Lecture 05
Wireless Communication

I-Hsiang Wang
ihwang@ntu.edu.tw
National Taiwan University

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Wireless Communication

- Wireless is a shared medium, inherently different from wireline
  - More than one pairs of Tx/Rx can share the same wireless medium
    $\Longrightarrow$ can support **more users**, but also **more interference**
  - Signals: broadcast at Tx, superimposed at Rx
    $\Longrightarrow$ **more paths from Tx to Rx** (variation over frequency)
  - Mobility of Tx and Rx
    $\Longrightarrow$ **channel variation over time**
  - **Fading**: the scale of variation over time and frequency matters

- Key challenges: **interference** and **fading**

- Look at point-to-point communication and focus on **fading**
  - Where does fading come from?
  - How to combat fading?

Outline

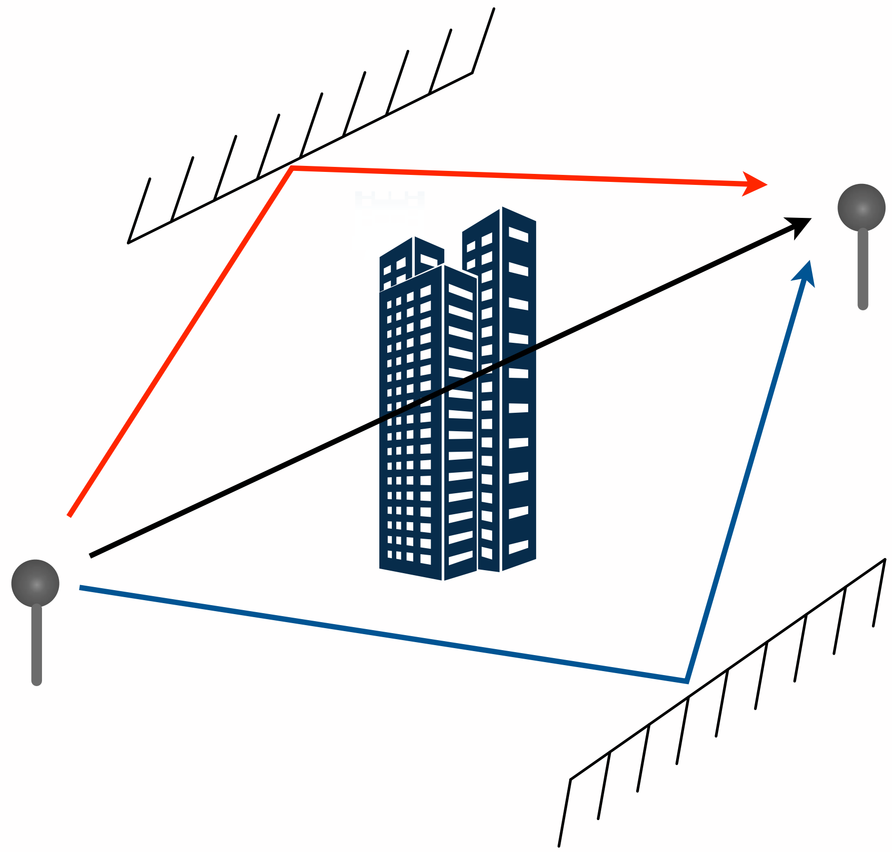
- Modeling of wireless channels
  - Physical modeling
  - Time and frequency coherence
  - Statistical modeling

- Fading and diversity
  - Impact of fading on signal detection
  - Diversity techniques

Part I. Modeling Wireless Channels

Physical Models; Equivalent Complex Baseband Discrete-Time Models; Stochastic Models

Multi-Path Physical Model



- Signals are transmitted using EM waves at a certain frequency f_c
- Far-field assumption:

  speed of light

  Tx-Rx distance $\gg \lambda_c \triangleq \frac{c}{f_c}$

- Approximate EM signals as rays under the far-field assumption. Each path corresponds to a ray.

- The input-output model of the wireless channel (neglect noise)

$$y(t) = \sum_i a_i(t) x\left(t - \tau_i(t)\right)$$

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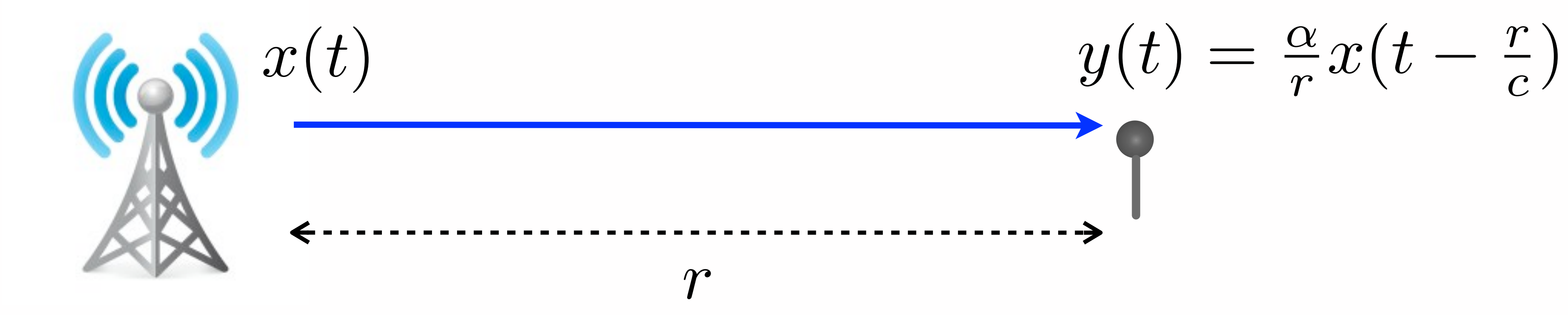
- For path i :

  $a_i(t)$: channel gain (attenuation) of path i

  $\tau_i(t)$: propagation delay of path i

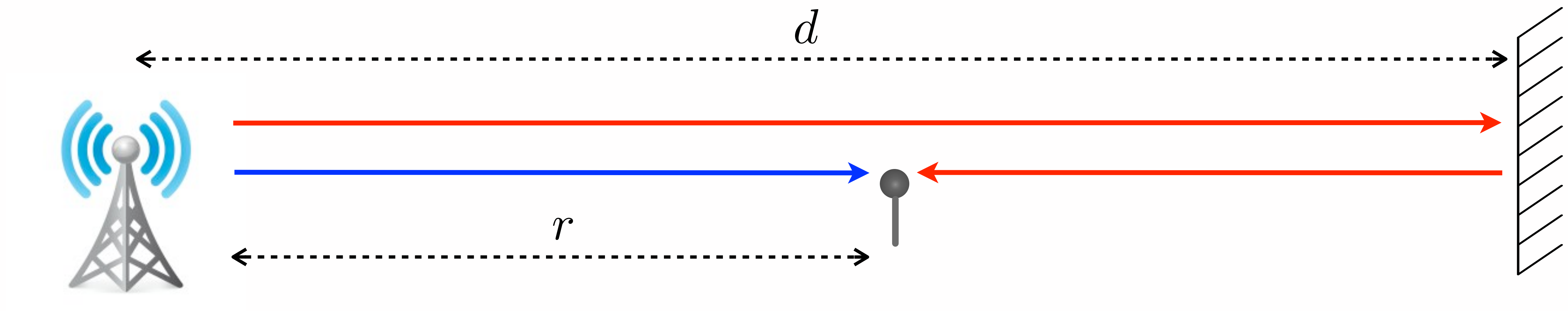
- Simplest example: single line-of-sight (LOS)



$$a(t) = \frac{\alpha}{r} \quad \text{(free space);} \quad \tau(t) = \frac{r}{c}$$

$$y(t) = \sum_i a_i(t) x\left(t - \tau_i(t)\right)$$

- Example: single LOS with a reflecting wall

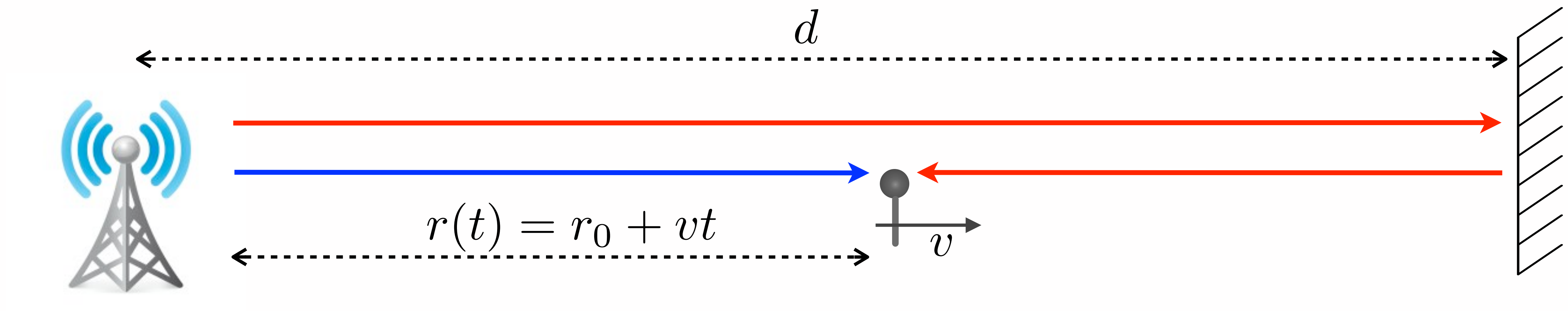


Path 1: $a_1(t) = \frac{\alpha}{r}; \quad \tau_1(t) = \frac{r}{c}$

Path 2: $a_2(t) = -\frac{\alpha}{2d-r}; \quad \tau_2(t) = \frac{2d-r}{c}$

$$y(t) = \sum_i a_i(t) x\left(t - \tau_i(t)\right)$$

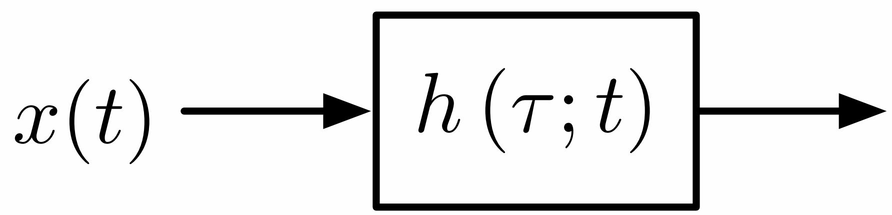
- Example: single LOS with a reflecting wall and moving Rx



Path 1: $a_1(t) = \frac{\alpha}{r_0 + vt}$; $\tau_1(t) = \frac{r_0 + vt}{c}$

Path 2: $a_2(t) = -\frac{\alpha}{2d - r_0 - vt}$; $\tau_2(t) = \frac{2d - r_0 - vt}{c}$

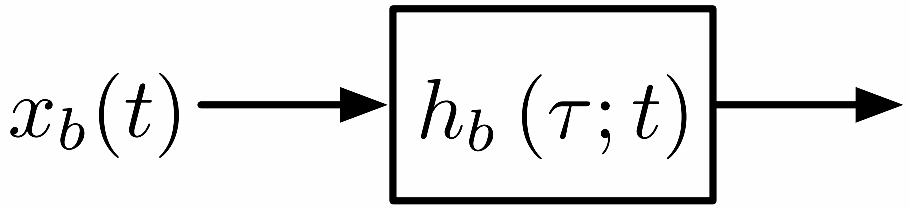
Linear Time Varying Channel Model



$$y(t) = \sum_i a_i(t) x\left(t - \tau_i(t)\right)$$

- Impulse response: $$h(\tau; t) = \sum_i a_i(t) \delta\left(\tau - \tau_i(t)\right)$$
- Frequency response: $$\breve{h}(f; t) = \sum_i a_i(t) e^{-j2\pi f \tau_i(t)}$$
- Equivalent baseband model can be derived, similar to the derivation in wireline communication

