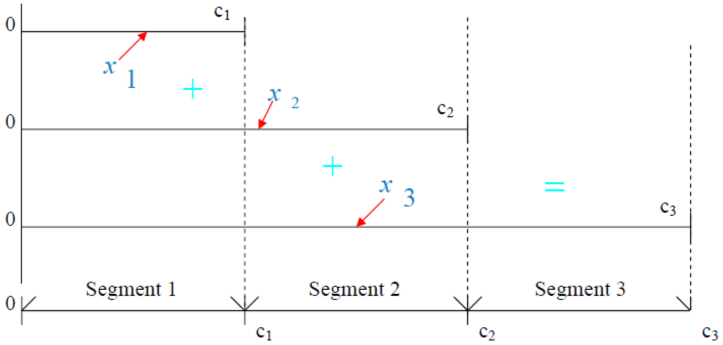
# The model: formal definition

$$\varphi\left(N \equiv \{1,2,3\}, c \equiv (c_1, c_2, c_3) \in \mathbb{R}_+^N\right)$$

$$= (x_1, x_2, x_3) \in \mathbb{R}^N \text{s.t. for each } i \in N, 0 \leq x_i \leq c_i \text{ and } \sum_{i \in N} x_i = \max_{j \in N} c_j.$$

- The property, $0 \leq x_i \leq c_i$, is referred to as reasonableness and says that agent $i$ should not receive a subsidy and should not contribute more than his cost parameter (the stand-alone cost).
- The property, $\sum_{i \in N} x_i = \max_{j \in N} c_j$, is referred to as efficiency and says that a rule should collect the exact amount of money to complete the work.

# The model: geometric representation




$$(x_1, x_2, x_3) = \varphi\left(\{1,2,3\}; c_1, c_2, c_3\right)$$

# TU game solutions applied to the problems

- In the literature, there are several ways to create cost sharing rules. One can take advantage of the theory of TU games to create rules.
- How to do it?

# The model: transformation

Airport Problem → TU Game

(N, C)

For each S in N,
$V_{(N,C)}(S)=\max\{C_i\}$

$(N, V_{(N,C)})$

$\varphi$ is a cost sharing
rule generated by $\eta$
and $\varphi(N,C)$ is the
choice for (N,C)

Apply a
TU game
solution $\eta$

$\varphi(N, C) = \eta(N, V_{(N,C)})$

# The egalitarian solution vs the nucleolus

Thanks to the linear structure of the nested cost sharing problems, the allocation chosen by the egalitarian solution can be obtained by the following formula (Aadland and Kolpin, 1998).

**Egalitarian solution**, $\boldsymbol{E}$: For each $(N, c) \in \mathcal{A}$,

$$\begin{aligned} E_1(N,c) \equiv & \quad \min_{1 \le k \le n} \left\{ \frac{c_k}{k} \right\} \\ E_i(N,c) \equiv & \quad \min_{i \le k \le n} \left\{ \frac{c_k - \sum_{p=1}^{i-1} E_p(N,c)}{k-i+1} \right\}, \quad \text{where } 2 \le i \le n-1 \\ E_n(N,c) \equiv & \quad c_n - \sum_{p=1}^{n-1} E_p(N,c). \end{aligned}$$

# The egalitarian solution vs the nucleolus

The formula of the egalitarian solution can be understood by the following algorithm. Start by requiring that all agents in $N$ should contribute equally until there are $\lambda^1 \in \mathbb{R}_+$ and a group of agents $\{1, \cdots, l^1\}$ such that $\lambda^1 l^1 = c_{l^1}$. Therefore, each agent in $\{1, \cdots, l^1\}$ then contributes $\lambda^1$. The algorithm next requires that all agents in $\{l^1 + 1, \cdots, n\}$ should contribute equally until there are $\lambda^2 \in \mathbb{R}_+$ and a group of agents $\{l^1 + 1, \cdots, l^2\}$ such that $\lambda^2 (l^2 - l^1) = c_{l^2} - c_{l^1}$. Therefore, each agent in $\{l^1, \cdots, l^2\}$ then contributes $\lambda^2$. Continue this process until the total cost $c_n$ is covered.

