The Seller's Listing Strategy in Online Auctions*

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Abstract

This paper proposes a unified framework to completely characterize the sellers' optimal listing strategy in the online auction as a function of their rates of time impatience. Specifically, the fixed-price listing, the regular auction, and the buy-it-now auction are each a solution of the seller's single optimization problem under different values of the rate of intertemporal discount: the perfectly patient seller adopts the regular auction; the sellers with a medium range of time impatience adopt the buy-it-now auction; and the most impatient of sellers adopt the fixed-price listing. Moreover, in the regular and buy-it-now auctions, the optimal reserve price is inversely related to the seller's intertemporal discount factor, either within or across listing types. This implies that the optimal reserve price for the buy-it-now auction is greater than that of the regular auction. These predictions offer clear empirical implications.

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1 Introduction

Although the Revenue Equivalence Theorem has identified a very general condition under which all auction formats yield exactly the same expected revenue to the seller, in reality the sellers have many other considerations than expected revenue itself. In the online auctions, their considerations are even more diverse. This is because the online auction offers the sellers a varieties of formats to list their items, which affect other dimensions of the auction's outcomes than revenue. One important consideration, which is the main concern of this paper, is that unlike the traditional virtual auctions whose bidding process rarely takes more than one hour, the duration of an online auction generally lasts for several days. Therefore, transaction rules which help to satisfy the seller's or the buyer's time preference are important.

Depending on the difference in the bidding rules, there are three major listing formats in the online auctions. The first is the usual ascending price auction. The seller lists an item by posting a fixed bidding duration (hereafter called "posted duration"), either with or without a reserve price.[1] The highest bidder at the end of the posted duration is the winner. To clearly distinguish between auctions of different formats, in this paper we will call it the "regular auction." Since most auction sites adopt the proxy bid rule, the online regular auction is essentially a second-price auction, and almost all the literature treats the online regular auctions as such. The second format is the fixed-price listing: An item is listed for a specific posted duration with a fixed posted price, and the bidders who is willing to pay this price can place an order at any time within the posted duration to win the item. The third format, which is a specific online auction innovation, is the buy-it-now (BIN) auction. On top of the regular auction, the seller sets a price, called the buy-it-now price (BIN price), under which the bidder has the additional option to win the item immediately by agreeing to pay that price any time within the posted duration. In the eBay auction (which is the main concern in this paper), the BIN option is temporary: If a bidder places a bid before any other bidder exercises BIN,

[1] In the online auction, there are two types of reserve price. The publicly observed reserve price is in the form of the starting (minimum) bid. The first bid must be at least as high as the starting bid to be eligible. Another type of reserve price, the secret reserve price, is one set by the seller which cannot be publicly observed. A bidder who bids below the secret reserve price will only be informed as such, without learning its value.

then the BIN option disappears, and the BIN auction reduces to a regular auction. Hacker and Sickles (2010) have shown in their eBay auction data that all three formats occupy significant portions among all listings.[2]

An obvious question to ask is: What determines a seller's choice of one format over the others? There is surprisingly little research that addresses this question. In a seminal paper, Wang (1993) compared the fixed-price sale and the regular auction. He showed that, if bidders' valuations are independent and they arrive randomly, the regular auction will be the more preferable to the seller relative to fixed-price sale, the steeper the buyer's marginal-revenue curve. His model compares only the regular auction and the fixed-price sale.

Wang et al. (2008) assumes that bidders have different costs of participating in a regular auction and a BIN auction. Depending on the relative cost, there exists a threshold value of the bidder's valuation above which he prefers exercising BIN to bidding. This in turns characterizes two cut-off values, one high and one low. If the BIN price is set to be higher than the high cut-off, no bidder will exercise BIN, and the BIN auction is essentially a regular auction. If the BIN price is set lower than the low cut-off, then all bidders prefer to exercise BIN rather than bidding, and the BIN auction is essentially a fixed-price posting. Only if the BIN price is set between the two cut-off values is the auction a true BIN auction. This model, however, implies that the fixed-price auction will result in the lowest price among all formats, essentially because its BIN price is so low that *every* bidder wants to exercise it. This not only is at odds with intuition, but is also inconsistent with the extant empirical literature, which consistently shows that the transaction price of the fixed-price listing is the highest among all formats (Hammond 2010, Chen et al. 2013b, and Einav et al. 2013).[3]

In this paper, we propose a simple theory to capture the sellers' strategic consideration in selecting the listing formats as a function of their time preferences. The difference in the

[2] Einav et al. (2013), however, had shown that the proportion of regulation auction (fixed-price listings) has substantially decreased (increased) in the past decade.

[3] Also, in the model of Wang et al. (2008), the seller's choice of format is not based on their own characteristics, but on their perception of the bidders' valuations. This implies that the sellers adopt different formats because of their difference in perceptions, which also implies that some sellers' perceptions are incorrect. In one word, their model is not an equilibrium model.

bidding rules for the three formats discussed above has strong implication for the expected length of time needed for completion of a transaction (hereafter called the "sale duration") in each format. This in turn provides the intuition on how each format can serve the needs of sellers with different time preferences. The regular auction needs to run the whole length of its posted duration to determine the highest bidder as winner. On the other hand, in a fixed-price listing, the transaction is completed as soon as the first bidder places an order. The BIN auction lies in between: If some bidder exercises BIN first, then transaction will be completed immediately. If, however, some bidder places a bid first, then the BIN auction reduces to a regular auction, and will then run until the end of the posted duration to complete the transaction. This implies that, given identical posted duration, the regular auction will takes the longest expected length of time to complete the transaction, followed by the BIN auction, with the fixed-price listing the shortest. Given that regular auction (with reserve price) is the optimal auction in terms of revenue (Myerson, 1981), the choice of the three formats is to balance the tradeoff between revenue and payoff delay.

The theoretical literature has also identified time preference as one of the reasons for the sellers to adopt BIN (e.g., Mathews, 2004).[4] Under this theory, the seller is time impatient and, other things being equal, prefers to reach a sale sooner than later. Using this framework, we endogenize the seller's optimal listing strategy in terms of their rates of time impatience. Specifically, in a two-stage bidding process, the seller sets the BIN price and the reserve price (or, equivalently, the minimum bid) to maximize discounted expected revenue.[5] Under this

[4] Another strand of literature rationalizes BIN by showing that it offers a chance to reduce price risk for either the buyer or the seller. See Budish and Takeyama (2001), Mathews and Katzman (2006), Hidvegi et al. (2006), Reynolds and Wooders (2009), and Chen et al. (2013).

[5] In the online auctions, the seller's publicly observed reserve price is implemented through the minimum (starting) bid that he must set when listing an item. In addition, the seller can also set a secret reserve price. In the standard price-clock model for auctions, price rises continuously until only one bidder remains. Given that the bidders bid their true valuations in the second-price auction, secret and public reserve prices are equivalent, either in bidder's strategy or seller's revenue. However, in reality they are not equivalent, and field experiments show mixed results. Bajari and Hortaçsu (2003) report that, relative to public reserve price, secret reserve price increases not only the number of bids, but also the seller's revenue. This is especially so for high-value items. On the other hand, Dewally and Ederington (2004) and Katar and Reiley (2005) show exactly the opposite. Secret reserve price is rarely used by sellers in most item categories; therefore we focus on the public reserve price in this paper.

formulation, the three listing formats are each a solution of the seller's maximization problem, depending on the value of the seller's intertemporal discount rate. The fixed-price listing is the corner solution in which the BIN price equals the reserve price; the regular auction is the corner solution in which the BIN price is so high that no bidder exercises it; and the BIN auction is the interior solution in which the BIN price is effective, and is strictly greater than the reserve price. We show that, when the seller is perfectly patient, BIN serves no purpose, and the regular auction is therefore the seller's optimal listing format. When the seller is not perfectly patient, there exists a threshold value of time impatience such that a seller less (more) patient than this threshold will adopt the fixed-price listing (BIN listing). We therefore completely characterize the sellers' listing strategy, together with the corresponding optimal reserve price, as a function of their degree of time impatience.

An important result implied by the characterization is that, under mild conditions, the seller's intertemporal discount factor has an inverse relationship with the optimal reserve price, either within or across formats. This result offers a clear empirical prediction of the theoretical model. Note that the minimum bid (also called the "starting bid") of an auction is equivalent to the publicly observed reserve price. Ideally, if a proxy for the degree of the seller's time preference can be identified in the auction data, then it should have an inverse relationship with the reserve price, regardless of auction formats. Even if such a proxy cannot be identified, we would still observe higher starting bids in the BIN auctions than in the regular auctions, after other heterogeneities are controlled for.

Some empirical literature, though not directly related to this paper, also investigates the reason behind the choice of format. Hammond (2010) compared the regular auctions and the fixed-price listings of CDs, and showed that the sellers who hold larger inventory are more likely to adopt the fixed-price listing than the regular auction. Moreover, regular auctions result in a higher sale rate but lower transaction prices. Einav et al. (2013) recorded that the proportion of regular auctions on eBay has substantially declined over the past ten years, while that of the fixed-price listings has greatly increased. This is mainly caused by the shift of buyers' preference toward the posted-price format. Bauner (2015) uses auction data of baseball tickets on eBay to show that, in addition to opportunity costs, buyer heterogeneity is also an important factor in explaining the simultaneous existence of auctions and fixed-price sales. This enables him to build up a structural model to investigate the welfare implications of BIN.

2 The Model

An item is auctioned among n bidders. The valuation of the item to each bidder i, v_i, is the bidder's private information, and is independently drawn according to a continuous density function $f(\cdot)$, with $f(v) > 0$ for all v in its support $[0, v_H]$.[6] Let $F(\cdot)$ be its distribution function. We make the usual assumption that $f(\cdot)$ has an increasing hazard rate, i.e., $f(v)/(1 - F(v))$ is strictly increasing in v. To list an item, the seller has two decision variables: One is the buy-it-now price B, and the other is the publicly observed reserve price r in the form of the starting (minimum) bid.[7] As a rule, the value of r must not be greater than B, i.e., it must be that $r \leq B$.

We are concerned with the eBay-type temporary BIN auction, in which BIN will disappear whenever a bidder places a bid.[8] We adopt the two-stage model first proposed by Mathews and Katzman (2006) and Reynolds and Wooders (2009) for the eBay BIN auctions.[9] In the first stage, every bidder can decide whether to purchase the item with the BIN price. If at least one bidder exercises BIN, it is sold with price B. In case more than one bidder is willing to do so, they win with equal probability. If no bidder is willing to buy out the item, then the auction enters the second stage, in which the bidders compete via a second-price auction with reserve price r.[10] The seller is time impatient, and discounts the second-stage payoff at a rate

[6] The support of $f(\cdot)$ can be extended to be any interval $[v_L, v_H]$, $0 < v_L < v_H$, with simple adaptation. We use $[0, v_H]$ to economize the use of notations.

[7] See also footnote 5.

[8] eBay has changed the rule for BIN frequently. At the time the paper was written, BIN disappears in the motors category only if the bid is greater than starting bid by a certain amount. In other categories BIN disappears as soon as some bidder places a bid.

[9] Both papers, however, assume risk-averse, rather than time-impatient sellers. Mathews (2004) considers time-impatient agents in a continuous time setting, but ignores the reserve price. All three papers are concerned with how BIN serves the sellers' risk aversion or time preference, not their choice of formats.

[10] The use of a proxy bid rule in the online auctions essentially makes the eBay ascending price auctions into

δ ($0 < \delta \leq 1$). Since a seller is characterized by his time discount rate, we call a seller with discount rate δ a"δ-seller".