Continuous-Time Baseband Model



$$y_b(t) = \sum_i a_i^b(t) x_b\left(t - \tau_i(t)\right)$$

- Impulse response: $$h_b(\tau; t) = h(\tau; t) e^{-\mathrm{j}2\pi f_c \tau}$$

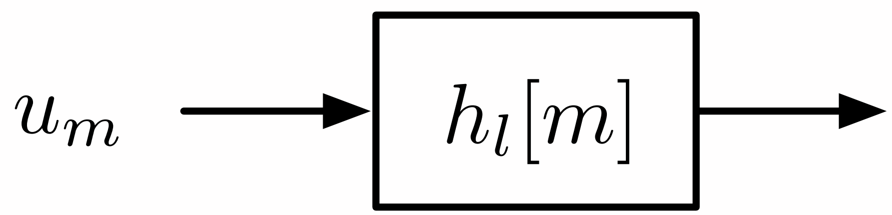
  $$a_i^b(t) \triangleq a_i(t) e^{-\mathrm{j}2\pi f_c \tau_i(t)}$$

  $$= \sum_i a_i^b(t) \delta\left(\tau - \tau_i(t)\right)$$

- Frequency response: $\breve{h}_b(f; t) = \breve{h}(f + f_c; t)$

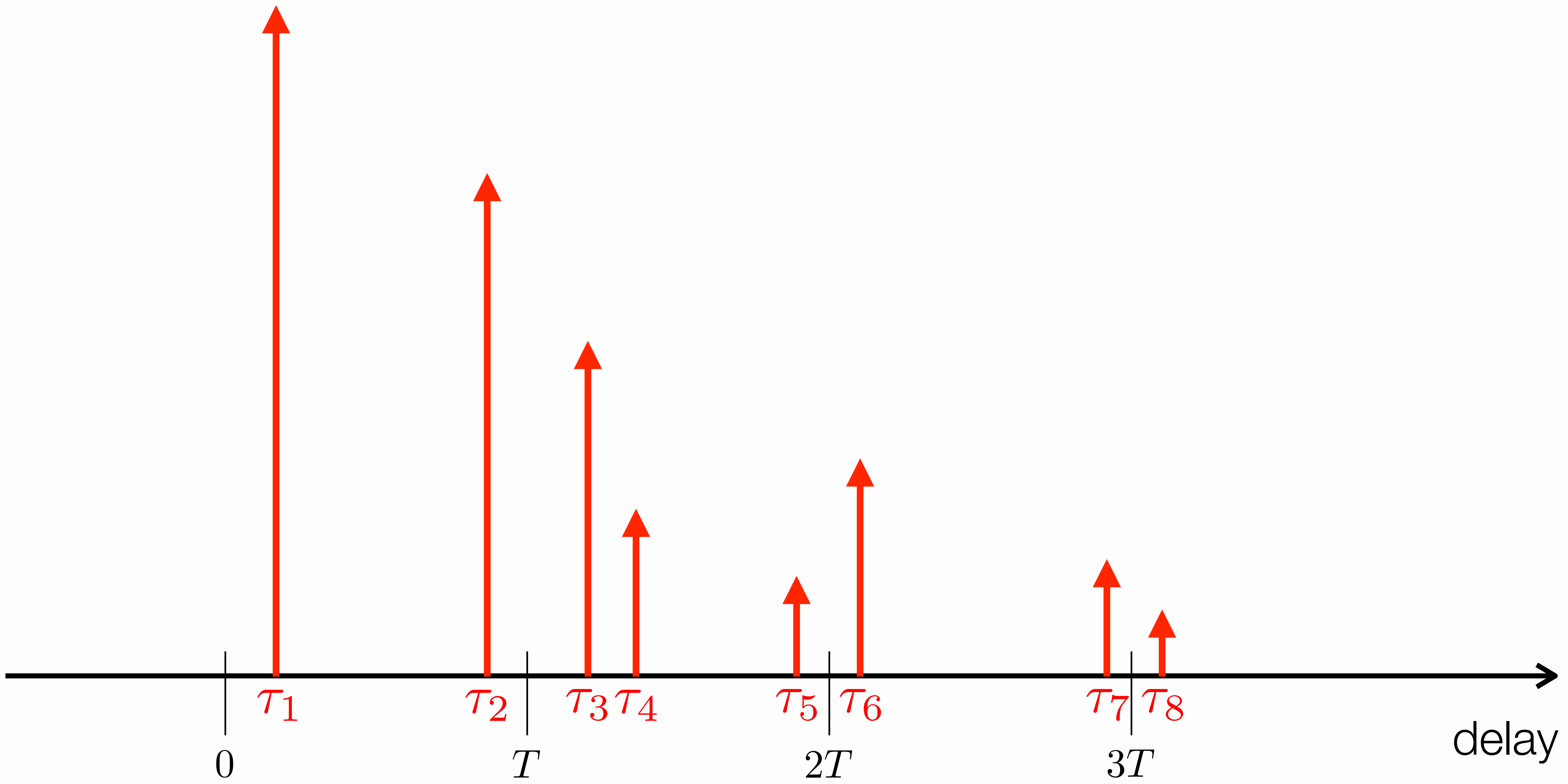
- The gain of each path is rotated with a phase

Discrete-Time Baseband Model



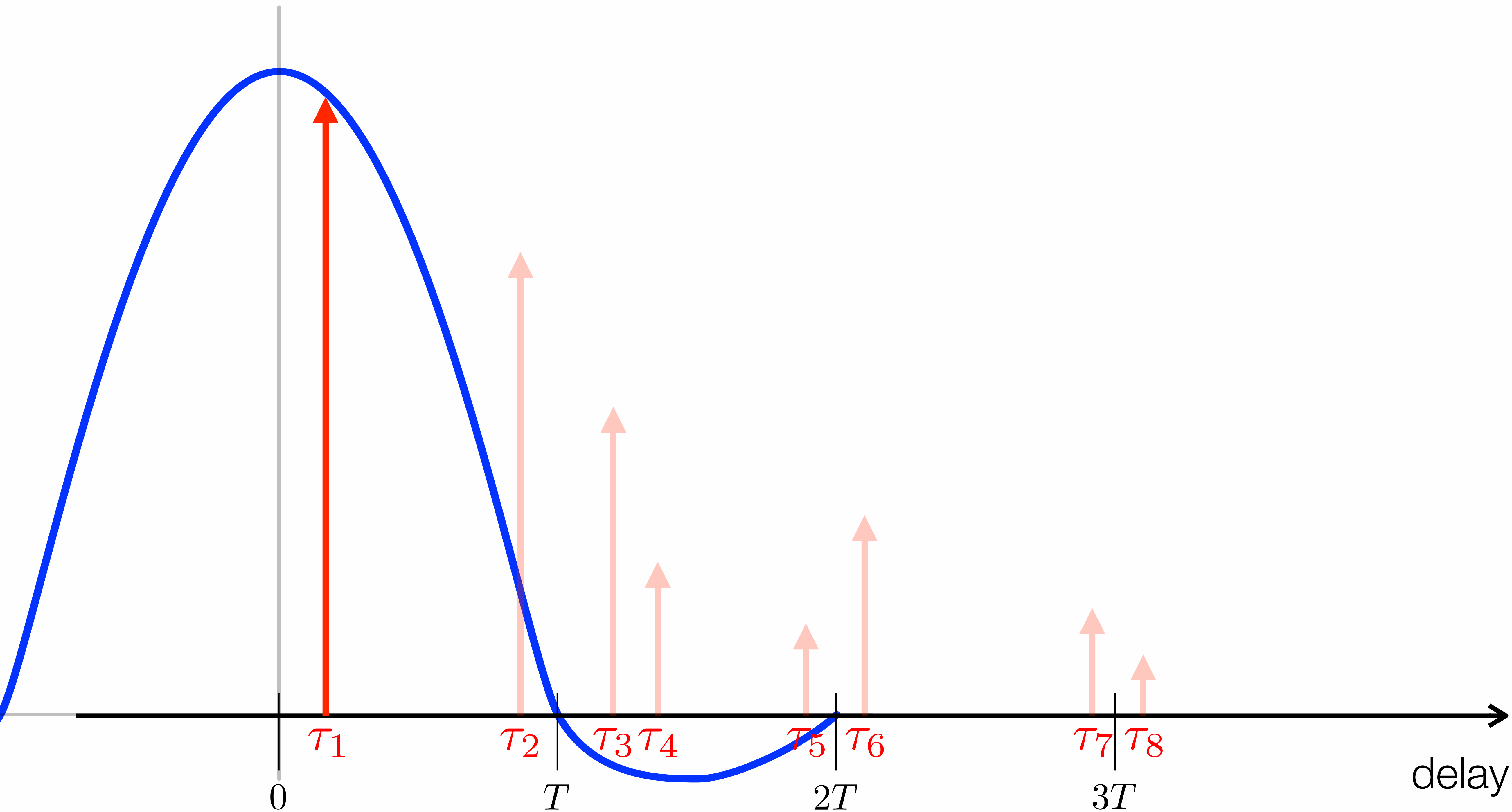
$$v_m = \sum_l h_l[m] u_{m-l}$$

- Impulse response: $$h_\ell[m] \triangleq \int_{-\infty}^{\infty} h_b(\tau; mT) g(\ell T - \tau)\,\mathrm{d}\tau$$ $$= \sum_i a_i^b(mT) g(\ell T - \tau_i(mT))$$
- Recall: $g(t)$ is the pulse used in pulse shaping
  examples: sinc pulse, raised cosine pulse, etc.
- Observation: The ℓ-th tap $h_\ell[m]$ majorly consists of the aggregation of paths with delay lying inside the “delay bin” $\tau_i(mT) \in \left[\ell T - \frac{T}{2}, \ell T + \frac{T}{2}\right]$