# The egalitarian solution vs the nucleolus

We exploit this algorithm to construct a two-stage extensive form game with finite rounds. Suppose that the game proceeds to Round $t$. Let $N^t$ be the set of the remaining agents in Round $t$. Let $c^t$ be the revised costs profile of $N^t$.

Stage 1: The first agent in Round $t$, say agent $p$, is the team maker (also a member of the team). He picks his teammates from $N^t$, say $S$, and proposes $|S|$ positive real numbers (potential contributions) for $S$, say $X_S \equiv \{x_1, \cdots, x_{|S|}\}$ with $\sum_{i=1}^{|S|} x_i = \max_{i \in S} c_i^t$, so that his teammates can select in the next stage.

# The egalitarian solution vs the nucleolus

Stage 2: Each of his teammate $j \in S\backslash\{p\}$ announces a permutation $\pi^j : S\backslash\{p\} \to S\backslash\{p\}$. The composition of all permutations determines the order of the teammates to respond the team maker's proposal $X_S$. When teammate $l$ is called up, he either accepts $X_S$ in which case he chooses a number from those numbers available to him, or rejects $X_S$ in which case he makes a counteroffer $a_l$ with $0 \leq a_l \leq \max X_S$ to the team maker.

# The egalitarian solution vs the nucleolus

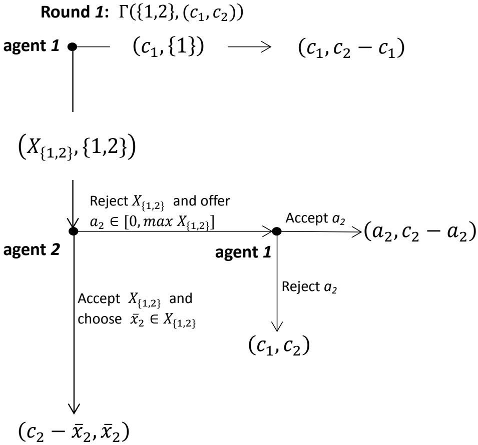
Stage 2: The team maker either accepts $a_l$ in which case he pays $a_l$ and leave the game, and the costs of all remaining agents are revised down by $a_l$ and play the same game without the team maker, or rejects $a_l$ in which case the team maker is ejected and pays $c_p$ (the stand alone cost of the team maker), and the costs of all remaining agents keep unchanged and play the same game without the team maker. Finally, when all teammates accept $X_S$, each teammate contributes his chosen amount and the team maker contributes the amount equal to the last number. The cost of each non-teammate is revised down by $\sum_{i=1}^{|S|} x_i$ and all non-teammates play the same game without $S$.

# The egalitarian solution vs the nucleolus

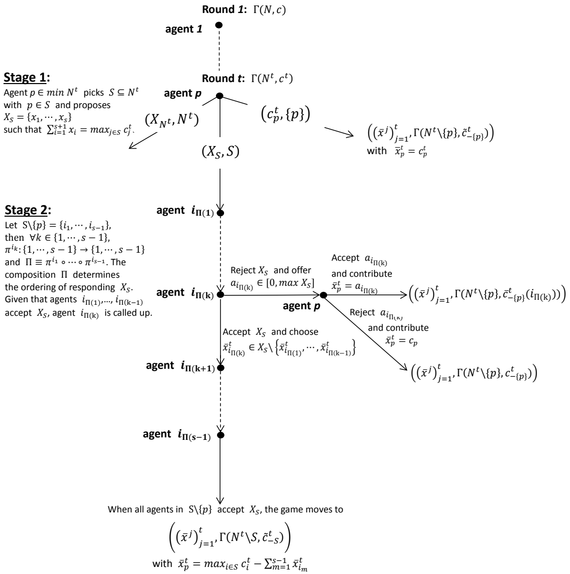
**Stage 1:**

Agent *1* picks $S \in \{\{1\},\{1,2\}\}$ and proposes $c_1$ if $S = \{1\}$, and $X_{\{1,2\}} = \{x_1, x_2\}$ such that $x_1 + x_2 = c_2$ if $S = \{1,2\}$.