The assumption regarding the two-stage transaction deserves further elaboration. In reality, neither the bidder who intends to place a bid nor one who intends to exercise BIN has the priority over the other: It is only a matter of who does it first. Therefore, the assumption that the bidder who intends to exercise BIN has priority in the first stage, while the bidder who intends to place a competitive bid can only do it in the second, seems artificial at first sight. However, this is actually a natural assumption. As mentioned, in an auction without BIN (or if the BIN price is so high that no bidder wants to exercise it), the seller has to wait until the end of the posted duration to determine the winner and complete the transaction. In a BIN auction, however, the bidders can end the auction anytime by exercising BIN. In others words, a regular auction necessarily goes through the entire posted duration (so the seller suffers discount) to complete the transaction, while the BIN auction needs not. Our assumption that the auction will end in the first stage if a bidder exercises BIN, and ends in the second if all bidders intend to wait and bid (so that no one exercises BIN), is meant to capture the nature of this important fact in the most parsimonious model that can be constructed.[11]

2.1 The Bidder's Strategy

We solve for the bidder's optimal strategy by backward induction. The optimal strategy for the second stage is simple: Since the second stage is a second-price auction, each bidder bids an amount equal to her valuation. In the first stage Lemma 1 shows that, a bidder's optimal strategy is a cut-off buyout strategy, i.e., to exercise BIN if her valuation is greater than some threshold value $\tilde{v}$; otherwise she waits until the second stage to place a bid. This result has been derived by Mathews and Katzman (2006) and Reynolds and Wooders (2009) in the context with risk-averse bidders. Our proof is a simple adaptation of their results to the time-impatient

second-price auctions.

[11] Our model rules out the possibility that a bidder, having low valuation, might place a bid in the first stage, only with the purpose of eliminating the BIN option. It is easy to show that this yields exactly the same expected payoff as doing nothing in the first stage, and waiting until the second stage to place a bid.

context, and is therefore omitted.[12]

Lemma 1. *Given B and r, there exists a symmetric Nash equilibrium which is characterized by a threshold value of valuation, $\tilde{v} \in [0, v_H]$, so that bidder i will exercise BIN in the first stage if $v_i > \tilde{v}$, and will refrain from exercising BIN and compete in the second stage if $v_i \leq \tilde{v}$. The value of the buyout threshold $\tilde{v}$ is determined by*

$$B = \tilde{v} - \left[\frac{n(1 - F(\tilde{v}))}{1 - F(\tilde{v})^n}\right] \int_r^{\tilde{v}} F(x)^{n-1} dx. \tag{1}$$

Lemma 1 says that once the seller sets the values of B and r, there is a threshold value $\tilde{v}$, characterized by (1), so that only the bidders whose valuations are greater than $\tilde{v}$ will exercise BIN.

To summarize a bidder's optimal strategy for the whole auction: Given r and B, in the first stage the bidder exercises BIN if her valuation is greater than $\tilde{v}$ as defined in (1), and waits until the second stage to bid if it is smaller. In the second stage, her bid equals her valuation.

2.2 The Seller's Optimal Choice

Given the bidder's optimal strategy characterized in Lemma 1, we now solve for the seller's optimal strategy, which is to choose B and r to maximize expected revenue:

$$\max_{B,r} \mathrm{E}\pi \equiv \int_{\tilde{v}}^{v_H} B dF(x)^n + n\delta \int_r^{\tilde{v}} \left(rF(r)^{n-1} + \int_r^x y dF(y)^{n-1} \right) dF(x),$$

$$s.t. \; v_H \geq B \geq r \geq 0,$$

where the value of $\tilde{v}$ is defined by (1). The first term in the objective function is the seller's expected revenue of some bidder exercising BIN in the first stage, and the second term is the seller's expected revenue of its ending with competitive bidding in the second stage.

We will make a transformation of the seller's decision variables that will substantially simplify the solution process. It is easy to see that the right-hand side of (1) is strictly increasing in $\tilde{v}$. Therefore, (1) implies a strictly monotonic relationship between the BIN price B and the

[12] For completeness, a proof is provided in the web appendix (Appendix B).

buyout threshold $\tilde{v}$. Given this 1-1 correspondence, we can assume that the seller's decision variables are $\tilde{v}$ and r, instead of B and r. In this case we denote $B(\tilde{v}, r)$ as the value of the BIN price the seller has to set, if he chooses the bidder's buyout threshold to be $\tilde{v}$ and the reserve price to be r. Note that, given $\tilde{v}$ and r, the value of $B(\tilde{v}, r)$ is exactly the right-hand side of (1).

Note that B and r must satisfy the constraint $v_H \geq B \geq r \geq 0$. Therefore, after the transformation from (B, r) to $(\tilde{v}, r)$, we also need to identify the corresponding constraint on $\tilde{v}$ and r. It turns out that the constraint for B and r transports naturally to $\tilde{v}$ and r, which is $v_H \geq \tilde{v} \geq r \geq 0$:

Lemma 2. *$B > r$ if and only if $\tilde{v} > r$, and $B = r$ if and only if $\tilde{v} = r$.*

Proof. Given $\tilde{v}$ and r, let $B = B(\tilde{v}, r)$. If $B \geq r$ but $\tilde{v} < B$, then $B = B(\tilde{v}, r) < B(B, r) \leq B$, a contradiction, implying that $\tilde{v} \geq B \geq r$. Conversely, if $\tilde{v} \geq r$, then $B = B(\tilde{v}, r) \geq B(r, r) = r$. We have therefore shown that $B \geq r$ if and only if $\tilde{v} \geq r$. Finally, $B = r$ if and only if $B = B(B, r)$, which is equivalent to $B = \tilde{v}$, which in turn is equivalent to $\tilde{v} = r$. □

Given Lemma 2, the seller's optimization problem can be rewritten as

$$\max_{\tilde{v},\, r} E\pi, \quad s.t.\ v_H \geq \tilde{v} \geq r \geq 0. \tag{2}$$

The Lagrangian is therefore

$$L = \mathrm{E}\pi + \lambda(\tilde{v} - r) + \mu_1(v_H - \tilde{v}) + \mu_2 r,$$

where $\lambda, \mu_1, \mu_2 \geq 0$ are the Lagrange multipliers.

Before proceeding to solve for (2), here we put forth the key insight for our model. The seller's optimal choice of listing format crucially depends on the solution of (2). First, if the solution of (2) is a corner solution such that $\tilde{v} = r$ — or equivalently, $B = r$ by Lemma 2 — then the auction is essentially a fixed-price listing. This is because in this case the item can only be sold at one price, r (or, equivalently, B). Second, note that $B(v_H, r)$ is the value of

BIN price for which $\tilde{v} = v_H$; i.e., it is the minimum value of the BIN price for which even the highest-valuation bidder will not exercise it.[13] If the seller sets the BIN price B to be equal to or higher than $B(v_H, r)$, then the BIN option is redundant, because no bidder will exercise it. In that case, the BIN auction is essentially a regular auction. Therefore, if the solution of (2) is a corner solution such that $\tilde{v} = v_H$ — or, equivalently $B \geq B(v_H, r)$ by Lemma 2 — then the auction is a regular auction. In this paper, we will identify the sellers who set the reserve price as r and the BIN prices greater or equal to $B(v_H, r)$ as those who adopt the regular auction with reserve price r. This is a very reasonable classification. Since a seller is charged by eBay for posting the BIN option, if he finds it optimal to set a BIN price so high to prevent all bidders from exercising it, he will not post BIN at all. Finally, if the solution for (2) is an interior solution such that $r < \tilde{v} < v_H$ — or equivalently, $r < B < B(v_H, r)$ by Lemma 2 — then it is a BIN auction as defined in the usual sense.

Depending on which of the above three cases is the solution, the seller adopts one of the three formats. In other words, the sellers' adoption of listing formats is the solution of a single optimization problem. Since the sellers differ only in the values of their discount factor, our framework completely characterizes the seller's listing strategy in terms of time impatience.

With the above classification, we can now proceed to solve for the optimization problem (2) and, given the above classification, the seller's optimal listing strategy. The first-order conditions for (2) are

$$\begin{aligned}\frac{\partial L}{\partial \tilde{v}} =& n\delta r F(r)^{n-1} f(\tilde{v}) - nF(\tilde{v})^{n-1} f(\tilde{v}) B + [1 - F(\tilde{v})^n] \frac{\partial B}{\partial \tilde{v}} \\ &+ n\delta f(\tilde{v}) \int_r^{\tilde{v}} x dF(x)^{n-1} + \lambda - \mu_1 = 0; \qquad (3)\\ \frac{\partial L}{\partial r} =& n\delta F(r)^{n-1} [F(\tilde{v}) - F(r)] - n\delta r F(r)^{n-1} f(r) + [1 - F(\tilde{v})^n] \frac{\partial B}{\partial r} - \lambda + \mu_2 = 0. \qquad (4)\end{aligned}$$

[13] Note that $B(v_H, r) = v_H - \int_r^{v_H} F(x)^{n-1} dx < v_H$ if $r < v_H$. This means that the BIN price need not be as high as v_H to prevent all bidders from exercising it.

Plugging $\frac{\partial B(\tilde{v},r)}{\partial \tilde{v}}$ and $\frac{\partial B(\tilde{v},r)}{\partial r}$ (which can be calculated from (1)) into (3) and (4), we obtain

$$1 - nF(\tilde{v})^{n-1} + (n-1)F(\tilde{v})^n - n(1-\delta)f(\tilde{v})\left[F(\tilde{v})^{n-1}\tilde{v} - \int_r^{\tilde{v}} F(x)^{n-1}dx\right] + \lambda - \mu_1 = 0; \quad (5)$$

$$n\delta F(r)^{n-1}\left[1 - F(r) - rf(r) + (1 - F(\tilde{v}))\left(\frac{1-\delta}{\delta}\right)\right] - \lambda + \mu_2 = 0. \quad (6)$$

Lemma 3 in the following shows that the optimal reserve price must be strictly positive, i.e., $\mu_2 = 0$, which in turn implies $\mu_1 = 0$ as well.

Lemma 3. *For any $\delta \in (0, 1]$, $r > 0$ and $\mu_1 = \mu_2 = 0$.*

Proof. Assume that $r = 0$, then $F(r) = 0$ and from (6) we know that $\lambda = \mu_2$. Now if $\tilde{v} = 0$ as well, then $\mu_1 = 0$. Plugging these into (5) we have $1 + \lambda = 0$, a contradiction. Therefore, $\tilde{v} > r = 0$. Therefore, in a neighbourhood of $r = 0$, the constraint $\tilde{v} \geq r$ takes strict inequality, implying that $\lambda = \mu_2 = 0$. In that case we can increase the value of r in this neighborhood without violating the constraint. However, since $\lambda = \mu_2 = 0$, the left-hand side of (6) become positive. This implies the seller can raise the value of r from 0 to increase his expected revenue. This contradicts $r = 0$ being the solution. Hence, we must have $r > 0$, implying $\mu_2 = 0$.

Next, assume that, contrary to our claim, $\mu_1 > 0$, then, $\tilde{v} = v_H$. Plugging this into (5), we have $\lambda \geq \mu_1 > 0$, implying that $r = \tilde{v} = v_H$. Plugging these again into (6), we have $\lambda \leq 0$, a contradiction. Hence, $\mu_1 = 0$. □

Lemma 3 simply says that the optimal reserve price is never as low as zero to render it worthless, and never as high as the valuation's upper bound v_H to exclude all bidders from bidding. It also allows us to ignore μ_1 in (5), and simplify (6) into:

$$1 - F(r) - rf(r) + (1 - F(\tilde{v}))\left(\frac{1-\delta}{\delta}\right) - \frac{\lambda}{n\delta F(r)^{n-1}} = 0. \quad (7)$$

In what follows, our derivation will be based on (5) and (7). To emphasize the dependence of the seller's optimal strategy on the discount factor δ, we will denote the solutions of (5) and (7) as $\tilde{v}(\delta)$ and $r(\delta)$.

We have not characterized the types of seller who will adopt the fixed-price format yet, but for the convenience of exposition that follows, we will first derive the optimal posted price in case fixed-price listing is adopted, i.e., when the seller's optimal solution is such that $\tilde{v} = r$.

Lemma 4. *The optimal posted price for the fixed-price listing, $r_F \in (0, v_H)$, is unique and satisfies*

$$r_F = \frac{1 - F(r_F)^n}{nF(r_F)^{n-1} f(r_F)}. \tag{8}$$

Proof. The probability that the item is sold, when r_F is the posted price in a fixed-price listing, is $1-F(r_F)^n$. Therefore, the seller's expected profit is $(1-F(r_F)^n)r_F$. The first-order condition for r_F is then

$$\begin{aligned} 0 &= 1 - nF(r_F)^{n-1} f(r_F) r_F - F(r_F)^n \\ &= -nF(r_F)^{n-1} f(r_F) \left\{ r_F - \left[\frac{1 - F(r_F)}{f(r_F)}\right] \left[\frac{1 + F(r_F) + \cdots + F(r_F)^{n-1}}{nF(r_F)^{n-1}}\right] \right\} \\ &\equiv -nF(r_F)^{n-1} f(r_F) h(r_F). \end{aligned}$$

Note that we can place $F(r_F)$ in the denominator because Lemma 3 has shown that $r_F > 0$ (and therefore $F(r_F) > 0$). Moreover, since $F(r_F) > 0$, the first-order condition for r_F is equivalent to $h(r_F) = 0$.