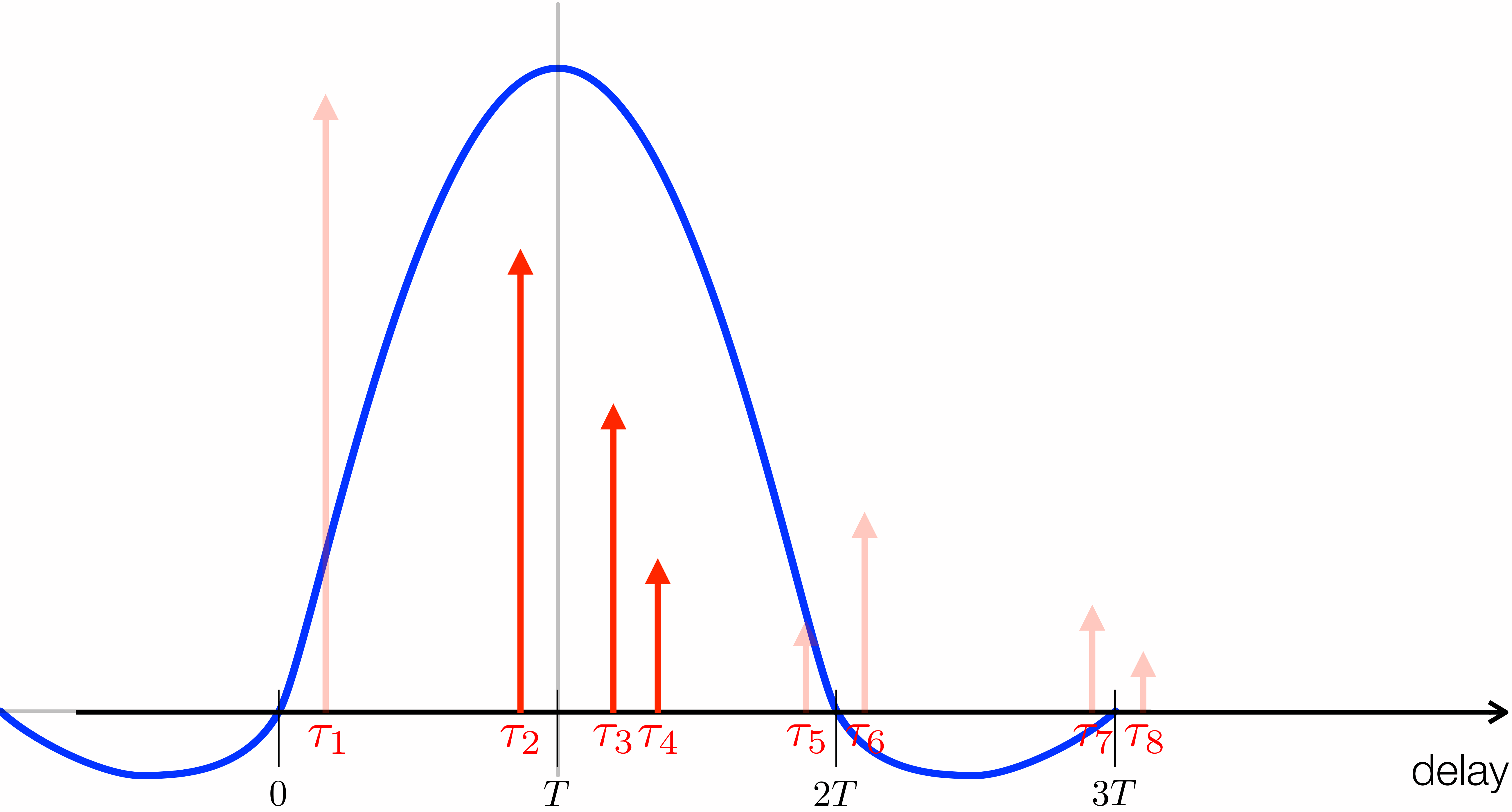


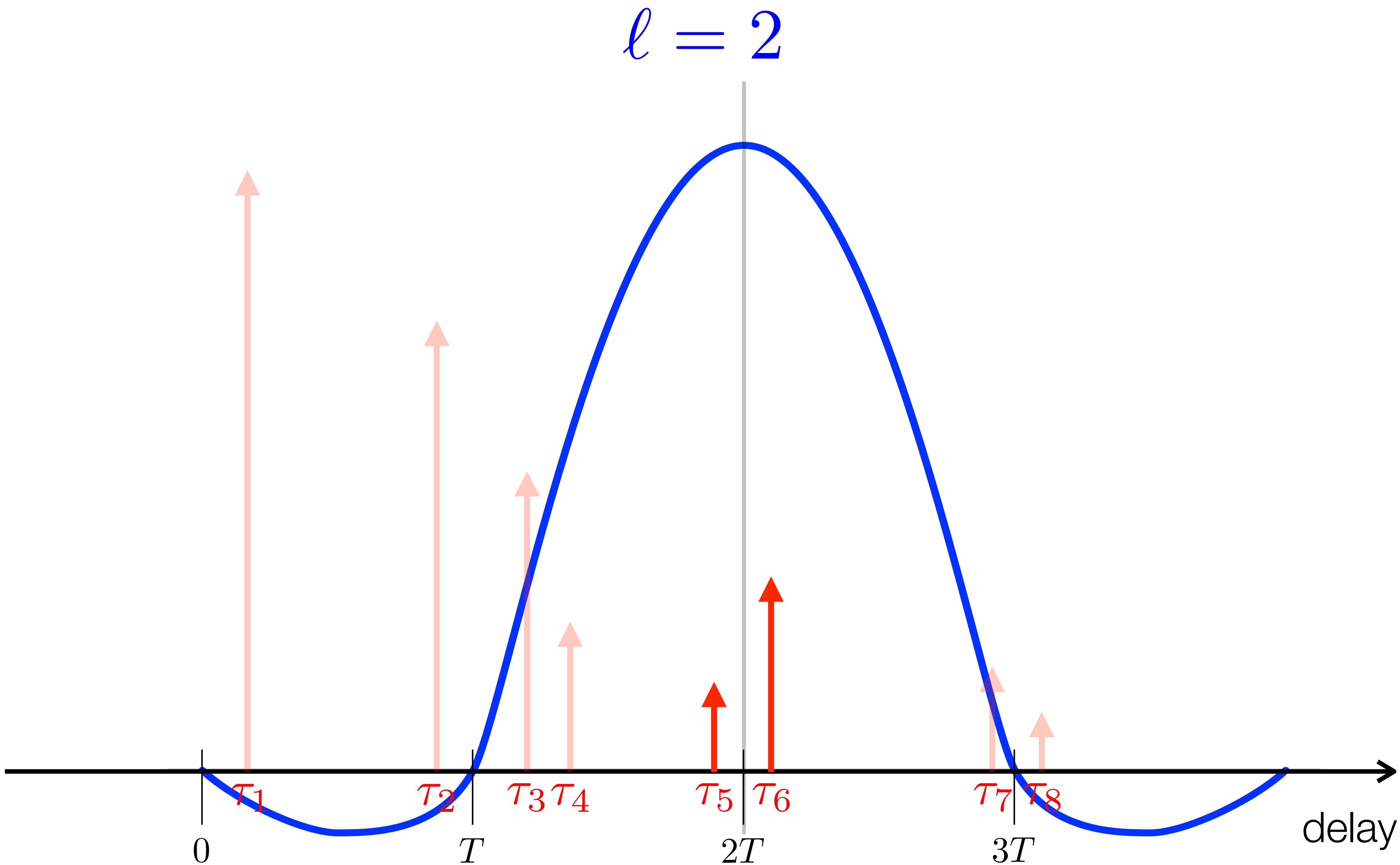
τ_1
τ_2
τ_3
τ_4
τ_5
τ_6
τ_7
τ_8
delay
0
T
$2T$
$3T$