**Stage 2:**



# The egalitarian solution vs the nucleolus



# The egalitarian solution vs the nucleolus

We show that

**Theorem:** (Existence result) There exists a subgame perfect equilibrium of $\Gamma_E(N, c)$ with outcome $E(N, c)$.

**Theorem:** (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma_E(N, c)$ is $E(N, c)$.

# The egalitarian solution vs the nucleolus

We next introduce the nucleolus. For general games, the payoff vector chosen by the nucleolus is difficult to compute since it involves a sequence of linear programs. However, for the cost sharing problem, its contributions vector can be obtained by Sönmez (1994)'s formula defined next.

**Nucleolus**, $\boldsymbol{Nu}$: For each $(N, c) \in \mathcal{A}$,

$$\begin{aligned} Nu_1(N,c) \equiv &\quad \min_{1 \le k \le n-1} \left\{ \frac{c_k}{k+1} \right\} \\ Nu_i(N,c) \equiv &\quad \min_{i \le k \le n-1} \left\{ \frac{c_k - \sum_{p=1}^{i-1} Nu_p(N,c)}{k-i+2} \right\}, \quad \text{where } 2 \le i \le n-1 \\ Nu_n(N,c) \equiv &\quad c_n - \sum_{p=1}^{n-1} Nu_p(N,c). \end{aligned}$$

# The egalitarian solution vs the nucleolus

What makes the nucleolus different from the egalitarian solution from strategic perspective. As krishina and Serrano (1996) suggest, the properties of a solution can be used as guides to design a non-cooperative game that implements the solution. It is well-known that the nucleolus satisfies the property of last-agent cost additivity but the egalitarian solution does not. We exploit this difference to obtain a new game from the one implementing the egalitarian solution. How to do it? We propose to exclude the participation of the last agent in Stage 1 of each round and after collecting all other other agents' contributions, agent $n$ contributes the residual cost (the difference between the total cost and the total contribution already made).

# The egalitarian solution vs the nucleolus

Inspired by the formulae of the nucleolus and the egalitarian solution, another difference between the two solutions is: the denominator of each term in the formula of the nucleolus is incremented by one, compared to the denominator of the corresponding term in the formula of the egalitarian solution. This suggests that the last agent plays a role of helper in Stage 2 of each round. What intuition behind this suggestion.

# The egalitarian solution vs the nucleolus

Thanks to the nested cost structure again, the nucleolus formula can be understood by the following algorithm, which shares a similar spirit with the one for the egalitarian solution. The algorithm starts by requiring that all agents in $N \backslash \{n\}$ should contribute equally until there are $\beta^1 \in \mathbb{R}_+$ and a group of agents $\left\{1, \cdots, p^1\right\}$ such that $\beta^1 \left(p^1 + 1\right) = c_{p^1}$. Therefore, each agent in $\left\{1, \cdots, p^1\right\}$ then contributes $\beta^1$. The algorithm next requires that all agents in $\left\{p^1 + 1, \cdots, n - 1\right\}$ should contribute equally until there are $\beta^2 \in \mathbb{R}_+$ and a group of agents $\left\{p^1 + 1, \cdots, p^2\right\}$ such that $\beta^2 \left(p^2 - p^1 + 1\right) = c_{p^2} - \beta^1 p^1$. Therefore, each agent in $\left\{p^1, \cdots, p^2\right\}$ then contributes $\beta^2$. Continue this process until $c_n$ is covered.

# The egalitarian solution vs the nucleolus

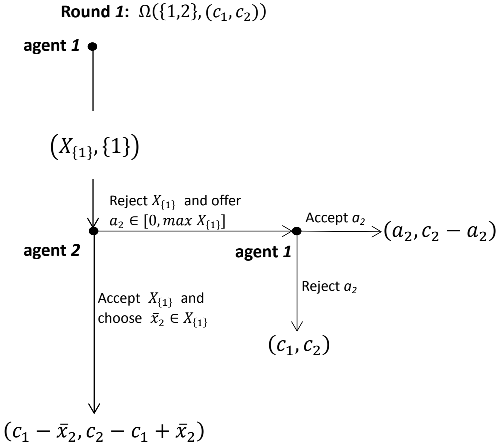
We revise the game that implements the egalitarian solution based on the two differences as follows.