Since $(1 - F(r_F))/f(r_F)$ is strictly decreasing (by the increasing hazard rate assumption) and $(1 + F(r_F) + \cdots + F(r_F)^{n-1})/nF(r_F)^{n-1}$ is strictly decreasing, $h(r_F)$ is strictly increasing. Moreover, $\lim_{r_F \to 0^+} h(r_F) = -\infty$ and $\lim_{r_F \to v_H^-} h(r_F) = v_H > 0$. Therefore, the equation $h(r_F) = 0$ has a unique solution, which, by solving the first equality of the first-order condition above, satisfies (8). Obviously, $0 < r_F < v_H$. □

Although sellers who adopt the fixed-price format might have different time-discount factors, Lemma 4 shows that their optimal posted prices are identical, i.e., independent of their discount factors. The reason is that the fixed-price listing must end in the first stage. Therefore the value of the time-discount rate has no influence on the determination of the optimal price. Because of this, all sellers who adopt the fixed-price format also have identical expected revenue.

The following proposition derives an upper bound for the values of the discount factor under which the seller will adopt the fixed-price format.

Proposition 1. *There exists a* $\delta^1 \equiv \frac{1-F(r_F)}{r_F f(r_F)} \in (0,1)$ *such that no seller with discount rate* $\delta \in (\delta^1, 1]$ *will adopt the fixed-price format.*

Proof. We know that $r = \tilde{v} = r_F$ when the seller adopts the fixed-price format. Plugging this into (7) we have

$$1 - F(r_F) - \delta r_F f(r_F) = \frac{\lambda}{nF(r_F)^{n-1}} \geq 0,$$

which implies

$$\delta \leq \frac{1 - F(r_F)}{r_F f(r_F)} \equiv \delta^1.$$

Hence, the necessary condition for a seller to adopt the fixed-price listing is that his discount factor is less than or equal to δ^1. To show that $\delta^1 \in (0,1)$, plug the value of r_F into δ^1, which implies

$$\delta^1 = \frac{1 - F(r_F)}{\frac{1-F(r_F)^n}{nF(r_F)^{n-1}}} = \frac{nF(r_F)^{n-1}}{1 + F(r_F) + F(r_F)^2 + \cdots + F(r_F)^{n-1}} \in (0,1).$$

□

Since the fixed-price listing does not allow for competitive bidding in the second stage, the item remains unsold if no bidder exercises BIN in the first stage. Proposition 1 simply says that, for sellers patient enough, the cost of sale delay is never great enough to foreclose the possibility of sales in the second stage. By switching to either the regular or BIN auction, the benefit in the increase of sale probability outweighs the loss of transaction delay.

Note that δ^1 in Proposition 1 is only an upper bound, and is not a threshold which separates the adoption between auction (regular or BIN) and fixed-price sale. The following theorem, one of the main results of this paper, characterizes the seller's optimal listing strategy as a function of his degree of time impatience.

Theorem 1. *The seller adopts the regular auction if and only if he is perfectly patient* ($\delta = 1$). *For sellers who are not perfectly patient* ($\delta < 1$), *there exists a* $\delta^* \in (0, \delta^1]$, *so that those with discount rate* $\delta \in (0, \delta^*)$ *adopt the fixed-price listing; while those with discount rate* $\delta \in (\delta^*, 1)$ *adopt the BIN auction. The* δ^**-seller, being indifferent, will adopt either a fixed-price listing or a BIN auction.*

Proof. See Appendix A.

Theorem 1 completely characterizes the seller's optimal listing of the seller: The perfectly patient seller adopts the regular auction, the sellers with a medium range of time impatience adopt the BIN auction, and the most impatient of sellers list the items with a fixed price. The characterization is not only simple, but also conforms to our intuition on how various formats can satisfy the seller's time preference.

Although we cannot pin down the exact value of δ^*, we know that its upper bound is δ^1 as given in Proposition 1. Morover, from the definition of δ^1, it can be shown that $\lim_{n\to\infty} \delta^1 = 0$. This is essentially because r_F will approach v_H when n is large. Since only the bidders with discount factors smaller than δ^* will adopt a fixed-price format and $\delta^* \leq \delta^1$, we know that almost no bidder will adopt the fixed-price listing when the number of bidders is large:

Corollary 1. *When the number of bidders is large, almost no seller will adopt the fixed-price format.*

3 Comparison of Reserve Prices

The framework we propose not only can characterize the seller's listing strategy, but also has a strong implication on the optimal reserve price. Since only the perfectly patient seller will adopt the regular auction, its optimal reserve price, denoted by r_A, is unique, and can be easily derived. Note that, for the regular auction, $\delta = 1$ (by Theorem 1) and $\tilde{v} > r$ (by our classification of formats). The latter implies $\lambda = 0$. Plugging the two values into (7), it is easy to see that r_A must satisfy

$$r_A = \frac{1 - F(r_A)}{f(r_A)}.$$

Note that this is exactly the determination of the optimal reserve price for a traditional auction (e.g., Myerson 1981).

Although all sellers with discount factors in $(0, \delta^*)$ will adopt the fixed price listing, Lemma 4 has shown that their optimal posted price, r_F, is identical, and is given by (8). Therefore, only the reserve price for the BIN auction can depend on the value of δ. Let $r_B(\delta)$ be the optimal reserve price for a δ-seller who adopts the BIN auction. Note that $r_B(\delta)$ is defined only on $[\delta^*, 1)$.

3.1 Main Results

The following proposition shows that, among all types of listing, the regular auction has the lowest reserve price.

Proposition 2. *$r_B(\delta) > r_A$ for all $\delta \in [\delta^*, 1)$, and $r_F > r_A$.*

Proof. From (7) we know that the optimal reserve price for the BIN auction must satisfy (by plugging in $\lambda = 0$)

$$r_B(\delta) = \frac{1 - F(r_B(\delta))}{f(r_B(\delta))} + \frac{1 - F(\tilde{v})}{f(r_B(\delta))} \left(\frac{1 - \delta}{\delta} \right). \tag{9}$$

Suppose, contrary to our first assertion, that $r_B(\delta) \leq r_A$ for some $\delta \in [\delta^*, 1)$. Then

$$r_A = \frac{1 - F(r_A)}{f(r_A)} \leq \frac{1 - F(r_B(\delta))}{f(r_B(\delta))} < \frac{1 - F(r_B(\delta))}{f(r_B(\delta))} + \frac{1 - F(\tilde{v})}{f(r_B(\delta))} \left(\frac{1 - \delta}{\delta} \right) = r_B(\delta) \leq r_A,$$

a contradiction. The first inequality above comes from the increasing hazard rate assumption. This proves the first assertion. For the second assertion, note that, from (8),

$$r_F = \frac{1 - F(r_F)^n}{nF(r_F)^{n-1} f(r_F)} > \frac{1 - F(r_F)}{f(r_F)},$$

Where the inequality comes from the last line of the proof of Proposition 1. The increasing hazard rate assumption and the fact that $r_A = \frac{1-F(r_A)}{f(r_A)}$ imply $r_F > r_A$.

□

It is important to note that the determination of $r_B(\delta)$ as defined in (9) does not ensure it being a function, so $r_B(\delta)$ might be a multi-valued correspondence. However, Proposition 2 shows that, regardless of the discount rate and whether $r_B(\delta)$ is a function, the optimal reserve price for any BIN auction is greater than that for the regular auction. The optimal reserve price, $r(\delta)$, can therefore be written as

$$r(\delta) = \begin{cases} r_F & \text{for } 0 < \delta < \delta^*; \\ r_B(\delta) & \text{for } \delta \in [\delta^*, 1); \\ r_A & \text{for } \delta = 1. \end{cases}$$

The correspondence $r(\delta)$ is the solution of (7) which maps a seller's discount factor to the optimal reserve price, taking into consideration his optimal choice of listing format. By the Maximum Theorem (see, e.g., Sundaram, 1996), we know that $r(\delta)$, and therefore $r_B(\delta)$, is upper semi-continuous. By imposing an additional assumption, $r_B(\delta)$ can be shown to be decreasing :

A1. *$F(v) + vf(v)$ is strictly increasing in v for $v \in [r_A, v_H]$.*

Assumption **A1** is highly related to the second-order conditions for the seller's optimization problem. Recall that the second-order conditions for (2) are $E\pi_{rr} < 0$ and $E\pi_{\tilde{v}\tilde{v}} < 0$ at the optimal points. In particular, $E\pi_{rr} < 0$ is equivalent to $F(r) + rF(r)$ being increasing at the point of the optimal reserve price. Since the optimal reserve price must move across $[r_A, v_H]$ to approach v_H as the number of bidders, n, grows large, **A1** simply requires that the second-order condition for r hold for all possible number of bidders.[14]

Proposition 3. *Assuming* **A1**, *then $r_B(\delta)$ is decreasing in δ on $[\delta^*, 1)$.*

Proof. See Appendix A.

Although we know from Proposition 2 that both $r_B(\delta)$ and r_F are greater than r_A, there is no definite relationship between r_F and $r_B(\delta^*)$. Figure 1 plots a typical graph of $r(\delta)$ when **A1** is imposed, so that it is strictly decreasing on $(\delta^*, 1]$. Since we do not know the relative value between r_F and $r_B(\delta^*)$, there can be two possible cases. Either $r_F > r_B(\delta^*)$ (the r_F in

[14] To give some idea of how strong a restriction **A1** is, for the Beta density function $f(v) = \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)}$ on $[0, 1]$, $\alpha, \beta > 0$, **A1** requires that $\beta \leq 1$. For the truncated exponential distribution with the density function $f(v) = \frac{k \exp(-kv)}{1-\exp(-k)}$ on $[0, 1]$, **A1** requires that $k \leq 2$.

Figure 1), or $r_F < r_B(\delta^*)$ (the r'_F in Figure 1).[15] Proposition 2, however, shows that both r_F and r'_F are greater than r_A. While not a main interest of the paper, we show in Appendix B that if an additional condition **A2** holds, then $r_F > r_B(\delta^*)$. In that case the reserve price will be in inverse relationship with the discount factor in the whole range $(0, 1]$. Appendix B also shows that **A2**, similar to **A1**, is also highly related to the second-order condition (for $\tilde{v}$). It essentially requires the second-order condition for $\tilde{v}$ to hold, similar to **A1**, on $[r_F, v_H]$.