$$\ell = 0$$



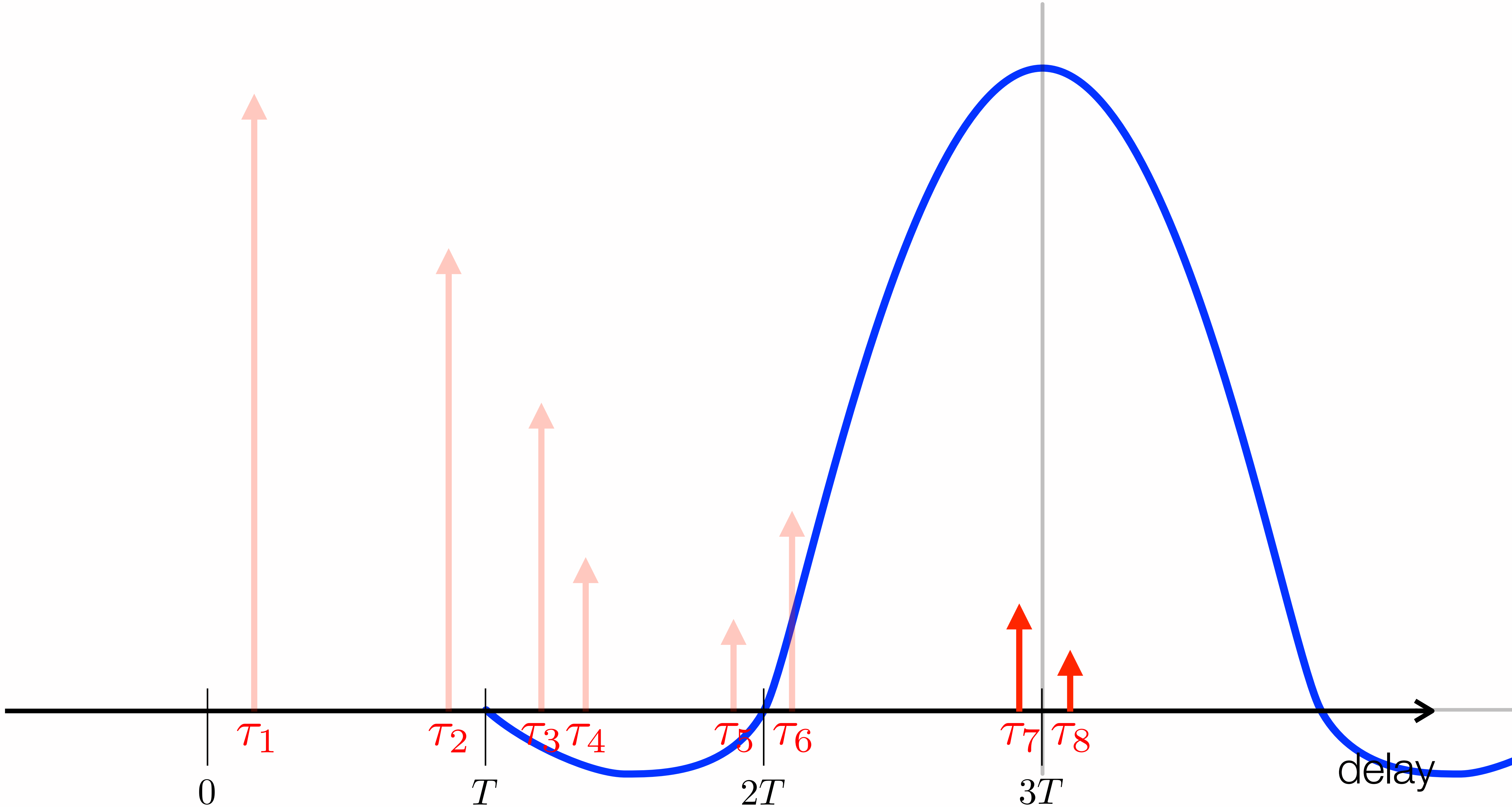
$$\ell = 1$$



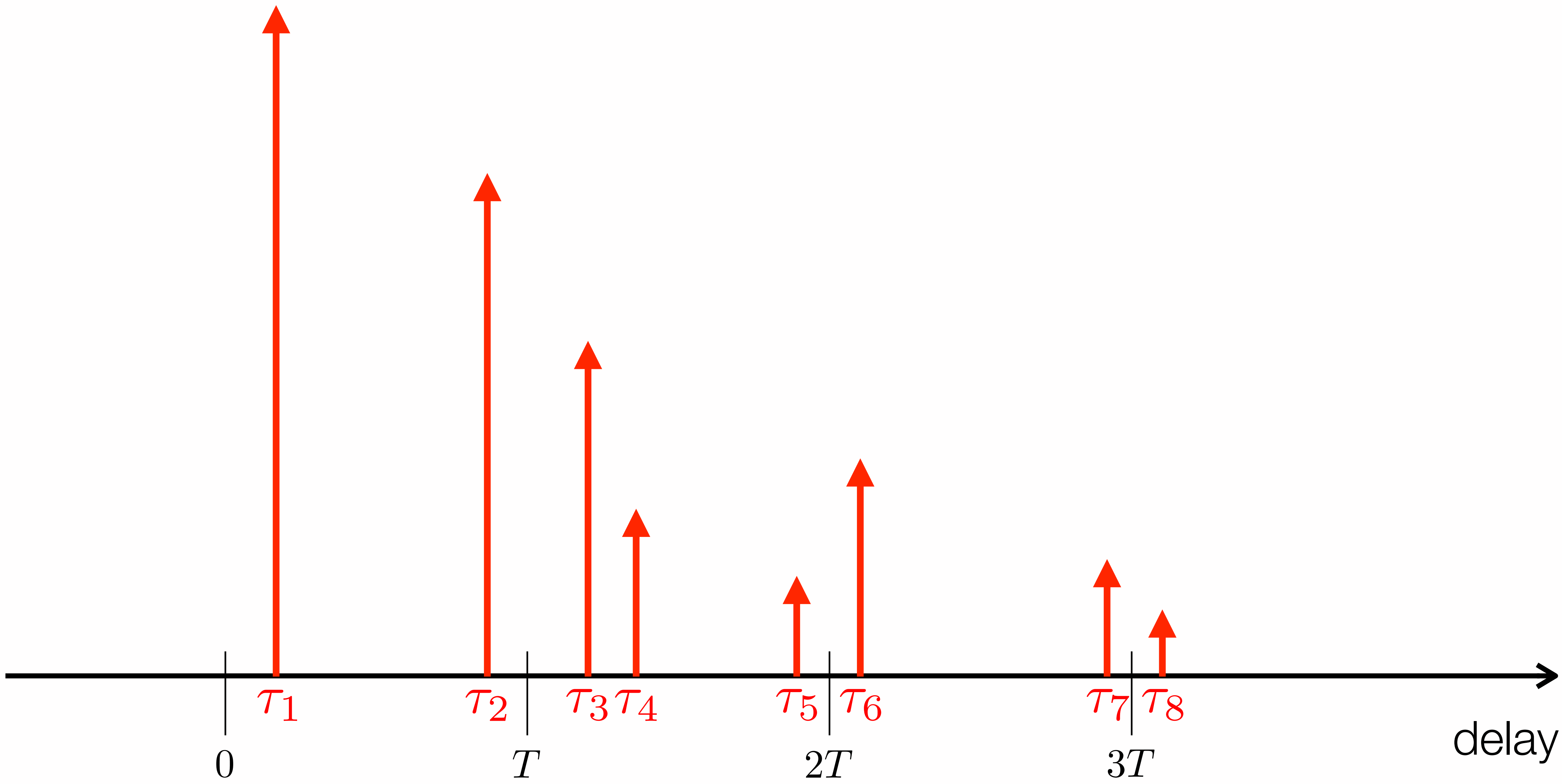


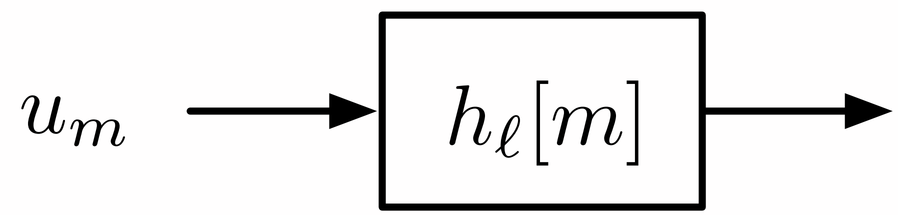
$\ell = 2$
τ_1 τ_2 τ_3 τ_4 τ_5 τ_6 τ_7 τ_8
0
T
$2T$
$3T$
delay

$\ell = 3$



Path resolution capability depends on the operating bandwidth





$$v_m = \sum_\ell h_\ell[m] u_{m-\ell}$$

$$h_\ell[m] = \sum_i a_i^b(mT) g(\ell T - \tau_i(mT))$$

$$= \sum_i a_i(mT) e^{-\mathrm{j}2\pi f_c \tau_i(mT)} g(\ell T - \tau_i(mT))$$

$$\approx \sum_{i \in \ell\text{-th delay bin}} a_i(mT) e^{-\mathrm{j}2\pi f_c \tau_i(mT)}$$

Difference in phases (over the paths that contribute significantly to the tap), causes variation of the tap gain

Large-scale Fading

- Path loss and Shadowing
  - In free space, received power $\propto r^{-2}$
  - With reflections and obstacles, can attenuate faster than r^{-2}

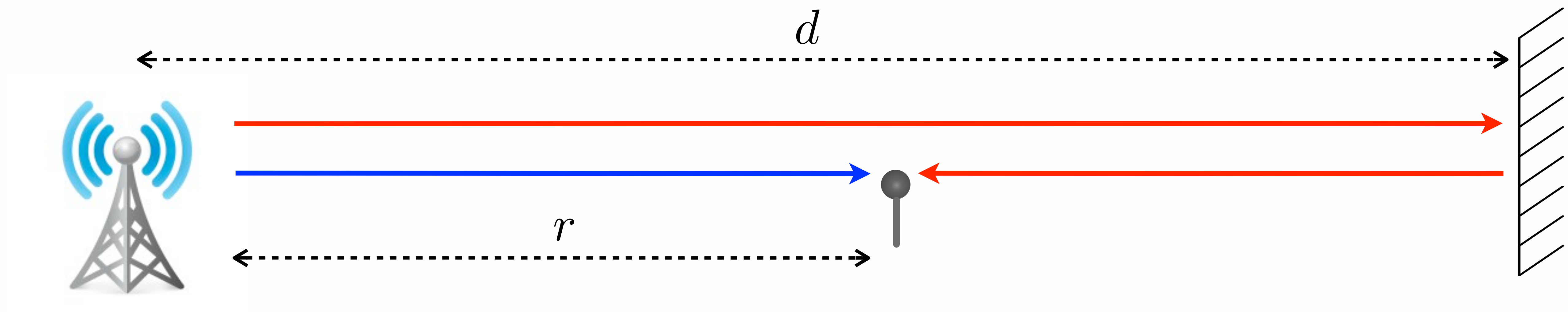
- Variation over time: very slow, order of seconds

- Critical for coverage and cell-cite planning

Multi-path (Small-scale) Fading

- Due to constructive and destructive interference of the waves
- Channel varies when the mobile moves a distance of the order of the carrier wavelength λ_c
  - Typical carrier frequency ~ 1GHz $\Longrightarrow \lambda_c \approx c/f_c = 0.3\text{m}$
- Variation over time: order of hundreds of microseconds
- Critical for design of communication systems

Fading over Frequency



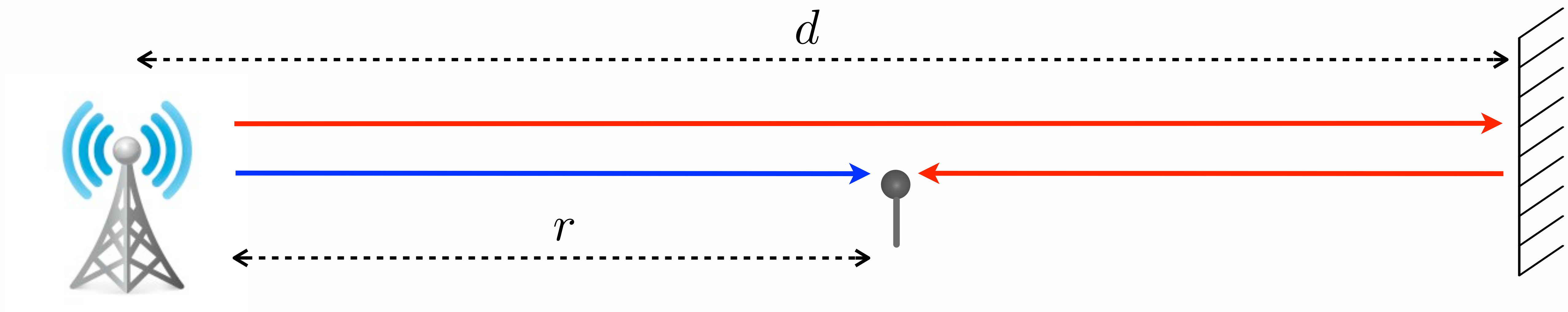
Transmitted Waveform (electric field): $\cos 2\pi f t$

Received Waveform (path 1): $\frac{\alpha}{r} \cos 2\pi f \left(t - \frac{r}{c}\right)$

Received Waveform (path 2): $-\frac{\alpha}{2d - r} \cos 2\pi f \left(t - \frac{2d-r}{c}\right)$

$\Longrightarrow$ Received Waveform (aggregate):

$$\boxed{\frac{\alpha}{r} \cos 2\pi f \left(t - \frac{r}{c}\right) - \frac{\alpha}{2d - r} \cos 2\pi f \left(t - \frac{2d-r}{c}\right)}$$



Transmitted Waveform (electric field): $\cos 2\pi f t$

Received Waveform (aggregate):

$$\boxed{\frac{\alpha}{r}\cos 2\pi f\left(t-\frac{r}{c}\right)-\frac{\alpha}{2d-r}\cos 2\pi f\left(t-\frac{2d-r}{c}\right)}$$

Phase Difference between the two sinusoids:

T_{d} **Delay Spread**

delay differences

$$\Delta\theta=\left\{\frac{2\pi f(2d-r)}{c}+\pi\right\}-\frac{2\pi f r}{c}=2\pi\frac{(2d-r)-r}{c}f+\pi$$

$$=\begin{cases}2n\pi, & \text{constructive interference}\\(2n+1)\pi, & \text{destructive interference}\end{cases}$$

Variation in Frequency Domain

neglect dependency on time

$$x(t) \longrightarrow \boxed{h(\tau; \cancel{t})} \longrightarrow y(t) = \sum_i a_i(\cancel{t})\, x\left(t - \tau_i(\cancel{t})\right)$$

- Frequency response: $\breve{h}(f) = \sum_i a_i e^{-\mathrm{j}2\pi\tau_i f}$
- Frequency variation causes variation in phase shift. Phase difference causes constructive or destructive interference.
- Phase difference: $2\pi f \max_{i \neq \tilde{i}} |\tau_i - \tau_{\tilde{i}}| \triangleq 2\pi f$ Delay Spread

  $T_{\mathrm{d}} \triangleq \max_{i \neq \tilde{i}} |\tau_i - \tau_{\tilde{i}}|$
- Frequency change by $\frac{1}{2T_{\mathrm{d}}}$, channel changes drastically!

Coherence Bandwidth

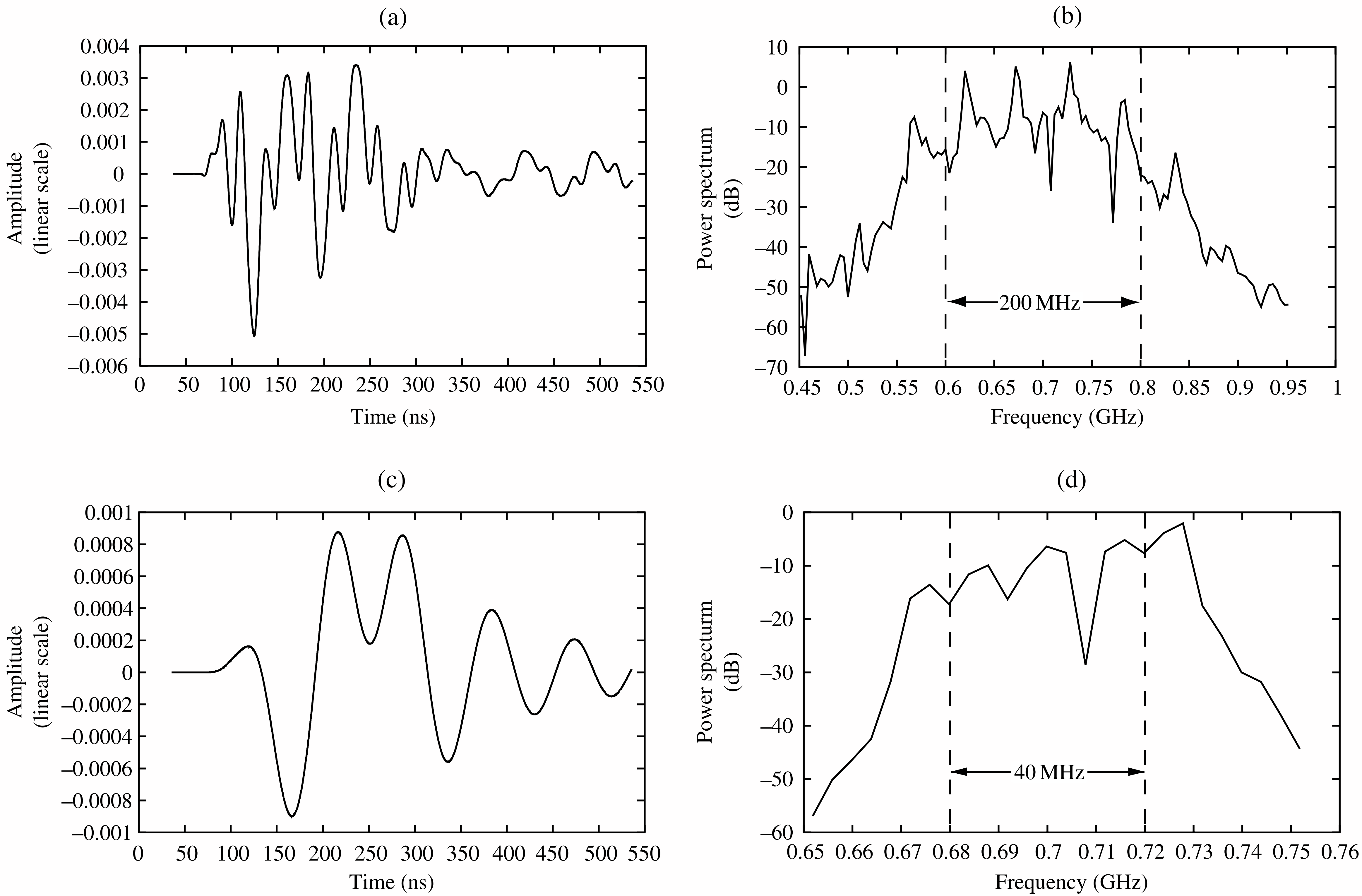
- Coherence bandwidth: $W_{\mathrm{c}} \sim \frac{1}{T_{\mathrm{d}}}$