- In Stage 1: The last agent is never a teammate in Stage 1 of each round. Instead,
- In Stage 2: The last agent plays in Stage 2 of each round as a role of helper to reduce the total contribution of the team in each round. However, the last agent's contribution is determined after collecting all other agents‘ contributions. Namely, he contributes the residual cost.

# The egalitarian solution vs the nucleolus

**Stage 1:**
Agent *1* picks $\{1\}$ and proposes $X_{\{1\}} = \{x_1, x_2\}$ such that $x_1 + x_2 = c_1$.

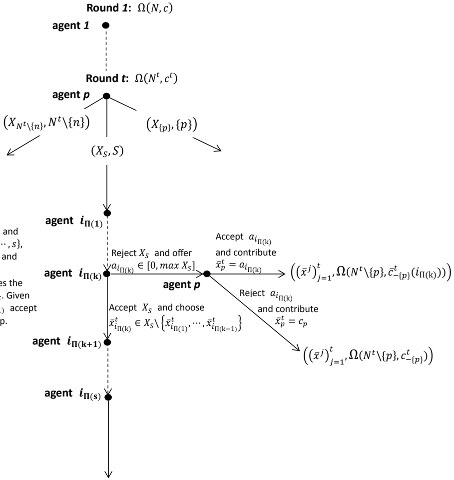
**Stage 2:**



# The egalitarian solution vs the nucleolus

**Stage 1:**
Agent $p \in min\, N^t$ picks $S \subseteq N^t \backslash \{n\}$ with $p \in S$, and proposes $X_S = \{x_1, \cdots, x_{s+1}\}$ such that $\sum_{i=1}^{s+1} x_i = max_{j \in S}\, c_j^t$.

**Stage 2:**
Let $S \backslash \{p\} = \{i_1, \cdots, i_{s-1}\}$, and let $i_s = n$, then $\forall k \in \{1, \cdots, s\}$, $\pi^{i_k}: \{1, \cdots, s\} \to \{1, \cdots, s\}$ and $\Pi \equiv \pi^{i_1} \circ \cdots \circ \pi^{i_s}$. The composition $\Pi$ determines the ordering of responding $X_S$. Given that agents $i_{\Pi(1)}, \ldots, i_{\Pi(k-1)}$ accept $X_S$, agent $i_{\Pi(k)}$ is called up.




When all agents in $(S \cup \{n\}) \backslash \{p\}$ accept $X_S$, the game moves to

$$\left((\bar{x}^j \backslash \{\bar{x}_n^j\})_{j=1}^t, \Omega(N^t \backslash S, \hat{c}_{-S}^t)\right)$$

with $\bar{x}_p^t = max_{i \in S}\, c_i^t - \sum_{m=1}^{s} \bar{x}_{i_m}^t$

where $\bar{c}_{-\{p\}}^t(j) \equiv \left(max\{c_i^t - a_j, 0\}\right)_{i \in N^t \backslash \{p\}}$ and $\hat{c}_{-S}^t \equiv (max\{c_i^t - \sum_{k \in S} \bar{x}_k^t, 0\})_{i \in N^t \backslash S}$.

# The egalitarian solution vs the nucleolus

We show that

**Theorem:** (Existence result) There exists a subgame perfect equilibrium of $\Gamma_{Nu}(N, c)$ with outcome $Nu\,(N, c)$.

**Theorem:** (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma_{Nu}(N, c)$ is $Nu\,(N, c)$.

# The egalitarian solution vs the nucleolus

We do not only offer strategic justifications of the egalitarian solution and the nucleolus but also show that assigning different roles to the last agent leads to implementing different solutions. The results point out the difference between the two solutions from strategic perspective and establish a strategic comparison between the solutions. This is the first paper in the non-cooperative implementation literature to observe such surprising phenomenon.

Thank you!!