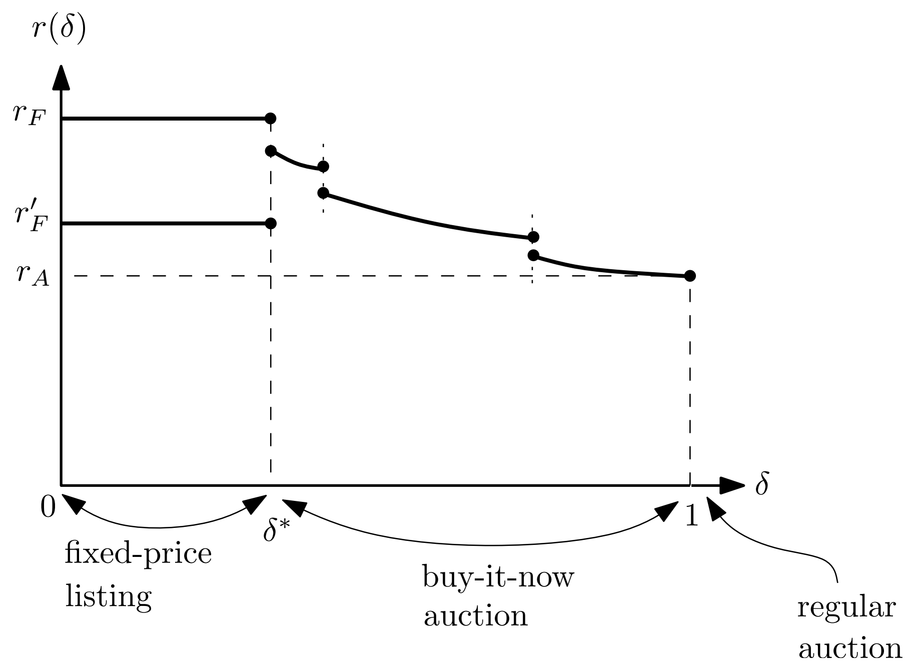

Figure 1: Optimal Listing Format and Reserve Price

Finally, in order for $r(\delta)$ to be single-valued and continuous, more restrictive conditions need to be imposed.[16] We can show that if (i) $f(v)$ is log-concave on $[r_A, r_F]$, and (ii) $f(v)$ is increasing in $[r_F, v_H]$, then not only is $r(\delta)$ a continuous function, but also the value of the discount factor that separates the BIN auction from the fixed-price listing, δ^*, is exactly δ^1

[15] Though unintuitive, the second case is theoretically possible. The function $f(\cdot)$ might be such that the sale probability under fixed price r_F is much greater than the sale rate of the BIN auction $(r_B(\delta^*), B(v(\delta^*), \delta^*)$. Therefore, the former has a much higher sale rate but lower price, while the latter is the opposite. However, they yield the same expected revenue.

[16] In fact, if $r(\delta)$ is single-valued, then $\delta^1 = \delta^*$.

defined in Proposition 1.[17] As an example, let $f(\cdot)$ be a uniformly density on $[0, v_H]$. Then $\delta^1 = \delta^* = \sqrt[n]{n+1} - 1$, and $r(\delta)$ is plotted in Figure 2. As a verification of Corollary 1, note that as $n \to \infty$, δ^* will approach 1, confirming the corollary.

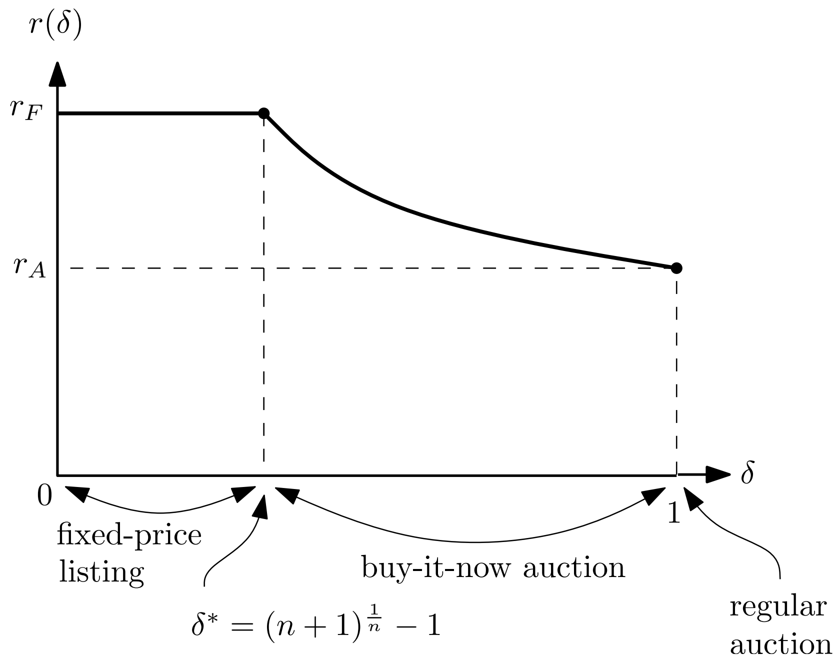

Figure 2: Optimal Listing Format and Reserve Price: Uniform Distribution Example

3.2 Intuition for the Reserve Price in BIN Auction

Our result that the reserve price in the BIN auction is higher than that of a regular auction is somewhat counter-intuitive. Since the optimal reserve price of an auction generally has a positive relation with the upper bound of the possible transaction prices (i.e., v_H in our model),[18] and since a BIN price essentially lowers this upper bound (from v_H to B), we would expect the optimal reserve price to be lower, relative to the regular auction, to accommodate the lower upper bound in the BIN auction. However, our result shows otherwise. Instead of being

[17] Please see Appendix B for proof.

[18] For example, if $F(\cdot)$ is the uniform distribution on $[0, v_H]$, then $r_A = v_H/2$, so that r_A is positively related with v_H.

lower, the BIN auction's reserve price is higher than in a regular auction, further narrowing the window of possible transaction prices. The reason for this is as follows.

As is well known in auction theory, the value of the reserve price in the regular auction is to balance the tradeoff between transaction rate and transaction price. An increase in reserve price increases the transaction price, if the item is sold, at the cost of lowing the possibility that it is sold. However, only a perfectly patient seller adopts the regular auction. When his discount factor decreases from unity, Theorem 2 shows that he will switch to the BIN auction. This opens up the possibility that the item is sold in the first stage (with BIN price B). In a BIN auction, the above-mentioned tradeoff remains true, but there is an additional source of benefit when the reserve price is increased: since it reduces the payoff of bidding in the second stage relative to direct buyout in the first, the bidders are more willing to exercise BIN. Therefore, part of the cost in the loss of transition rate when reserve price increases is compensated for by an increase of the probability that the item is sold under B in the BIN auction, a benefit that is absent in the regular auction. This implies that the seller's marginal benefit in increasing the reserve price is higher for the BIN auction than the regular auction. Consequently, the optimal reserve price must be higher for the BIN auction.

This reasoning can be seen clearly from the first-order condition for the reserve price. Since $\lambda = 0$ (the constraint $\tilde{v} \geq r$ is slack) for both the BIN and regular auctions, (7) becomes

$$[1 - F(r) - rf(r)] + (1 - F(\tilde{v}))\left(\frac{1-\delta}{\delta}\right) = 0. \tag{10}$$

Equation (10) indicates that the reserve price serves two purposes. First, as captured by the first term in (10), it balances the well-known tradeoff between transaction price and sale probability for the regular auction.[19] Second, in our framework, it also balances the tradeoff between transaction in the first stage (with BIN price) and the second stage (with winning bid). This is captured by the second term in (10). When $\delta = 1$ (under which the regular auction is optimal), the second term of (10) vanishes, and the first-order condition for the optimal reserve price is identical to that for a traditional auction. When δ decreases a little from unity, by

[19] The first term of (10) equaling 0 is exactly the first-order condition for r_A.

our characterization the seller switches to the BIN auction. This creates a positive probability that the item is sold in the first stage under the BIN price. The second term in (10) captures exactly this part of marginal gain when the seller increases the reserve price. Since this term is always positive, the marginal benefit of increasing the reserve price is always greater for the BIN auction than the regular auction. Consequently, the optimal reserve price for the BIN auction must be higher.

Equation (10) also provides intuition for the relationship between reserve price and discount factor in Proposition 3. When discount factor increases, the expected payoff of the regular auction relative to BIN will also increase (as the payoff from regular auction is less discounted). In response to this, the seller will raise the buyout threshold $\tilde{v}$ to force more bidders to enter the second stage. This implies that the second term of (10) will decrease as δ increases. In order to maintain the optimality condition (10), its first term must increase. However, in (**C1**) of the proof of Proposition 3 (Appendix A), we show that r and $\tilde{v}$ are strategic substitutes.[20] Therefore, the value of r should decrease when $\tilde{v}$ is increased (implying the inverse relationship between r and δ stated in Proposition 3. This is possible only if $F(r)+rf(r)$ must be increasing in r, so that $1 - F(r) - rf(r)$ increases when r is reduced. This is exactly what **A1** requires.

4 Conclusion

This paper adopts the time impatience framework to characterize the optimal listing strategy for the sellers. It is shown that only the sellers who are perfectly patient will adopt the regular auction. If the sellers discount the future payoffs, then those who discount the future relatively lightly will adopt the BIN auction, while the least patient among them will adopt the fixed-price listing. Not only is the characterization simple and intuitive, it also offers a clear empirical prediction regarding the relative value of reserve prices for the BIN and regular auctions. Specifically, the optimal reserve price for the BIN auction must be greater than that for the regular auction. This is a prediction ready for empirical verification. Furthermore,

[20] See Bulow et al. (1983). In the proof, we show that $\tilde{G}_{\tilde{v}\tilde{r}} > 0$, which implies $\tilde{G}_{\tilde{v}r} < 0$.

the optimal reserve price is inversely related to the seller's time-discount factor (even across listings). This prediction is also ready for empirical verification if an ideal proxy for the seller's time impatience can be identified.

Although our model assumes that the seller is time impatient while the bidders are not, all results can be extended to the case when both are time impatient, if the bidders are more patient than the seller. When the seller is more patient, the characterization of listing stays the same, while the comparison of reserve price need only be slightly adjusted.[21]

As mentioned in the Introduction, another strand of literature rationalizes BIN by showing how it can reduce transaction risk for either the bidders or the sellers. It will be a worthwhile endeavour to characterize a seller's optimal listing strategy as a function of his degree of risk aversion. Note that there are two types of risk in auctions: sales risk and price risk. While the regular auction has the lowest sales risk, it has the highest price risk. On the other hand, the fixed-price format has the lowest price risk but the highest sales risk. In other words, we cannot expect a monotonic ranking of the three formats in terms of their riskiness. Therefore the characterization of the seller's optimal listing strategy as a function of risk aversion rather than time impatience will likely be much more complicated, but can greatly enhance our understanding of the sellers' strategic listing behavior in online auctions.

[21] In the web appendix (Appendix B), we consider the case when the bidders also discount the second-stage payoff at rate δ_b ($0 < \delta_b \leq 1$). All the results go through as long as $\delta \leq \delta_b$.

Appendix A. Omitted Proofs

Proof of Theorem 1

Let $\pi^*(\delta)$ denote the optimal expected revenue of the δ-seller. By the Envelope Theorem, we have:[22]

$$\begin{aligned}\pi^*(\delta) &= \pi^*(0) + \int_0^\delta \frac{\partial \pi(\tilde{v}(\delta'), r(\delta'))}{\partial \delta'} d\delta' \\ &= \pi^*(0) + n \int_0^\delta \int_r^{\tilde{v}} \left(r dF(r)^{n-1} + \int_r^x y dF(y)^{n-1} \right) dF(x) d\delta'.\end{aligned}$$

Therefore, $\pi^*(\delta)$ is increasing in δ, and is strictly increasing when $\tilde{v}(\delta) > r(\delta)$.

Let $S = \{\delta |$ The fixed-price listing is optimal for the δ-seller$\}$. First, we argue that $S \neq \emptyset$. We prove this by showing that, for δ sufficiently small (but positive), $\tilde{v}(\delta) = r$. Suppose, on the contrary, that there exists a strictly decreasing sequence of discount rates, $\{\delta_k\}_{k=1}^\infty$, that converges to zero, such that all the δ_k-sellers adopt either the BIN or regular auction. Then the corresponding Lagrange multipliers λ_k equal 0 for all k. Since $(1-\delta_k)/\delta_k$ approaches infinity, the only way for (7) to hold for all k is for $1 - F(\tilde{v}(\delta_k))$ to approach 0. That is, $\tilde{v}(v_k)$ must approach v_H. In that case $1 - nF(\tilde{v}(\delta_k))^{n-1} + (n-1)F(\tilde{v}(\delta_k))^n \to 0$, and (5) implies that $n(1-\delta_k) f(\tilde{v}(\delta_k)) \left[F(\tilde{v}(\delta_k))^{n-1} \tilde{v}(\delta_k) - \int_r^{\tilde{v}(\delta_k)} F(x)^{n-1} dx \right] \to 0$, which in turn implies its limits, $nf(v_H) \left(v_H - \int_r^{v_H} F(x)^{n-1} dx \right) = 0$, a contradiction.