- From the perspective of the equivalent discrete-time model, for a system with operating (one-sided) bandwidth W :

$$W_{\mathrm{c}} \gg 2W \implies \text{single tap, flat fading}$$
$$W_{\mathrm{c}} < 2W \implies \text{multiple taps, frequency-selective fading}$$

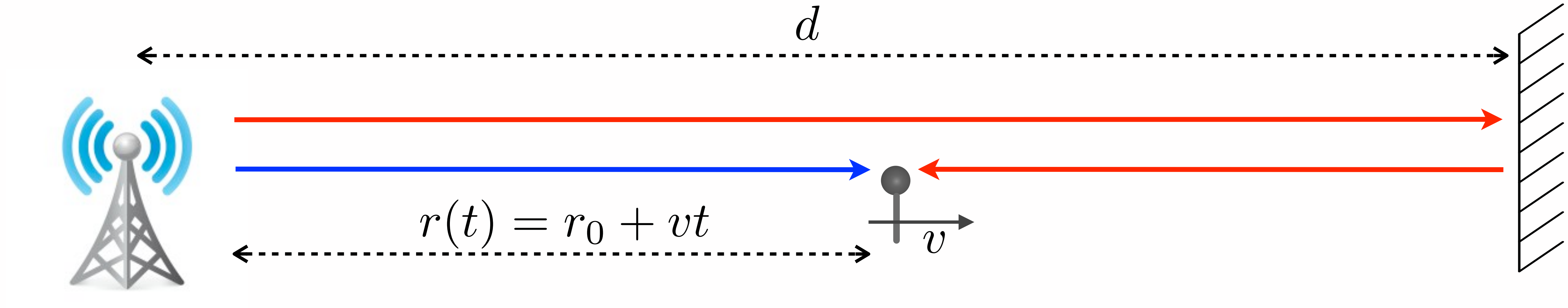
- Note: this is a rough qualitative classification

Same channel, different operating bandwidth



Larger bandwidth, more paths can be resolved

Fading over Time



Transmitted Waveform (electric field): $\cos 2\pi f t$

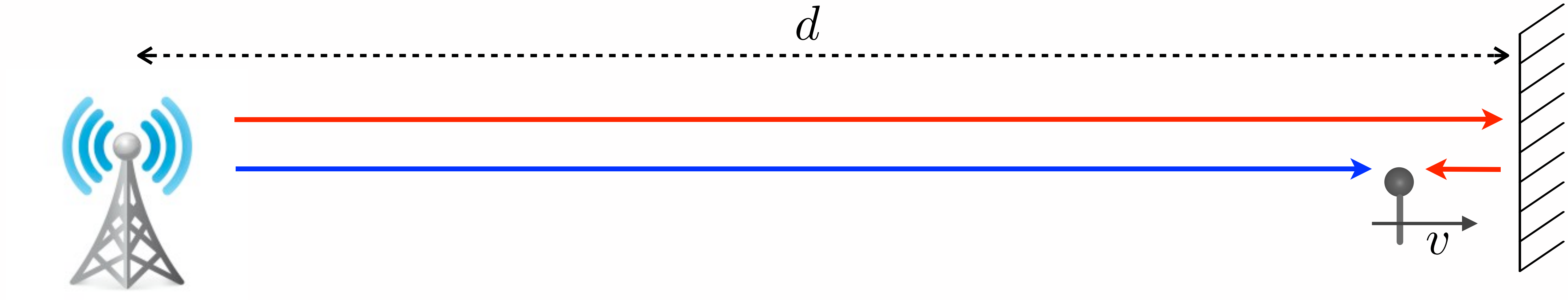
Received Waveform (path 1): $\dfrac{\alpha}{r(t)} \cos 2\pi f \left(t - \dfrac{r(t)}{c}\right)$

Received Waveform (path 2): $-\dfrac{\alpha}{2d - r(t)} \cos 2\pi f \left(t - \dfrac{2d - r(t)}{c}\right)$

$\Longrightarrow$ Received Waveform (aggregate):

$$\frac{\alpha}{r(t)} \cos 2\pi f \left(t - \frac{r(t)}{c}\right) - \frac{\alpha}{2d - r(t)} \cos 2\pi f \left(t - \frac{2d - r(t)}{c}\right)$$

$$= \boxed{\frac{\alpha}{r_0 + vt} \cos 2\pi f \left[\left(1 - \frac{v}{c}\right) t - \frac{r_0}{c}\right] - \frac{\alpha}{2d - r_0 - vt} \cos 2\pi f \left[\left(1 + \frac{v}{c}\right) t - \frac{2d - r_0}{c}\right]}$$



Approximation: distance to mobile Rx $\ll$ distance to Tx

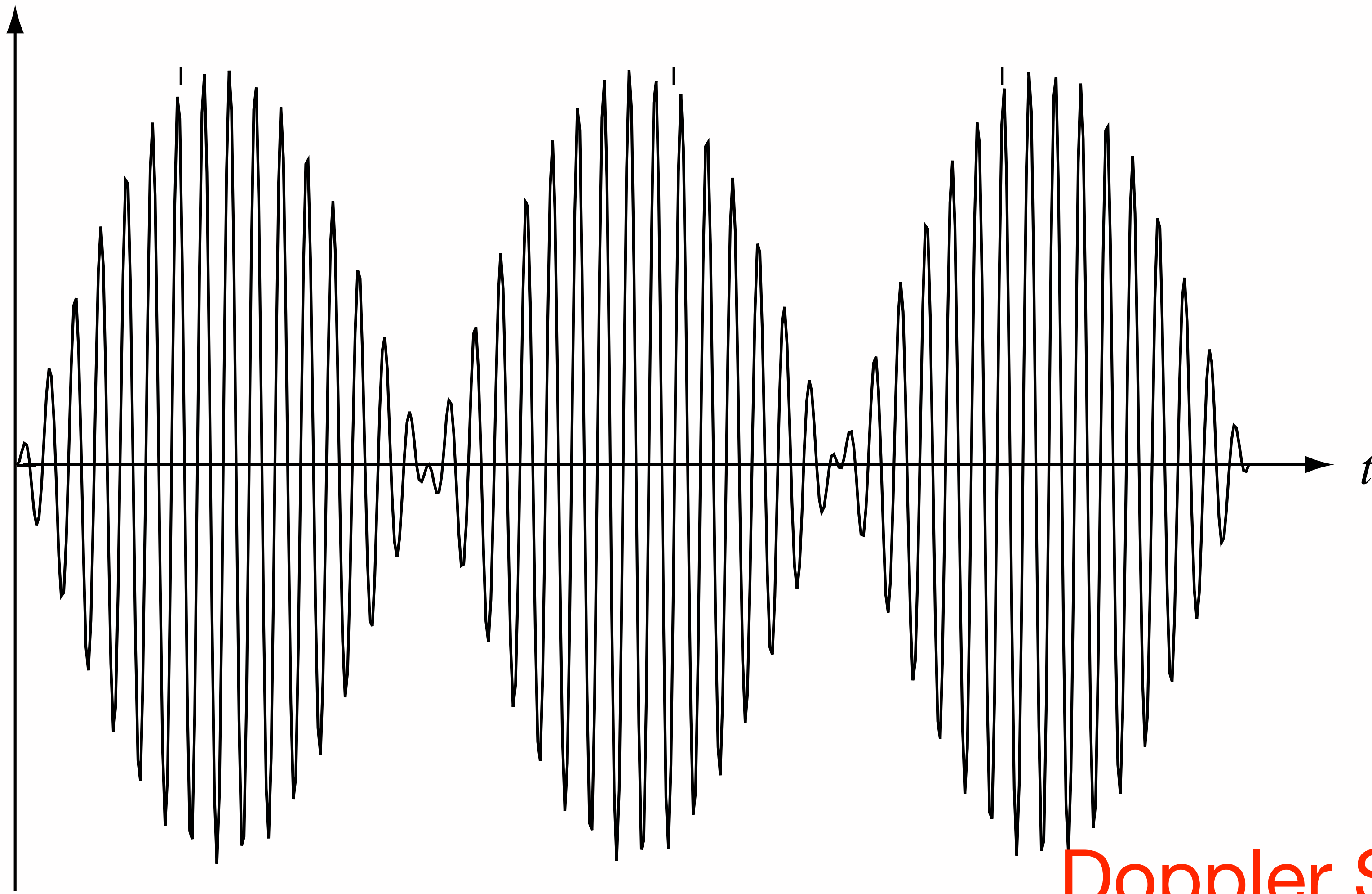
$\Longrightarrow$ Received Waveform (aggregate):

$$= \frac{\alpha}{r_0 + vt} \cos 2\pi f \left[\left(1 - \frac{v}{c}\right) t - \frac{r_0}{c} \right] - \frac{\alpha}{2d - r_0 - vt} \cos 2\pi f \left[\left(1 + \frac{v}{c}\right) t - \frac{2d - r_0}{c} \right]$$

$$\approx \frac{2\alpha}{r_0 + vt} \sin 2\pi f \left(\frac{vt}{c} + \frac{r_0 - d}{c} \right) \sin 2\pi f \left(t - \frac{d}{c} \right)$$

Time-varying amplitude

Time-invariant shift of the original input waveform



Doppler Spread

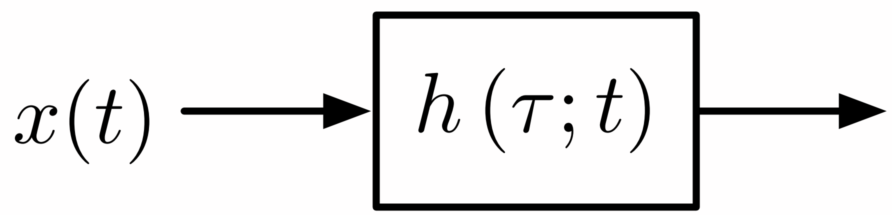
Time-varying envelope $\frac{2\alpha}{r_0 + vt} \sin 2\pi f \left(\frac{vt}{c} + \frac{r_0 - d}{c} \right)$

$D_s \triangleq \frac{2fv}{c}$

Time-variation scale: r_0/v (seconds or minutes), much smaller than that of the second term

Difference of the Doppler shifts of the two paths, cause this variation over time. Time-variation scale: c/fv (ms)