Let $\delta^* = \sup S$. Proposition 1 and the fact that $S \neq \emptyset$ together imply $\delta^* \in (0, \delta^1]$. Also, by the continuity of $\pi^*(\delta)$, $\delta^* \in S$.[23] We shall show that, for all $\delta \in (0, \delta^*)$ the δ-seller will adopt the fixed price format. If not, i.e. if the optimal strategy of a δ_0-seller is not the fixed-price listing for some $\delta_0 \in (0, \delta^*)$, then $\tilde{v} > r$ for that seller. As we have shown above, $\pi^*(\delta)$ must be strictly increasing at $\delta = \delta_0$. This implies that $\pi^*(\delta^*) > \pi^*(\delta_0)$. Since $\delta^* \in S$, the δ^*-seller can guarantee an expected revenue of $\pi^*(\delta^*)$ by adopting the fixed-price listing. Moreover, the seller's revenue in a fixed-price listing is independent of his discount factor. Then, the δ_0-seller

[22] See, for example, Milgrom and Segal (2002, Theorem 2).

[23] Note that this does not imply that the δ^*-seller will necessarily adopt the fixed-price listing. Some other format might result in exactly the same expected revenue, and also be optimal. Actually, as we will show, the δ^*-seller is indifferent between the BIN auction and the fixed-price listing.

can improve his payoff (from $\pi^*(\delta_0)$ to $\pi^*(\delta^*)$) by switching to the fixed-price format. This is a contradiction.

It remains to be shown that, a seller adopts the regular auction if only if he is perfectly patient ($\delta = 1$). Proposition 1 has shown that a δ-seller with $\delta \in (\delta^1, 1]$ will not adopt the fixed-price listing. This implies that $\tilde{v} > r$, and hence $\lambda = 0$, for $\delta = 1$. (5) further implies that

$$1 - nF(\tilde{v})^{n-1} + (n-1)F(\tilde{v})^n = 0. \tag{11}$$

Note that the left-hand side of (11) is strictly decreasing in $F(\tilde{v})$, and equals 0 when $\tilde{v} = v_H$. Therefore, (11) holds only if $\tilde{v} = v_H$, i.e., only when the solution is a regular auction. Conversely, if the seller adopts the regular auction, the left-hand side of (5), when valued at v_H, is zero. This implies that $n(1-\delta)f(v_H)\left[v_H - \int_r^{v_H} F(x)^{n-1}dx\right] = 0$. Since the term in the bracket is strictly greater than 0, it implies that $\delta = 1$.

Proof of Proposition 3

In order to simplify the proof, denote $\tilde{r} = -r \in [-\tilde{v}, -r_A]$. Also, to emphasize its dependence on the choice variables and the parameters, let $\tilde{G}(\tilde{v}, \tilde{r}; \delta)$ denote the seller's expected revenue, Eπ, on the domain $\mathcal{D} = \{(\tilde{v}, \tilde{r}) \in \mathbb{R}^2 | r_A \leq \tilde{v} \leq v_H, -\tilde{v} \leq \tilde{r} \leq -r_A\}$.[24] Let $\mathbf{x} = (\tilde{v}, \tilde{r})$, and $\mathbf{x}(\delta) \equiv (\tilde{v}(\delta), \tilde{r}(\delta)) \in \arg\max_{\mathbf{x}\in\mathcal{D}} \tilde{G}(\mathbf{x}; \delta)$. Then $r(\delta) = -\tilde{r}(\delta)$. In what follows, we use the subscript to denote the partial derivative, e.g., $\tilde{G}_{\tilde{v}} = \frac{\partial \tilde{G}}{\partial \tilde{v}}$, $\tilde{G}_{\tilde{r}} = \frac{\partial \tilde{G}}{\partial \tilde{r}} = -\frac{\partial E\pi}{\partial r}$.

The strategy of the proof is first to show that $\tilde{G}(\cdot)$ is supermodular (**C1**). Then we show that $\mathbf{x}(\delta)$ is increasing locally (**C2**). Third, we show that this can be extended globally, albeit only with weak inequality (**C3**). Finally, in (**C4**) we show that the strict inequality also holds. Since $r(\delta) = -\tilde{r}(\delta)$, a strictly increasing $\mathbf{x}(\delta)$ (and therefore $\tilde{r}(\delta)$) implies a strictly decreasing $r(\delta)$ on $(\delta^*, 1)$, which proves Proposition 3.

(**C1**) $\tilde{G}(\mathbf{x}; \delta)$ *is strictly supermodular in* $\mathbf{x}$ *for any* $\delta \in (0, 1)$. *Moreover,* $\tilde{G}(\cdot)$ *satisfies strictly increasing differences in* $(\mathbf{x}, \delta)$*; i.e.,*$\tilde{G}(\mathbf{x}, \delta) - \tilde{G}(\mathbf{x}', \delta) > \tilde{G}(\mathbf{x}, \delta') - \tilde{G}(\mathbf{x}', \delta')$ *if* $\mathbf{x} > \mathbf{x}'$ *and* $\delta > \delta'$.

[24] $\mathcal{D}$ is actually the restrictions on r and $\tilde{v}$ in (2).

Proof. For all $\delta \in (0,1)$, we can show that

$$\begin{aligned}
\tilde{G}_{\tilde{v}\tilde{r}} &= n\delta F(r)^{n-1} f(\tilde{v})(\frac{1-\delta}{\delta}) > 0; \\
\tilde{G}_{\tilde{v}\delta} &= n f(\tilde{v}) \left[F(\tilde{v})^{n-1}\tilde{v} - \int_r^{\tilde{v}} F(x)^{n-1} dx \right] > 0; \\
\tilde{G}_{\tilde{r}\delta} &= nF(r)^{n-1} f(r) \left[r - \frac{1-F(r)}{f(r)} + \frac{1-F(\tilde{v})}{f(r)} \right] \\
&\geq nF(r)^{n-1} f(r) \left[r_A - \frac{1-F(r_A)}{f(r_A)} \right] = 0;
\end{aligned}$$

where the final inequality follows from the increasing hazard rate assumption which, additionally, holds for equality only at $\mathbf{x} = (v_H, -r_A)$. The signs of these second-order derivatives imply that $\tilde{G}(\mathbf{x}; \delta)$ is strictly supermodular (Sundaram 1996, Theorem 10.4), and satisfies strictly increasing difference on $(\mathbf{x}, \delta)$ (Sundaram, 1996, Theorem 10.12).[25] ☐

(**C2**) *For any* $\delta \in (\delta^*, 1)$*, there exists an* $\varepsilon_\delta > 0$*, such that for every* $\delta' \in (\delta - \varepsilon_\delta, \delta + \varepsilon_\delta) \equiv \mathcal{B}_\delta \subset (\delta^*, 1)$*, we have* $\mathbf{x}(\delta) \leq \mathbf{x}(\delta')$ *if* $\delta < \delta'$*, and* $\mathbf{x}(\delta) \geq \mathbf{x}(\delta')$ *if* $\delta > \delta'$*.*

Proof. We only consider the case in which $\delta' > \delta$. The proof for $\delta' < \delta$ is identical.

First, we argue that $\tilde{v}(\delta) \geq r(\delta')$ and $\tilde{v}(\delta') \geq r(\delta)$ for sufficiently small ε_δ. Suppose to the contrary. Then there exists a sequence $\{\delta'_k\}_{k=1}^\infty$, with limit δ, such that for all k and the corresponding maximizers $\{(\tilde{v}(\delta'_k), r(\delta'_k))\}_{k=1}^\infty$, either $\tilde{v}(\delta) < r(\delta'_k)$ or $\tilde{v}(\delta'_k) < r(\delta)$. Since $\delta' > \delta^*$ and $\delta > \delta^*$, we have $\tilde{v}(\delta) > r(\delta)$ and $\tilde{v}(\delta') > r(\delta')$. Therefore, either (i) $\tilde{v}(\delta'_k) > r(\delta'_k) > \tilde{v}(\delta) > r(\delta)$, or (ii) $\tilde{v}(\delta) > r(\delta) > \tilde{v}(\delta'_k) > r(\delta'_k)$. Assume that $(\tilde{v}^*, r^*)$ is an accumulation point of $\{(\tilde{v}(\delta'_k), r(\delta'_k))\}_{k=1}^\infty$. Then the upper semi-continuity property of maximizers (the Maximum Theorem; see Sundam 1996) implies that $(\tilde{v}^*, r^*)$ maximizes $\mathrm{E}\pi(\delta)$. For case (i), we have $\tilde{v}^* \geq r^* \geq \tilde{v}(\delta) > r(\delta)$, and **A1** implies that $F(r^*) + r^* f(r^*) \geq F(\tilde{v}(\delta)) + \tilde{v}(\delta) f(\tilde{v}(\delta)) > F(r(\delta)) + r(\delta) f(r(\delta))$. Then it follows from (7) that

[25] If we do not require $\tilde{v} \geq r$, then $\mathcal{D}$ will be a sublattice. In that case we can greatly simplify the proof by directly applying Theorem 10.7 in Sundaram (1996) to show that $\mathbf{x}(\delta)$ is weakly increasing in δ. Our proof can be regarded as an extension on that theorem without the sublattice assumption, but with an additional assumption **A1**.

$$\begin{aligned}0 &= 1 - F(r^*) - r^* f(r^*) + (1 - F(\tilde{v}^*)) \left(\frac{1-\delta}{\delta}\right) \\ &< 1 - F(r(\delta)) - r(\delta) f(r(\delta)) + (1 - F(\tilde{v}(\delta))) \left(\frac{1-\delta}{\delta}\right),\end{aligned}$$

a contradiction to the fact that $(\tilde{v}(\delta), r(\delta))$ itself is a solution, and must satisfy (7). The proof for case (ii) is similar.

We have up to now shown that, if $(\tilde{v}(\delta), \tilde{r}(\delta))$ and $(\tilde{v}(\delta'), \tilde{r}(\delta'))$ are both in $\mathcal{D}$, it must also be the case that $(\tilde{v}(\delta), \tilde{r}(\delta')), (\tilde{v}(\delta'), \tilde{r}(\delta)) \in \mathcal{D}$.[26] Hence, both $\mathbf{x}(\delta) \vee \mathbf{x}(\delta')$ and $\mathbf{x}(\delta) \wedge \mathbf{x}(\delta')$ must also belong to $\mathcal{D}$.[27] It follows that

$$\begin{aligned}0 &\leq \tilde{G}(\mathbf{x}(\delta'); \delta') - \tilde{G}(\mathbf{x}(\delta) \vee \mathbf{x}(\delta'); \delta') && \text{(by optimality of } \mathbf{x}(\delta') \text{ at } \delta') \\ &\leq \tilde{G}(\mathbf{x}(\delta) \wedge \mathbf{x}(\delta'); \delta') - \tilde{G}(\mathbf{x}(\delta); \delta') && \text{(by supermodularity of } \tilde{G}(\cdot) \text{ in } \mathbf{x}) \\ &\leq \tilde{G}(\mathbf{x}(\delta) \wedge \mathbf{x}(\delta'); \delta) - \tilde{G}(\mathbf{x}(\delta); \delta) && \text{(by increasing differences of } \tilde{G}(\cdot) \text{ in } (\mathbf{x}, \delta)) \\ &\leq 0. && \text{(by optimality of } \mathbf{x}(\delta) \text{ at } \delta)\end{aligned}$$

Therefore, equality must hold for every inequality in the above string. If it were not the case that $\mathbf{x}(\delta) \leq \mathbf{x}(\delta')$, then $\mathbf{x}(\delta) \vee \mathbf{x}(\delta') > \mathbf{x}(\delta')$ and $\mathbf{x}(\delta) \wedge \mathbf{x}(\delta') < \mathbf{x}(\delta)$. Since $\tilde{G}$ satisfies *strictly* increasing differences in $(\mathbf{x}, \delta)$, the third inequality in the string becomes strict, a contradiction. □

(**C3**) $\mathbf{x}(\delta)$ *is weakly increasing on* $(\delta^*, 1)$*, i.e., if* $\delta^* < \delta_1 < \delta_2 < 1$*, then* $\mathbf{x}(\delta_1) \leq \mathbf{x}(\delta_2)$.