Variation in Time Domain



$$y(t) = \sum_i a_i(t) x\left(t - \tau_i(t)\right)$$

- Frequency response: $\check{h}(f; t) = \sum_i a_i(t) e^{-j2\pi f \tau_i(t)}$

- Phase shift changes over time at a rate $2\pi f \tau_i'(t)$
  Doppler shift (shift in frequency) of path i : $\delta_i \triangleq f \tau_i'(t)$

- Phase difference changes over time at a rate

$$2\pi f \max_{i \neq \tilde{i}} \left| \tau_i'(t) - \tau_{\tilde{i}}'(t) \right|$$

Doppler spread: $D_\mathrm{s} \triangleq f_\mathrm{c} \max_{i \neq \tilde{i}} \left| \tau_i'(t) - \tau_{\tilde{i}}'(t) \right|$

Coherence Time

- Coherence time: $T_{\mathrm{c}} \sim \dfrac{1}{D_{\mathrm{s}}}$

- For a system with latency requirement T:

$$T_{\mathrm{c}} \gg T \implies \text{slow fading}$$
$$T_{\mathrm{c}} < T \implies \text{fast fading}$$

- Note: this is a rough qualitative classification

Parameters of Wireless Channels

Key channel parameters and time-scales	Symbol	Representative values
Carrier frequency	f_c	1 GHz
Communication bandwidth	W	1 MHz
Distance between transmitter and receiver	d	1 km
Velocity of mobile	v	64 km/h
Doppler shift for a path	$D = f_c v/c$	50 Hz
Doppler spread of paths corresponding to a tap	D_s	100 Hz
Time-scale for change of path amplitude	d/v	1 minute
Time-scale for change of path phase	$1/(4D)$	5 ms
Time-scale for a path to move over a tap	$c/(vW)$	20 s
Coherence time	$T_c = 1/(4D_s)$	2.5 ms
Delay spread	T_d	1 μs
Coherence bandwidth	$W_c = 1/(2T_d)$	500 kHz

Types of Wireless Channels

Types of channel	Defining characteristic
Fast fading	$T_c \ll$ delay requirement
Slow fading	$T_c \gg$ delay requirement
Flat fading	$W \ll W_c$
Frequency-selective fading	$W \gg W_c$
Underspread	$T_d \ll T_c$

- Typical channels are **underspread**
- Coherence time T_c depends on carrier frequency and mobile speed, of the order of ms or more
- Delay spread T_d depends on distance to scatters and cell size, of the order of ns (indoor) to μs (outdoor)

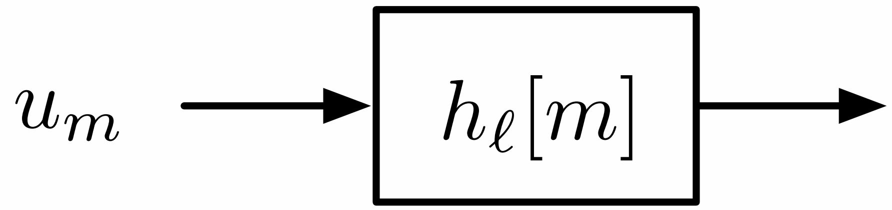
Fading: Short Summary

- In wireless communications, channel coefficients can have a widely varying magnitude. They change over time as well.

- As for the effective discrete-time LTV model:
  - The number of taps depends on the coherence bandwidth W_{c} and the operating bandwidth W
  - The tap coefficient changes over time at a scale of the coherence time T_{c}

- The tap coefficients can be tracked, but due to the widely varying range and the variation over time and frequency, it is beneficial to model them as random processes

Stochastic Modeling of Fading



$$V_m = \sum_\ell h_\ell[m] u_{m-\ell} + Z_m$$

$$h_\ell[m] \approx \sum_{i \in \ell\text{-th delay bin}} a_i(mT) e^{-\mathrm{j}2\pi f_c \tau_i(mT)}$$

- Additive noise Z_m
  - Essentially completely random, no correlation over time
  - Largely depends on nature
  - Can be dealt with using wireline communication techniques
- Filter taps $h_\ell[m]$
  - Varying over time and frequency
  - Largely depends on nature
  - Why not use stochastic models for taps as well?

Modeling Philosophy

- Simple models may not fit the practical scenarios perfectly
- Complicated models can be established by extensive measurement
- But simple models make analysis tractable and generate insights for system design

- So it is better to develop new systems based on simple yet representative models, and validate the design over-the-air or through simulation on complicated models

- We will focus on a classical model, Rayleigh fading, to model a single tap
- Then we discuss about modeling the variation over time by using WSS random processes

Rayleigh Fading

- Many small scattered paths for each tap (no dominant path):
  - Phase of each path is **uniformly distributed** over $[0, 2\pi]$

$$h_\ell[m] \approx \sum_{i \in \ell\text{-th delay bin}} a_i(mT) e^{-\mathrm{j} 2\pi f_c \tau_i(mT)}$$

  - For each path it is a **circular symmetric** random variable

$$X : \text{circular symmetric} \iff X \overset{\mathrm{d}}{=} X e^{\mathrm{j}\phi}, \ \forall \phi$$

- Each tap: sum of many small indep. **circular symmetric** r.v.'s
  - By Central Limit Theorem (CLT), we can model $H_\ell[m] \sim \mathcal{CN}\left(0, \sigma_\ell^2\right)$
  - Zero-mean because of rich scattering

$$H \sim \mathcal{CN}(0, \sigma^2) \iff |H|^2 \sim \mathrm{Exp}(\sigma^{-2}), \ \angle H \sim \mathrm{Unif}[0, 2\pi]$$

Time and Frequency Coherence

- Model $\{H_\ell[m] \mid m \in \mathbb{Z}\}$ as a WSS random process $\forall$ tap ℓ
- Processes $\{H_\ell[m] \mid m \in \mathbb{Z}\}$ are independent across ℓ
- Tap gain auto-correlation function:

$$\mathsf{R}_{H_\ell}[k] \triangleq \mathsf{E}\left[H_\ell[m+k]H_\ell^*[m]\right]$$

- Observe that $\mathsf{R}_{H_\ell}[0] = \mathsf{E}\left[|H_\ell[m]|^2\right]$: energy at the ℓ-th tap
- Delay spread: $T \times$ the range of ℓ that contain most energy
- Coherence time: $T \times$ the largest value of k such that

  $\mathsf{R}_{H_\ell}[k]$ is very different from $\mathsf{R}_{H_\ell}[0]$

Part II. Fading and Diversity

Impact of Fading in Detection; Time Diversity; Antenna Diversity; Frequency Diversity

Simplest Model: Single-Tap Rayleigh Fading

- Flat fading: single-tap Rayleigh fading

$$V = Hu + Z, \quad H \sim \mathcal{CN}(0,1),\ Z \sim \mathcal{CN}(0, N_0)$$

- Detection:

$$V = Hu + Z$$

$$\hat{\Theta} \qquad \hat{u} = a_{\hat{\theta}}$$

$$u = a_\theta \in \mathcal{A} \triangleq \{a_1, \ldots, a_M\}$$

- Detector (Rx) may or may not know the channel coefficients

<u>Coherent Detection</u>: Rx knows the realization of H

<u>Noncoherent Detection</u>: Rx does not know the realization of H

Coherent Detection of BPSK

$$V = Hu + Z \longrightarrow \boxed{\text{Detection}} \longrightarrow \hat{\Theta} = \phi(V, H) \quad \hat{u} = a_{\hat{\theta}}$$

$$u \in \{\pm\sqrt{E_s}\} \qquad a_0 = +\sqrt{E_s},\ a_1 = -\sqrt{E_s}$$

$$H \sim \mathcal{CN}(0, 1),\ Z \sim \mathcal{CN}(0, N_0)$$

- Likelihood function:

$$f_{V,H|\Theta}(v, h|\theta) = f_{V|H,\Theta}(v|h, \theta) f_H(h) \propto \boxed{f_{V|H,\Theta}(v|h, \theta)}$$

- The detection problem is equivalent to binary detection in

$$\tilde{V} = u + \tilde{Z}, \quad \tilde{V} \triangleq V/h,\ \tilde{Z} \triangleq Z/h \sim \mathcal{CN}(0, N_0/|h|^2)$$

- Probability of error conditioned on the realization of $H = h$:

$$\mathsf{P_e}(\phi_{\mathrm{ML}}; H = h) = \mathrm{Q}\left(\frac{2\sqrt{E_s}}{2\sqrt{N_0/(2|h|^2)}}\right) = \mathrm{Q}\left(\sqrt{\frac{2|h|^2 E_s}{N_0}}\right)$$

- Probability of error:

$$\mathsf{P_e}(\phi_{\mathrm{ML}}; H = h) = \mathrm{Q}\left(\sqrt{\frac{2|h|^2 E_{\mathrm{s}}}{N_0}}\right)$$

$$\mathsf{P_e}(\phi_{\mathrm{ML}}) = \mathsf{E}_{H\sim\mathcal{CN}(0,1)}\left[\mathsf{P_e}(\phi_{\mathrm{ML}}; H)\right]$$

$$= \mathsf{E}_{H\sim\mathcal{CN}(0,1)}\left[\mathrm{Q}\left(\sqrt{\frac{2|H|^2 E_{\mathrm{s}}}{N_0}}\right)\right]$$

$$\leq \mathsf{E}_{|H|^2\sim\mathrm{Exp}(0,1)}\left[\tfrac{1}{2}\exp(-|H|^2\mathsf{SNR})\right]$$

$$= \int_0^\infty \frac{1}{2} e^{-t\mathsf{SNR}} e^{-t}\,\mathrm{d}t = \boxed{\frac{1}{2(1+\mathsf{SNR})}}$$

Impact of Fading

- Let us explore the impact of fading by comparing the performance of coherent BPSK between AWGN and single-tap Rayleigh fading

- The average received SNRs are the same:

$$\mathsf{E}_{H\sim\mathcal{CN}(0.1)}\left[|H|^2\mathsf{SNR}\right] = \mathsf{SNR}$$

- AWGN: probability of error decays exponentially fast:

$$\mathsf{P_e}(\phi_{\mathrm{ML}}) = \mathrm{Q}\left(\sqrt{2\mathsf{SNR}}\right) \le \tfrac{1}{2}\exp(-\mathsf{SNR})$$