Proof. For all $h \in [\delta_1, \delta_2]$, let $\mathcal{B}_h = (h - \varepsilon_h, h + \varepsilon_h)$, where every ε_h is chosen to satisfy the requirement in (**C2**). Then $\{\mathcal{B}_h\}_{h \in [\delta_1, \delta 2]}$ is an open cover of the compact interval $[\delta_1, \delta_2]$. Therefore, there exists a finite subcover of $[\delta_1, \delta_2]$, denoted by $\{B_{h_i}\}_{i=1}^{I+1}$, such that $\delta_1 \leq h_1 < h_2 < \cdots < h_{I+1} \leq \delta_2$. For $i = 1, 2, \cdots, I$, take $k_i \in (h_i, h_{i+1}) \cap \mathcal{B}_{h_i} \cap \mathcal{B}_{h_{i+1}}$. Then (**C2**) implies

[26] Recall that $\tilde{r}(\delta) = -r(\delta)$.

[27] $\mathbf{x}(\delta) \vee \mathbf{x}(\delta') = (\max\{\tilde{v}(\delta), \tilde{v}(\delta')\}, \max\{\tilde{r}(\delta), \tilde{r}(\delta')\})$, and $\mathbf{x}(\delta) \wedge \mathbf{x}(\delta') = (\min\{\tilde{v}(\delta), \tilde{v}(\delta')\}, \min\{\tilde{r}(\delta), \tilde{r}(\delta')\})$. See Sundaram (1996), p.254.

that $\mathbf{x}(k_i) \le \mathbf{x}(h_{i+1}) \le \mathbf{x}(k_{i+1})$ for $i = 1, 2, \cdots, I$, and that $\mathbf{x}(\delta_1) \le \mathbf{x}(k_1)$ and $\mathbf{x}(k_I) \le \mathbf{x}(\delta_2)$. This in turn implies that $\mathbf{x}(\delta_1) \le \mathbf{x}(k_1) \le \mathbf{x}(k_2) \le \cdots \le \mathbf{x}(k_I) \le \mathbf{x}(\delta_2)$, as desired. □

(**C4**) $r_B(\delta)$ is strictly decreasing on $[\delta^*, 1)$.

Proof. We already know from (**C3**) that $\tilde{r}(\delta)$ is weakly increasing, and therefore $r(\delta)$ is weakly decreasing, on $[\delta^*, 1)$.[28] Suppose, however, that it is not strictly decreasing. Then we have $r(\delta_1) = r(\delta_2)$ for some (δ_1, δ_2) with $\delta^* \le \delta_1 < \delta_2 < 1$.

Since $\lambda = 0$ for the BIN auction, it follows from (7) that

$$1 - F(r(\delta_1)) - r_1 f(r(\delta_1)) + (1 - F(\tilde{v}(\delta_1))) \left(\frac{1 - \delta_1}{\delta_1} \right)$$

$$= 0 = 1 - F(r(\delta_2)) - r(\delta_2) f(r(\delta_2)) + (1 - F(\tilde{v}(\delta_2))) \left(\frac{1 - \delta_2}{\delta_2} \right).$$

Our assumption that $r(\delta_1) = r(\delta_2)$ implies that

$$(1 - F(\tilde{v}(\delta_1))) \left(\frac{1}{\delta_1} - 1 \right) = (1 - F(\tilde{v}(\delta_2))) \left(\frac{1}{\delta_2} - 1 \right).$$

However, this equality cannot possibly hold because (i) (**C3**) also implies that $\tilde{v}(\delta_1) \le \tilde{v}(\delta_2)$, so that $1 - F(\tilde{v}(\delta_1)) \ge 1 - F(\tilde{v}(\delta_2)) > 0$; and that (ii) $\frac{1}{\delta_1} - 1 > \frac{1}{\delta_2} - 1 > 0$. □

[28] For the point $\delta = \delta^*$, the arguments in (**C2**) and (**C3**) are also applicable since the key point is $\lambda = 0$.

References

[1] Bajari, P. and A. Hortaçsu (2003), "Winner's Curse, Reserve Price and Endogenous Entry: Empirical Insights from eBay Auctions," *RAND Journal of Economics*, 329-55.

[2] Bauner, C. (2015), "Mechanism Choice and the Buy-it-Now Auction: A Structural Model of Competing Buyers and Sellers," *International Journal of Industrial Organization*, 38, 19-31.

[3] Budish, E. and L. Takeyama (2001), "Buy Prices in Online Auctions: Irrationality on the Internet?" *Economic Letters*, 73, 325-33.

[4] Bulow, J.I., J.D. Geanakoplos, and P.D. Klemperer (1985), "Multimarket Oligopoly: Strategic Substitutes and Complements," *Journal of Political Economy*, 93, 488-511.

[5] Chen, J.-R., K.-P. Chen, C.-F. Chou and C.-I. Huang (2013a), "A Dynamic Model of Auctions with Buy-it-Now: Theory and Evidence," *Journal of Industrial Economics*, 61, 393-429.

[6] Chen, K.-P., Y.-S. Liu and Y.-T. Yu (2013b), "The Seller's Listing Strategy in Online Auctions: Evidence from eBay," working paper, Academia Sinica.

[7] Dewally, M. and L.H Ederington (2005), "What Attracts Bidders to Online Auctions and What is their Incremental Price Impact?" mimeo.

[8] Einav, L., C. Farronato, J. Levin and N. Sundaresan (2013), "Sales Mechanisms in Online Markets: What Happened to Internet Auction?" working paper, Stanford University.

[9] Hammond, R.G. (2010), "Comparing Revenue from Auctions and Posted Prices," *International Journal of Industrial Organization*, 28, 1-9.

[10] Hasker, K. and R. Sickles (2010), "eBay in the Economic Literature: Analysis of an Auction Marketplace," *Review of Industrial Organization*, 37, 3-42.

[11] Hidvegi, Z., W. Wang and A.B. Whinston (2006), "Buy-Price English Auction," *Journal of Economic Theory*, 129, 31-56.

[12] Katar, R. and D.H. Reiley (2006). "Public versus Secret Reserve Price in eBay Auctions: Result from a Pokèmon Field Experiment," *Advances in Economic Analysis and Policy*, 6, Article 7.

[13] Mathews, T. (2004), "The Impact of Discounting on an Auction with a Buy-out Option: a Theoretical Analysis Motivated by eBay's Buy-It-Now Feature," *Journal of Economics*, 81, 25-52.

[14] Mathews, T. and B. Katzman (2006), "The Role of Varying Risk Attitudes in an Auction with a Buyout Option," *Economic Theory*, 27, 597-613.

[15] Milgrom P. and I. Segal (2002), "Envelope Theorem for Arbitrary Choice Sets," *Econometrica*, 70, 583-601.

[16] Myerson, R. (1981), "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58-73.

[17] Reynolds, S. and J. Wooders (2009), "Auctions with a Buy Price," *Economic Theory*, 38, 9-39.

[18] Sundaram, R.K. (1996), *A First Course in Optimization Theory*, Cambridge University Press, Cambridge.

[19] Wang, R. (1993), "Auctions versus Posted-Price Selling," *American Economic Review*, 83, 838-51.

[20] Wang, X., A. L. Montgomery and K. Srinivasan (2008), "When Auction Meets Fixed Price: A Theoretical and Empirical Examination of Buy-It-Now Auctions," *Quantitative Marketing and Economics*, 6, 339-370.

Appendix B. (Not for Publication)

1. Proof of Lemma 1

We will prove that for every bidder to adopt the cut-off buyout strategy as stated in the lemma constitutes an Nash equilibrium. Consider the choice of bidder i with valuation v_i in the first stage, when every other bidder exercises BIN option if and only if her valuation exceeds $\tilde{v}$. It must be shown that the bidder i's optimal strategy is to follow the same strategy as well. For a bidder with $v_i < r$, it is trivial that her best strategy is not exercising the BIN option in stage 1 and truthfully bidding in stage 2, resulting in zero payoff. In the following, we need only discuss the case that $v_i \geq r$. First, note that, her expected payoff from *not* exercising the BIN option is

$$\begin{aligned} u_A(v_i) &= (v_i - r)F(r)^{n-1} + \int_r^{\min\{v_i,\tilde{v}\}} (v_i - x)dF(x)^{n-1} \\ &= (v_i - \min\{v_i,\tilde{v}\})\, F(\min\{v_i,\tilde{v}\})^{n-1} + \int_r^{\min\{v_i,\tilde{v}\}} F(x)^{n-1}dx. \end{aligned}$$

The first term following the first equality is bidder's payoff if no other bidders compete with her (all have valuations lower than the reserve price), so that she wins the item at the reserve price. The second term is her expected payoff when at least one competes with her. Next, note that, her expected payoff by exercising the BIN option is

$$u_B(v_i) = (v_i - B)\sum_{j=0}^{n-1} \frac{(1-F(\tilde{v}))^j F(\tilde{v})^{n-1-j}}{1+j}\binom{n-1}{j} = (v_i - B)\frac{1-F(\tilde{v})^n}{n(1-F(\tilde{v}))}.$$

It can be seen easily that $u_A(\tilde{v}) = u_B(\tilde{v})$ by (1). Moreover, for all $v_i \in [r, v_H)$,

$$\begin{aligned} &u_B'(v_i) - u_A'(v_i) = \frac{1-F(\tilde{v})^n}{n(1-F(\tilde{v}))} - F(\min\{v_i,\tilde{v}\})^{n-1} \\ &\geq \frac{1-F(\tilde{v})^n}{n(1-F(\tilde{v}))} - F(\tilde{v})^{n-1} = \frac{1}{n(1-F(\tilde{v}))}\left[1 - nF(\tilde{v})^{n-1} + (n-1)F(\tilde{v})^n\right] > 0. \end{aligned}$$

Since $u_B(v_i) - u_A(v_i)$ are strictly increasing and $u_B(v_i) = u_A(v_i)$ only when $v_i = \tilde{v}$, we know $u_B(v_i) \gtreqless u_A(v_i)$ if and only if $v_i \gtreqless \tilde{v}$, implying that for every bidder i to follow the cut-off buyout strategy constitutes a Nash equilibrium. QED

2. A sufficient condition for $r(\delta)$ being weakly decreasing

In this section, we impose an additional assumption which, combined with assumption **A1**, is essentially a sufficient condition that $\mathrm{E}\pi_{\tilde{v}\tilde{v}} < 0$ for all possible numbers of bidders.

A2. $f'(v) \geq -(n-1)f(v)^2$ *for* $v \in [r_F, v_H]$. [29]

We have the following proposition:

Proposition 4. *Under assumptions* **A1** *and* **A2**, $r_F > r_B(\delta)$ *for* $\delta \in [\delta^*, 1)$.

Our argument for Proposition 4 is divided into two parts: the case that $\delta^* = \delta^1$ (**C5**), and the case that $\delta^* < \delta^1$ (**C6**).

(**C5**) $r_F > r_B(\delta)$ for any $\delta \in [\delta^1, 1)$.[30]

Proof. From Proposition 1, if $\delta > \delta^1$, we have $\tilde{v}(\delta) > r(\delta)$, implying $\lambda = 0$. If $\delta = \delta^1$, then there are two possibilities: either $\tilde{v}(\delta) > r(\delta)$, or $\tilde{v}(\delta) = r(\delta)$. The former directly implies that $\lambda = 0$. In the latter case, from the definition of δ^1 and (7) we know that λ also equals 0. We therefore have $\lambda = 0$ for all $\delta \in [\delta^1, 1]$. Again, from (7),

$$\begin{aligned} r(\delta) &= \frac{1-F(r(\delta))}{f(r(\delta))} + \frac{1-F(\tilde{v}(\delta))}{f(r(\delta))}\left(\frac{1-\delta}{\delta}\right) \\ &\leq \frac{1-F(r(\delta))}{f(r(\delta))} + \frac{1-F(r(\delta))}{f(r(\delta))}\left(\frac{1-\delta}{\delta}\right) = \frac{1-F(r(\delta))}{\delta f(r(\delta))}. \end{aligned} \tag{12}$$

If, contrary to the claim of the lemma, $r_F \leq r_B(\delta)$, then

$$\frac{1-F(r(\delta))}{\delta f(r(\delta))} \leq \frac{1-F(r_F)}{\delta f(r_F)} \leq \frac{1-F(r_F)}{\delta^1 f(r_F)} = r_F; \tag{13}$$

where the first inequality is from the increasing hazard rate assumption. Hence, all equalities hold in (12) and (13), implying that $\tilde{v}(\delta) = r(\delta) = r_F$, that is, the BIN auction is indeed a fixed-price listing, which is a contradiction. □

[29] Assumptions **A1**, **A2** and the usual increasing hazard rate assumption are three restrictions on the rate of the decline of probability density $f(\cdot)$. They overlap but do not imply one another. The are satisfied by a broad class of density functions, for example, any (weakly) increasing density function $f(\cdot)$, including the uniform density function.

[30] This claim does not rely on the assumptions **A1** or **A2**.

(**C6**) If $\delta^* < \delta^1$, $r_F > r_B(\delta)$ for $\delta \in [\delta^*, 1)$.

Proof. We claim that there exists an $\tilde{\varepsilon} > 0$, such that for $\delta \in \mathcal{B}_{\delta^*} = [\delta^*, \delta^* + \tilde{\varepsilon}) \subset [\delta^*, 1)$, $r_F > r_B(\delta)$.[31] By combining this claim and Proposition 3, (**C6**) follows. In order to prove this claim, first by (7) we note that

$$\lambda = n\delta^* F(r_F)^{n-1} \left[1 - F(r_F) - r_F f(r_F) + (1 - F(r_F)) \left(\frac{1-\delta^*}{\delta^*} \right) \right] > 0. \qquad (14)$$

Suppose the claim were not true. Then, there exists a sequence $\{\delta_k\}_{k=1}^{\infty}$, with limit δ^*, such that the corresponding maximizers $(\tilde{v}(\delta_k), r(\delta_k), \lambda_k)$ have $\tilde{v}(\delta_k) > r(\delta_k) \geq r_F$ and $\lambda_k = 0$ for all k. Assume that $(\tilde{v}^*, r^*, \lambda)$ is an accumulation point of $\{(\tilde{v}_k, r_k, \lambda_k)\}_{k=1}^{\infty}$. Then, $\lambda = 0$, and the upper semi-continuity property of the maximizers implies that $(\tilde{v}^*, r^*)$ maximizes $\mathrm{E}\pi(\tilde{v}, r; \delta^*)$. Then, it follows $\tilde{v}^* > r^* \geq r_F$. (If $\tilde{v}^* = r^*$, we meet a fixed-price listing with $\lambda = 0$ at $\delta = \delta^*$, contradicting (14).) Let $\Delta\tilde{v} = r_F - \tilde{v}^* < 0$, and $\Delta r = r_F - r^* \leq 0$. By the Taylor's theorem, there exists a $\xi \in (0, 1)$, such that

$$\begin{aligned} \mathrm{E}\pi(r_F, r_F; \delta^*) =& \mathrm{E}\pi(\tilde{v}^*, r^*; \delta^*) + \mathrm{E}\pi_{\tilde{v}}(\tilde{v}^*, r^*; \delta^*)\Delta\tilde{v} + \mathrm{E}\pi_r(\tilde{v}^*, r^*; \delta^*)\Delta r \\ &+ \frac{1}{2} \left[\mathrm{E}\pi_{\tilde{v}\tilde{v}}(\tilde{v}_\xi, r_\xi; \delta^*)\Delta\tilde{v}^2 + 2\mathrm{E}\pi_{\tilde{v}r}(\tilde{v}_\xi, r_\xi; \delta^*)\Delta\tilde{v}\Delta r + \mathrm{E}\pi_{rr}(\tilde{v}_\xi, r_\xi; \delta^*)\Delta r^2 \right], \end{aligned}$$

where $\tilde{v}_\xi = \tilde{v}^* + \xi\Delta\tilde{v}$ and $r_\xi = r^* + \xi\Delta r$. Since $\mathrm{E}\pi(r_F, r_F; \delta^*) = \mathrm{E}\pi(\tilde{v}^*, r^*; \delta^*)$ and the first-order condition requires that $\mathrm{E}\pi_{\tilde{v}}(\tilde{v}^*, r^*; \delta^*) = \mathrm{E}\pi_r(\tilde{v}^*, r^*; \delta^*) = 0$, we have

$$\mathrm{E}\pi_{\tilde{v}\tilde{v}}(\tilde{v}_\xi, r_\xi; \delta^*)\Delta\tilde{v}^2 + 2\mathrm{E}\pi_{\tilde{v}r}(\tilde{v}_\xi, r_\xi; \delta^*)\Delta\tilde{v}\Delta r + \mathrm{E}\pi_{rr}(\tilde{v}_\xi, r_\xi; \delta^*)\Delta r^2 = 0.$$

Since $\mathrm{E}\pi_{\tilde{v}r}(\tilde{v}_\xi, r_\xi; \delta^*) < 0$ and the assumption (**A1**) implies that $\mathrm{E}\pi_{rr}(\tilde{v}_\xi, r_\xi; \delta^*) \leq 0$, we must

[31] This claim seems to be only an adaptation of (**C2**). However, it involves the case $\lambda > 0$, so the argument is different.

have $-\mathrm{E}\pi_{\tilde{v}\tilde{v}}(\tilde{v}_\xi, r_\xi; \delta^*) \le 0$. We directly calculate $-\mathrm{E}\pi_{\tilde{v}\tilde{v}}(\tilde{v}_\xi, r_\xi; \delta^*)$, as follows:

$$\begin{aligned}
-\mathrm{E}\pi_{\tilde{v}\tilde{v}}(\tilde{v}_\xi, r_\xi; \delta^*) =& n(n-1)F(\tilde{v}_\xi)^{n-2}f(\tilde{v}_\xi)[1 - F(\tilde{v}_\xi) + (1-\delta^*)f(\tilde{v}_\xi)\tilde{v}_\xi] \\
&+ (1-\delta^*)nf'(\tilde{v}_\xi)\left[F(\tilde{v}_\xi)^{n-1}\tilde{v}_\xi - \int_{r_\xi}^{\tilde{v}_\xi} F(x)^{n-1}dx\right] \\
>& (1-\delta^*)n(n-1)F(\tilde{v}_\xi)^{n-2}f(\tilde{v}_\xi)^2\tilde{v}_\xi \qquad \text{(by } F(\tilde{v}_\xi) < 1\text{)} \\
&- (1-\delta^*)n|f'(\tilde{v}_\xi)|\left[F(\tilde{v}_\xi)^{n-1}\tilde{v}_\xi - \int_{r_\xi}^{\tilde{v}_\xi} F(x)^{n-1}dx\right] \\
>& (1-\delta^*)nF(\tilde{v}_\xi)^{n-2}\tilde{v}_\xi[(n-1)f(\tilde{v}_\xi)^2 - |f'(\tilde{v}_\xi)|] \qquad \text{(by } \tilde{v}_\xi > r_\xi \text{ and } F(\tilde{v}_\xi) < 1\text{)} \\
\ge& 0,
\end{aligned}$$

where the final inequality follows from assumption (**A2**) that $f'(v) \ge -(n-1)f(v)^2$ for $v \in [r_F, \bar{v}]$. Hence, $-\mathrm{E}\pi_{\tilde{v}\tilde{v}}(\tilde{v}_\xi, r_\xi; \delta^*) > 0$, arriving at a contradiction. □

The following theorem concludes Proposition 3 and Proposition 4:

Theorem 2. *Under assumptions* **A1** *and* **A2***, the correspondence $r(\delta)$ is (weakly) decreasing on $(0, 1]$, and is strictly decreasing on $[\delta^*, 1]$. As a result, $r_F \ge r_B(\delta) \ge r_A$ for $\delta \in [\delta^*, 1)$.*

3. A sufficient condition for $r(\delta)$ being single-valued

Note that if the correspondence $r(\delta)$ is single-valued, i.e., $r(\delta)$ is a function, the upper semi-continuity property implies $r(\delta)$ must be a continuous function. This could be achieved by imposing the following assumptions:

A3. *$f(v)$ is log-concave on $[r_A, r_F]$.*

A4. *$f(v)$ is increasing on $[r_F, v_H]$.*

A3 can be replaced by a much weaker, but somewhat cumbersome assumption:

A3'. $\frac{2f(v)+vf'(v)}{F(v)} \ge \frac{f(r_F)}{1+F(r_F)}$ *for all $v \in [r_A, r_F]$.*

The reason is as follows. By Bagnoli and Bergstrom (2005), f being log-concave implies that F is log-concave, which in turn implies that $\frac{f(v)}{F(v)} \ge \frac{f(r_F)}{F(r_F)}$, for all $v \in [r_A, r_F]$. Combining this with $f(v) + vf'(v) \ge 0$ yields **A3'**. Then, we have the following proposition:

Proposition 5. *Under assumptions* **A3'** *and* **A4**, $\delta^* = \delta^1$, *and the correspondence* $r(\delta)$ *is single-valued on* $(0, 1]$. *Hence,* $r(\delta)$ *is a continuous function.*

Proof. We will first show that

$$\frac{(n-1)\delta^1 f(r_F) r_F}{1-\delta^1} > 1 + F(r_F). \tag{15}$$

Using (8) and the definition of δ^1 introduced in Proposition 1, (15) is equivalent to

$$J(F(r_F)) \equiv (n-2) - nF(r_F) + nF(r_F)^{n-1} - (n-2)F(r_F)^n > 0. \tag{16}$$

First note that $J(F(\bar{v})) = J(1) = 0$. Therefore, in order to prove (16), we only need to show that $J(x)$ is a strictly decreasing function on $[0, 1]$. For this purpose, first note that $J'(1) = 0$. Second, $J''(x)$ can be easily seen to be positive for all $x \in (0, 1)$, meaning that $J'(x)$ is an strictly increasing function on $[0, 1]$. Combining this with the fact that $J'(1) = 0$, we know that $J'(x) < 0$, which shows that $J(x)$ is indeed a strictly decreasing function, and (15) is proved.

Now we turn to our main proof. Our goal is to show that there exists exactly one optimal solution if $\delta \geq \delta^1$. First, note that by the argument in (**C5**), we have $\lambda = 0$. Plugging this into (7) yields

$$F(\tilde{v}) = \frac{1}{1-\delta} - \frac{\delta}{1-\delta}(rf(r) + F(r)). \tag{17}$$

Denote the right-hand side of (17) as $H_\delta(r)$. We can rewrite (17) as $\tilde{v} = F^{-1}(H_\delta(r))$. Moreover,

$$\frac{d\tilde{v}}{dr} = F^{-1}(H_\delta(r))' \cdot H'_\delta(r) = \frac{H'_\delta(r)}{f(\tilde{v})}. \tag{18}$$

The optimal revenue of the seller, when viewed as a function of r, satisfies

$$\frac{d\mathrm{E}\pi}{dr} = \left(\frac{\partial \mathrm{E}\pi}{\partial \tilde{v}}\frac{d\tilde{v}}{dr} + \frac{\partial \mathrm{E}\pi}{\partial r}\right)_{\tilde{v}=F^{-1}(H(r))} = \frac{\partial \mathrm{E}\pi}{\partial \tilde{v}}\Big|_{\tilde{v}=F^{-1}(H_\delta(r))} \cdot \frac{H'_\delta(r)}{f(\tilde{v})},$$

where the last equality comes from (18). Note that $H'_\delta(r) = -\frac{\delta}{1-\delta}\big(2f(r) + rf'(r)\big) < 0$, so that $H_\delta(r)$ is decreasing. Also note that $\tilde{v} \equiv \tilde{v}(\delta) \geq \tilde{v}(\delta^1) = F^{-1}(H_{\delta^1}(r(\delta^1)))$ (by $\delta \geq \delta^1$ and the argument in (**C3**)), and $r_F \geq r(\delta^1)$ (by Proposition 4), so we have $\tilde{v} \geq F^{-1}(H_{\delta^1}(r(\delta^1))) \geq F^{-1}(H(r_F)) = r_F$.