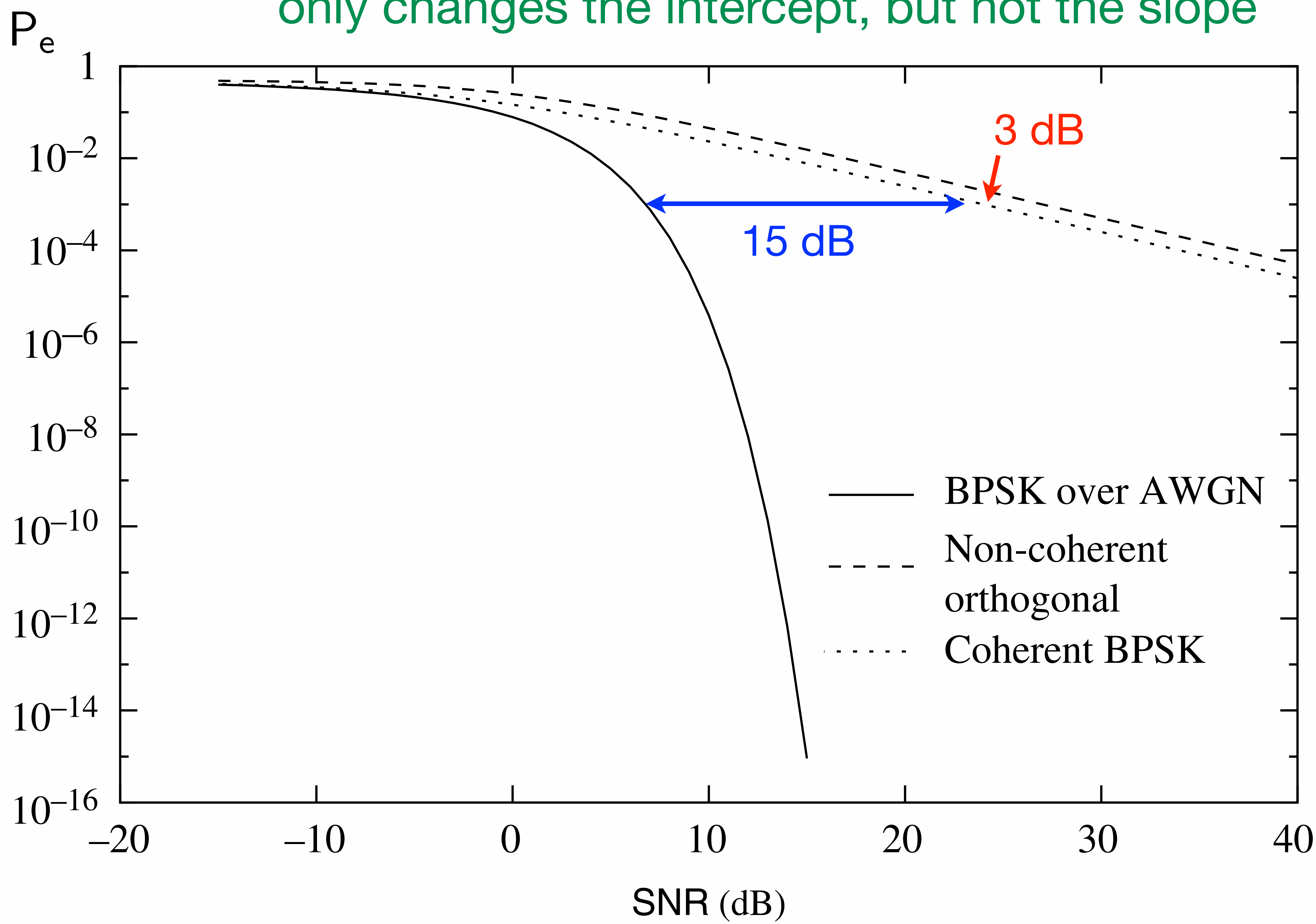
order: $e^{-\mathsf{SNR}}$

- Rayleigh fading: probability of error decays much slower:

$$\mathsf{P_e}(\phi_{\mathrm{ML}}) = \mathsf{E}_{H\sim\mathcal{CN}(0,1)}\left[\mathrm{Q}\left(\sqrt{2|H|^2\mathsf{SNR}}\right)\right] \le \tfrac{1}{2}\tfrac{1}{1+\mathsf{SNR}}$$

order: SNR^{-1}

Availability of channel state information (CSI) at Rx only changes the intercept, but not the slope



Coherent Detection of General QAM

- Probability of error for $M = 2^{2\ell}$ -ary QAM

$$\mathsf{P_e}(\phi_{\mathrm{ML}}; H = h) \le 4\mathrm{Q}\left(\sqrt{\frac{|h|^2 d_{\min}^2}{2N_0}}\right) = 4\mathrm{Q}\left(\sqrt{\tfrac{3}{M-1}|h|^2\mathsf{SNR}}\right)$$

$$\mathsf{P_e}(\phi_{\mathrm{ML}}) \le \mathsf{E}_{H\sim\mathcal{CN}(0,1)}\left[4\mathrm{Q}\left(\sqrt{\tfrac{3}{M-1}|H|^2\mathsf{SNR}}\right)\right]$$

$$\le \mathsf{E}_{|H|^2\sim\mathrm{Exp}(0,1)}\left[2\exp(-|H|^2\tfrac{3}{2(M-1)}\mathsf{SNR})\right]$$

$$= \frac{2}{1+\frac{3}{2(M-1)}\mathsf{SNR}} \approx \frac{4(M-1)}{3}\mathsf{SNR}^{-1}$$

- Using general constellation does not change the order of performance (the “slope” on the $\log \mathsf{P_e}$ vs. $\log \mathsf{SNR}$ plot)

- Different constellation only changes the intercept

Deep Fade: the Typical Error Event

- In Rayleigh fading channel, regardless of constellation size and detection method (coherent/non-coherent), $\mathsf{P_e} \sim \mathsf{SNR}^{-1}$
- This is in sharp contrast to AWGN: $\mathsf{P_e} \sim \exp(-c\mathsf{SNR})$

- Why? Let's take a deeper look at the BPSK case:

$$\mathsf{P_e}(\phi_{\mathrm{ML}}; H = h) = \mathrm{Q}\left(2|h|^2\mathsf{SNR}\right)$$

  - If $|h|^2\mathsf{SNR} \gg 1 \implies$ channel is good, error probability $\sim \exp(-c\mathsf{SNR})$
  - If $|h|^2\mathsf{SNR} < 1 \implies$ channel is bad, error probability is $\Theta(1)$

$$\begin{aligned}\mathsf{P_e} \equiv \mathsf{P}\{\mathcal{E}\} &= \mathsf{P}\left\{|H|^2 > \mathsf{SNR}^{-1}\right\}\mathsf{P}\left\{\mathcal{E} \mid |H|^2 > \mathsf{SNR}^{-1}\right\} \\ &\quad + \mathsf{P}\left\{|H|^2 < \mathsf{SNR}^{-1}\right\}\mathsf{P}\left\{\mathcal{E} \mid |H|^2 < \mathsf{SNR}^{-1}\right\} \\ &\lessapprox \mathsf{P}\left\{|H|^2 < \mathsf{SNR}^{-1}\right\} = 1 - e^{-\mathsf{SNR}^{-1}} \approx \boxed{\mathsf{SNR}^{-1}}\end{aligned}$$

- Deep fade event: $\left\{|H|^2 < \mathsf{SNR}^{-1}\right\}$

Diversity

$$V = Hu + Z \qquad \text{Deep fade event: } \{|H|^2 < \mathsf{SNR}^{-1}\}$$

- Reception only relies on a single “look” at the fading state H
- If H is in deep fade $\Longrightarrow$ big trouble (low reliability)

- Increase the number of “looks” $\Longleftrightarrow$ Increase **diversity**
  - If one look is in deep fade, other looks can compensate!
- If there are L indep. looks, the probability of deep fade becomes

$$\prod_{\ell=1}^{L} \mathsf{P}\{\text{Path } i \text{ in deep fade}\} \approx \mathsf{SNR}^{-L}$$

- Find **independent** “looks” over **time**, **space**, and **frequency** to increase diversity!

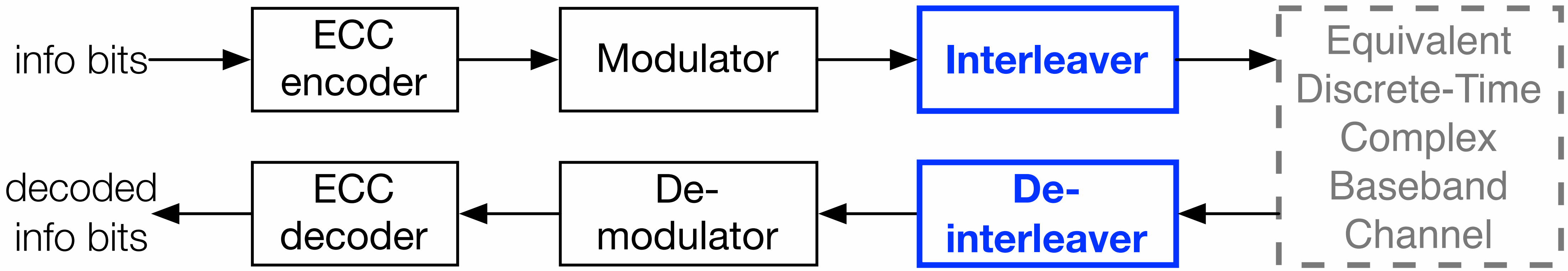
Time Diversity

- Channel varies over time, at the scale of coherence time T_c .

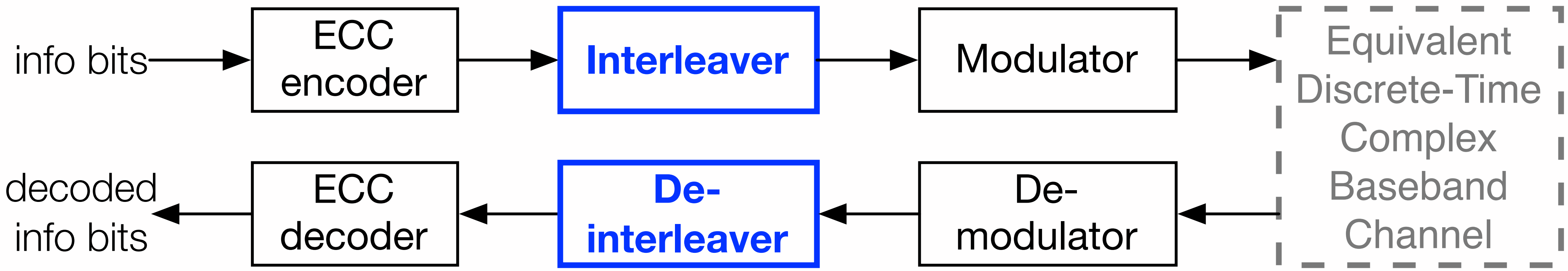
- Interleaving:
  - Channels within a coherence time are highly correlated
  - Realizations separated by several T_c's apart are roughly independent
  - Diversity is obtained if we spread the codeword across multiple coherence time periods

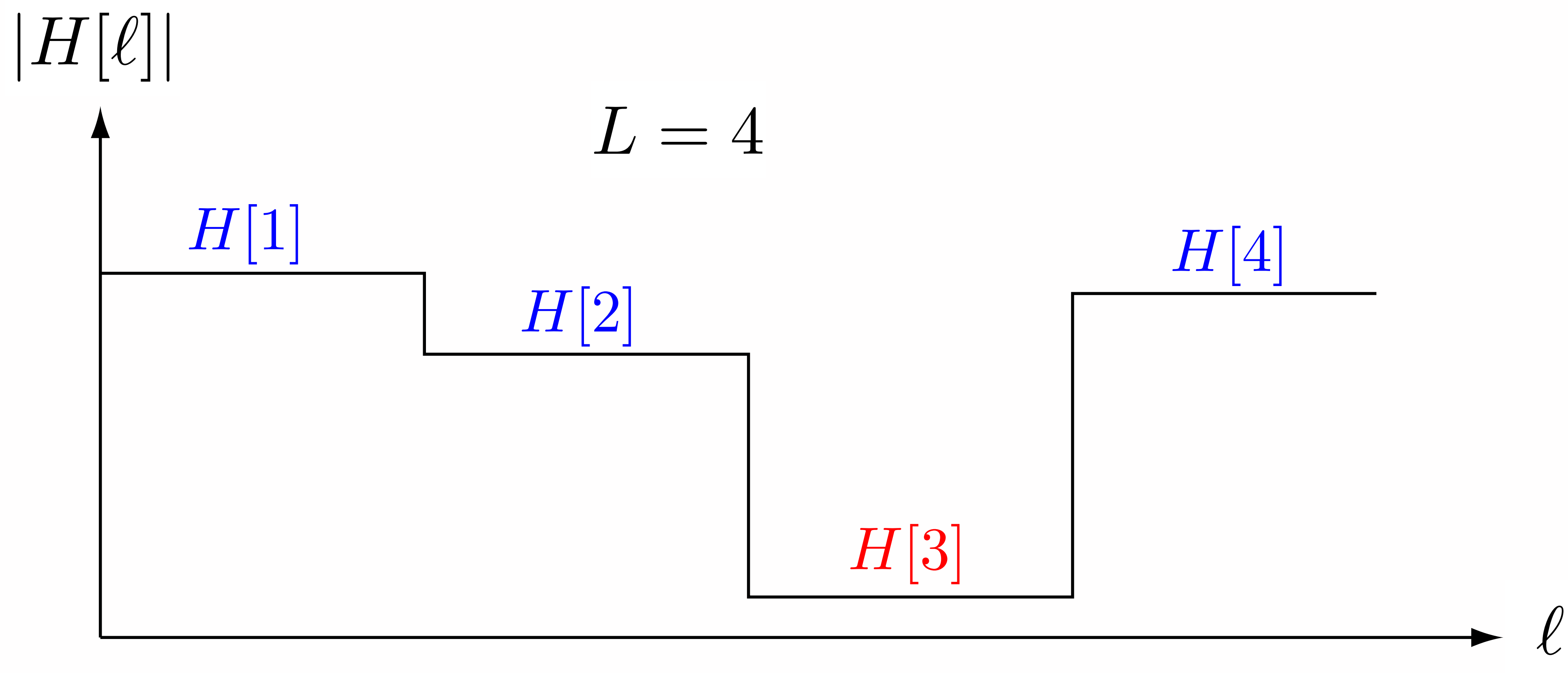
- Architecture(s):
  - bit-level interleaver: interleave before modulation
  - symbol-level interleaver: interleave after modulation