It suffices to show that the derivative against r of $\partial \mathrm{E}\pi/\partial\tilde{v}\big|_{\tilde{v}=F^{-1}(H(r))}$ is positive while $r \le r_F$ and $\delta \ge \delta^1$. Straightforward calculation shows that

$$\frac{\partial \mathrm{E}\pi}{\partial \tilde{v}}\Big|_{\tilde{v}=F^{-1}(H(r))} = 1 - nH(r)^{n-1} + (n-1)H(r)^n - (1-\delta)nf(\tilde{v})\left[F(\tilde{v})^{n-1}\tilde{v} - \int_r^{\tilde{v}} F(x)^{n-1}dx\right],$$

whose derivative against r is

$$-nH'(r)A(r) + n(1-\delta)f(\tilde{v})B(r),$$

where

$$A(r) = (n-1)F(\tilde{v})^{n-2}(1-F(\tilde{v})) + (1-\delta^1)\frac{f'(\tilde{v})}{f(\tilde{v})}I(r),$$

$$B(r) = -(n-1)H'(r)F(\tilde{v})^{n-2}\tilde{v} - F(r)^{n-1},$$

and $I(r) = F(\tilde{v})^{n-1}\tilde{v} - \int_r^{\tilde{v}} F(x)^{n-1}dx$. By assumption **A4**, $A(r) \ge 0$. Moreover,

$$\begin{aligned} B(r) =& F(\tilde{v})^{n-2}\left[\frac{\delta}{\delta^1}\cdot\frac{1-\delta^1}{1-\delta}\cdot\frac{(n-1)\delta^1 f(r_F)r_F}{1-\delta^1}\cdot\frac{2f(r)+rf'(r)}{f(r_F)}\cdot\frac{\tilde{v}}{r_F} - F(r)\Big(\frac{F(r)}{F(\tilde{v})}\Big)^{n-2}\right] \\ >& F(\tilde{v})^{n-2}\left[(1+F(r_F))\cdot\frac{2f(r)+rf'(r)}{f(r_F)} - F(r)\right] \ge 0, \end{aligned}$$

where the first inequality comes from (15), $\delta \ge \delta^1$, and $\tilde{v} \ge r_F \ge r$; and the second inequality comes from assumption **A3'**. Therefore, $-nH'(r)A(r) + n(1-\delta)f(\tilde{v})B(r) > 0$. Then, $r(\delta)$ is single-valued on $[\delta^1, 1]$.

Finally, note that both (5) and (6) hold when $\delta = \delta^1$ and $\tilde{v} = r = r_F$. This implies that $\tilde{v} = r = r_F$ satisfies the first-order conditions when $\delta = \delta^1$. Then, $\delta^* = \delta^1$. Since $r(\delta) = r_F$ on $(0, \delta^*)$, i.e., $r(\delta)$ is single-valued on $(0, \delta^1)$, combing this with the previous argument, we conclude that $r(\delta)$ is single-valued on $(0, 1]$. □

4. Both sellers and bidders are time impatient

In this appendix, we discuss the case that the bidders are also time-impatient. That is, similar to the seller, the bidders also discount the payoff in the second stage. Let δ_b be the discount rate of bidders. We will give a series of propositions parallel to those in the main text. Since all of the proofs follow a similar logic, we omit arguments in the proofs.

Lemma 1'. Given B and r, there exists a symmetric Nash equilibrium which is characterized by a threshold value of valuation, $\tilde{v} \in [0, v_H]$, so that, for any bidder i: if $v_i > \tilde{v}$, it is optimal to buy out the item in the first stage; if $v_i \leq \tilde{v}$, it is optimal to refrain from placing BIN and to compete in the second stage if $v_i \geq r$. The value of the buyout threshold $\tilde{v}$ is determined by

$$B = B(\tilde{v}, r) \equiv \tilde{v} - \delta_b \left[\frac{n(1 - F(\tilde{v}))}{1 - F(\tilde{v})^n} \right] \int_r^{\tilde{v}} F(x)^{n-1} dx, \tag{1'}$$

for $B \leq B(v_H, r) \equiv \lim_{\tilde{v} \to v_H^-} B(\tilde{v}, r) = v_H - \delta_b \int_r^{v_H} F(x)^{n-1} dx$; for $B > B(v_H, r)$, the BIN price is never exercised in the first stage, and the threshold value is $\tilde{v} = v_H$.

Lemma 2'. If the values of B, $\tilde{v}$ and r are such that $B = B(\tilde{v},\, r)$, then for r given, B and $\tilde{v}$ is one-one correspondence. Moreover, $B > r$ if and only if $\tilde{v} > r$, and $B = r$ if and only if $\tilde{v} = r$.

The first order conditions for the seller's optimization problem are as follows:

$$\begin{aligned} &1 - n\delta_b F(\tilde{v})^{n-1} + (n\delta_b - 1) F(\tilde{v})^n \\ &\quad - n f(\tilde{v}) \left[(1 - \delta) F(\tilde{v})^{n-1} \tilde{v} - (\delta_b - \delta) \int_r^{\tilde{v}} F(x)^{n-1} dx \right] + \lambda - \mu_1 = 0; \end{aligned} \tag{5'}$$

$$n\delta F(r)^{n-1} \left[1 - F(r) - r f(r) + (1 - F(\tilde{v})) \left(\frac{\delta_b - \delta}{\delta} \right) \right] - \lambda + \mu_2 = 0. \tag{6'}$$

Lemma 3'. For any $\delta \in (0, 1]$, $r > 0$ and $\mu_1 = \mu_2 = 0$.

From Lemma 3', (6') can be simplified as follows:

$$1 - F(r) - r f(r) + (1 - F(\tilde{v})) \left(\frac{\delta_b - \delta}{\delta} \right) - \frac{\lambda}{n\delta F(r)^{n-1}} = 0. \tag{7'}$$

Since the optimal posted-price is irrelevant to the bidders' discount rate, Lemma 4' and the optimal posted-price remain the same as Lemma 4.

Proposition 1'. There exists a $\delta^1 \equiv \frac{1 - F(r_F)}{r_F f(r_F)} \delta_b \in (0, \delta_b)$ such that any seller with discount rate $\delta \in (\delta^1, 1]$ will never adopt the fixed-price format.

Theorem 1'. The seller will optimally adopt the regular auction if and only if both the seller and the bidders are perfectly patient. Otherwise, there exists a $\delta^* \in (0, 1)$, such that a seller with discount rate $\delta \in (0, \delta^*)$ will adopt a fixed-price listing, and will adopt a BIN auction if $\delta \in (\delta^*, 1]$. The δ^*-seller will adopt either a fixed-price listing or a BIN auction.

Theorem 1' has a strong implication. It says that if the bidder is also time impatient ($\delta_b < 1$), then the seller will never adopt the regular auction, even if $\delta = 1$. Therefore, if we start out with a case with fixed δ_b, and investigate the seller's listing as a function of his time-discount factor, then *only two* possible listing formats will be adopted: the BIN auction (fixed-price listing) for more (less) patient sellers.

Proposition 2'. $r_A \lesseqgtr r_B(\delta)$ if and only if $\delta \lesseqgtr \delta_b$.[32]

Proposition 3'. Under assumption **A1**, $r_B(\delta)$ is strictly decreasing on $[\delta^*, \delta_b]$.

Proposition 4'. Under assumptions **A1** and **A2**, $r_F > r_B(\delta)$ for $\delta \in [\delta^*, \delta_b]$.

The following theorem concludes the above results.

Threorem 2'. Under assumptions **A1** and **A2**, $r_F \geq r_B(\delta_{s_1}) > r_A = r_B(\delta_b) > r_B(\delta_{s_2})$ for $\delta_{s_1} \in [\delta^*, \delta_b)$ and $\delta_{s_2} \in (\delta_b, 1]$. Moreover, $r_B(\delta)$ is strictly decreasing on $[\delta^*, \delta_b]$.

Theorem 2' establishes the ranking of reserve prices for the three types of listings: If the seller is more impatient than the bidder, then $r_A < r_B(\delta) < r_F$, i.e., the optimal posted price of the fixed-price listing is greater than the optimal reserve price of any BIN auction, which in turn is greater than that of the regular auction. In sum, the optimal reserve price is inversely related to the seller's discount factor; Otherwise, if the bidders are more impatient, the reserve price of any BIN auction is lower than that of that of the regular auction. See Figure B1.

Finally, Proposition 5' is analogous to Proposition 5.

Proposition 5'. Under assumptions **A3'** and **A4**, $\delta^* = \delta^1$, and the correspondence $r(\delta)$ is single-valued on $(0, \delta_b]$. Hence, $r(\delta)$ is a continuous function on $(0, \delta_b]$.

[32] If $\delta = \delta_b < 1$, the seller will optimally set an reserve price $r_B(\delta) = r_A$ in a BIN auction.

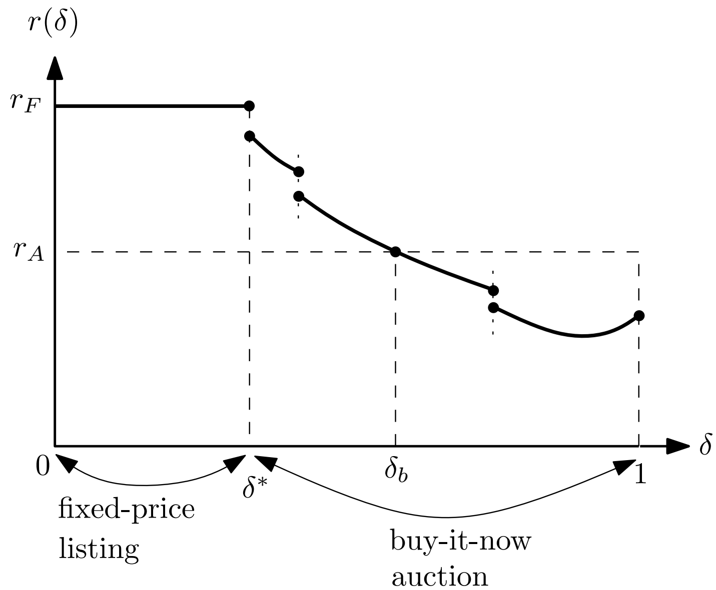

Figure A1: Optimal Listing Format and Reserve Price: Both Bidders and Seller Time Impatient

Remark. We can plot the optimal reserve prices across the values of discount rates. Figure A1 plots the general condition: $r(\delta)$ is (weakly) decreasing on $(0, \delta_b]$, and it always paths through (δ_b, r_A). As for $\delta_b < 1$, $r(\delta)$ in general is *not* monotone on $(\delta_b, 1]$. A heuristic view is offered as follows: Let $G(\tilde{v}, r; \delta)$ denote the seller's optimization function. For $\delta \in (\delta^*, 1]$, all the constraints are slack. Then, the first-order conditions $G_{\tilde{v}} = 0$ and $G_r = 0$ must be satisfied. Let $\mathbf{x}(\delta) = (\tilde{v}(\delta), r(\delta)) = \arg\max_{(\tilde{v},r)} \mathrm{E}\pi(\delta)$, if it is *continuous* at some δ, then it follows that

$$r'(\delta) = -\frac{\begin{vmatrix} G_{r\delta} & G_{r\tilde{v}} \\ G_{\tilde{v}\delta} & G_{\tilde{v}\tilde{v}} \end{vmatrix}}{\begin{vmatrix} G_{rr} & G_{r\tilde{v}} \\ G_{\tilde{v}r} & G_{\tilde{v}\tilde{v}} \end{vmatrix}}.$$

Note that the second-order conditions generically imply that the denominator is positive. As for the numerator: $G_{r\delta} = nF(r)^{n-1}f(r)\left[\frac{1-F(r)}{f(r)} - r - \frac{1-F(\tilde{v})}{f(r)}\right] < 0$ (by (7') with $\lambda = 0$), $G_{\tilde{v}\tilde{v}} < 0$ (by the second-order conditions), $G_{\tilde{v}\delta} = nf(\tilde{v})[F(\tilde{v})^{n-1}\tilde{v} - \int_r^{\tilde{v}} F(x)^{n-1}dx] > 0$, and $G_{r\tilde{v}} = n\delta F(r)^{n-1}f(\tilde{v})(\frac{\delta-\delta_b}{\delta}) > 0$ (by $\delta > \delta_b$). Then, the sign of the numerator is ambiguous.

Hence, we are not sure if $r(\delta)$ is decreasing near $\delta = 1$. However, $r(\delta)$ is decreasing if the sign of the numerator is positive, which can be achieved if $\delta \leq \delta_b$ or if δ is slightly greater than δ_b.

References

[1] Bagnoli M., and T. Bergstrom, (2005), "Log-concave probability and its applications," *Economic Theory*, 26, 445-469.