- Symbol-level interleaving



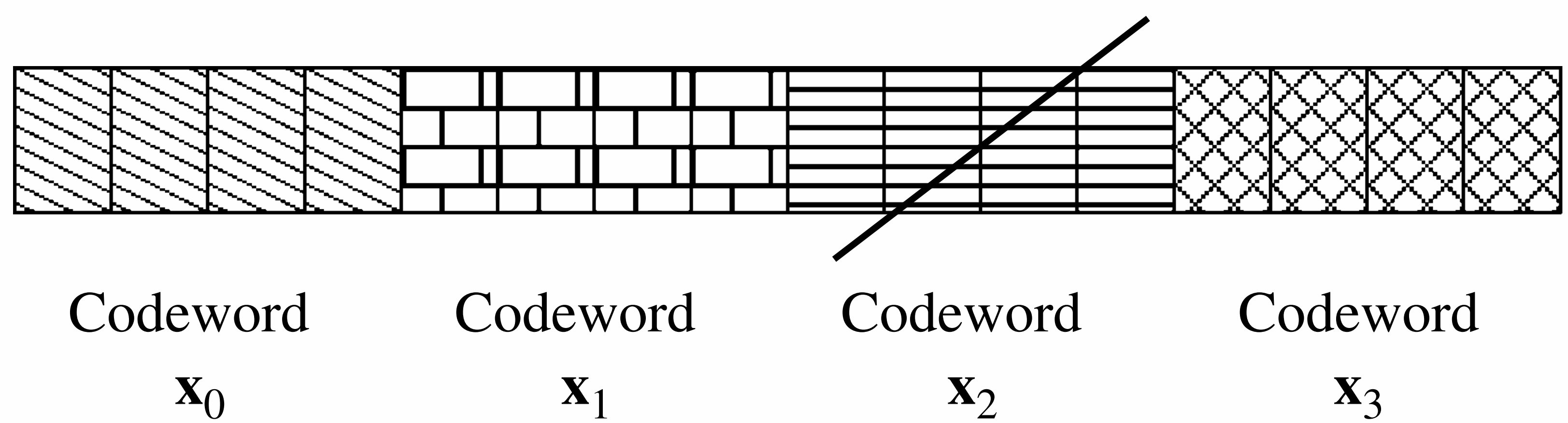
- Bit-level interleaving





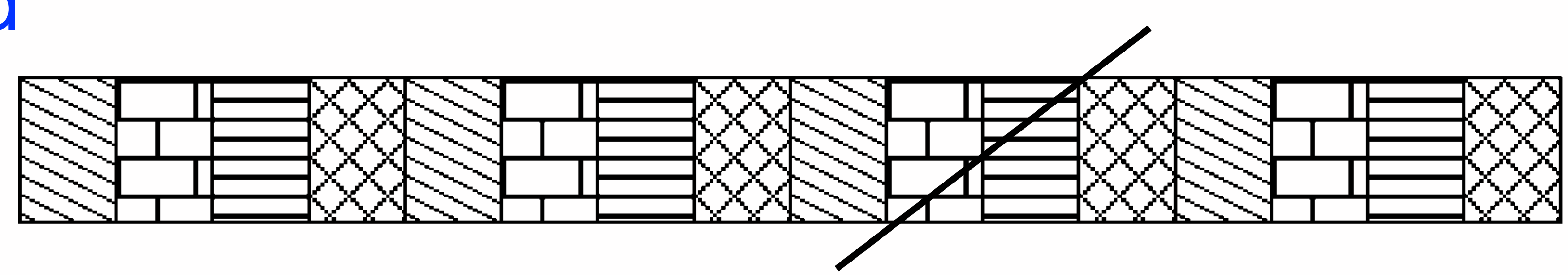
All are bad

No interleaving



Only one is bad

Interleaving



Repetition Coding + Interleaving

- Equivalent vector channel
  - Channel model: $V[\ell] = H[\ell]u[\ell] + Z[\ell],\ Z[\ell] \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_0), \quad \ell = 1, ..., L$
  - (sufficient) Interleaving $\implies \{H[\ell]\}_{\ell=1}^{L}$: i.i.d. $\mathcal{CN}(0,1)$
  - Repetition coding $\implies u[\ell] = u, \quad \ell = 1, ..., L$
  - Equivalent vector channel: $\boxed{\boldsymbol{V} = \boldsymbol{H}u + \boldsymbol{Z}}$

$$\boldsymbol{V} \triangleq \begin{bmatrix} V[1] & \cdots & V[L] \end{bmatrix}^{\mathsf{T}} \quad \boldsymbol{H} \triangleq \begin{bmatrix} H[1] & \cdots & H[L] \end{bmatrix}^{\mathsf{T}} \quad \boldsymbol{Z} \triangleq \begin{bmatrix} Z[1] & \cdots & Z[L] \end{bmatrix}^{\mathsf{T}}$$

- Probability of error analysis for BPSK:
  - Conditioned on $\boldsymbol{H} = \boldsymbol{h}$: $\mathsf{P}_{\mathsf{e}}(\phi_{\mathrm{ML}}; \boldsymbol{H} = \boldsymbol{h}) = \mathrm{Q}\left(2\left\|\boldsymbol{h}\right\|^2 \mathsf{SNR}\right)$
  - Average probability of error:

$$\mathsf{P}_{\mathsf{e}}(\phi_{\mathrm{ML}}) = \mathsf{E}_{\boldsymbol{H}}\left[\mathrm{Q}\left(\sqrt{2\left\|\boldsymbol{H}\right\|^2 \mathsf{SNR}}\right)\right] \le \mathsf{E}_{\boldsymbol{H}}\left[\tfrac{1}{2}\exp(-\left\|\boldsymbol{H}\right\|^2 \mathsf{SNR})\right]$$

$$= \frac{1}{2}\prod_{\ell=1}^{L} \mathsf{E}_{H_\ell}\left[\exp(-|H_\ell|^2 \mathsf{SNR})\right] = \boxed{\frac{1}{2}(1 + \mathsf{SNR})^{-L}}$$

order: SNR^{-L}

Probability of Deep Fade

- Deep fade event: $\{\|\boldsymbol{H}\|^2 < \mathsf{SNR}^{-1}\}$
  - "Equivalent squared channel" $\|\boldsymbol{H}\|^2$ is the sum of L i.i.d. $\mathrm{Exp}(1)$ r.v.:

$$f_{\|\boldsymbol{H}\|^2}(x) = \frac{1}{(L-1)!} x^{L-1} e^{-x}, \quad x \geq 0$$

  - Chi-squared distribution with $2L$ degrees of freedom: $\|\boldsymbol{H}\|^2 \sim \chi^2_{2L}$

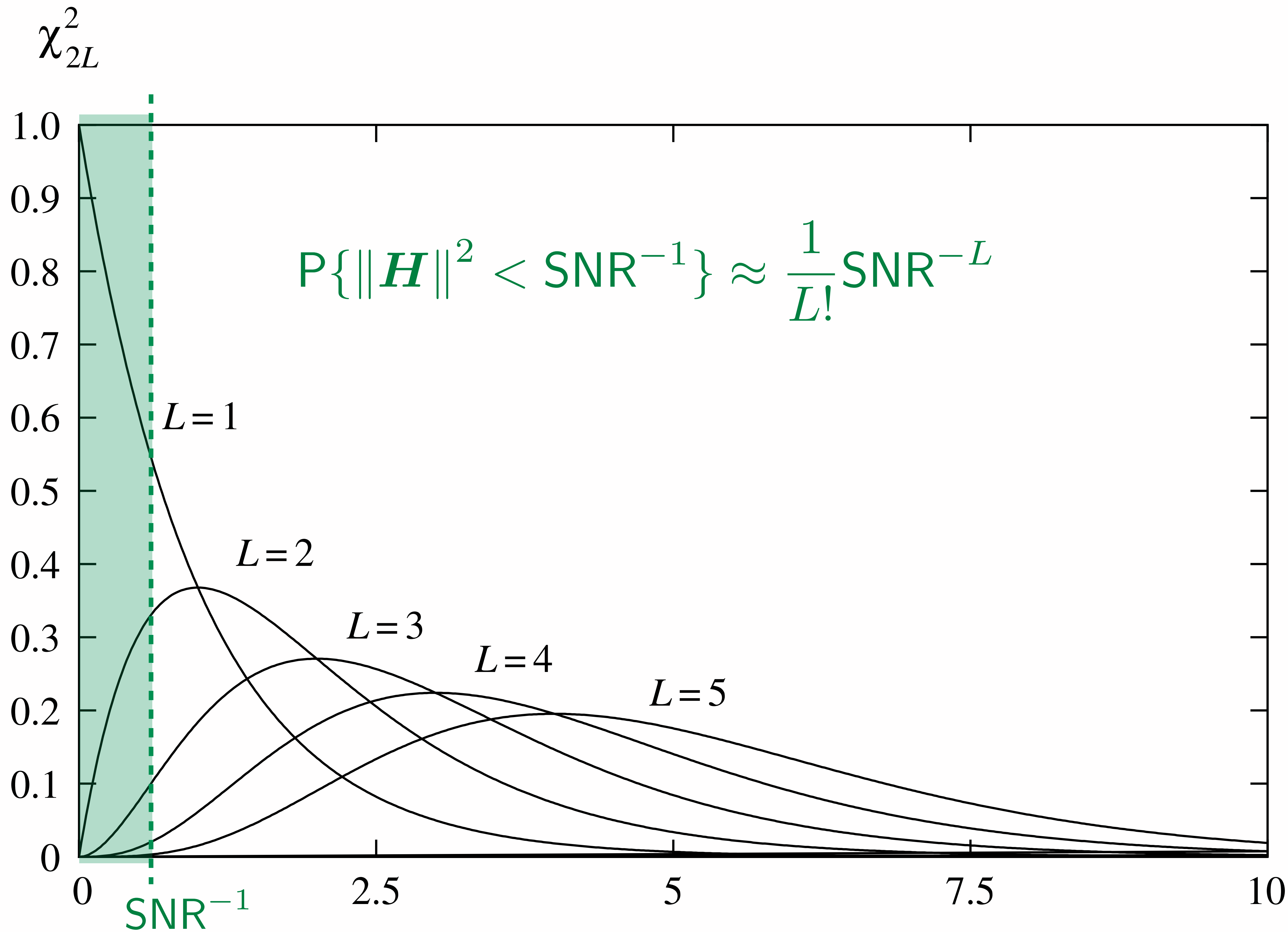
- Probability of deep fade:

$$\mathsf{P}\{\|\boldsymbol{H}\|^2 < \mathsf{SNR}^{-1}\} = \int_0^{\mathsf{SNR}^{-1}} \frac{1}{(L-1)!} x^{L-1} e^{-x} \, \mathrm{d}x$$

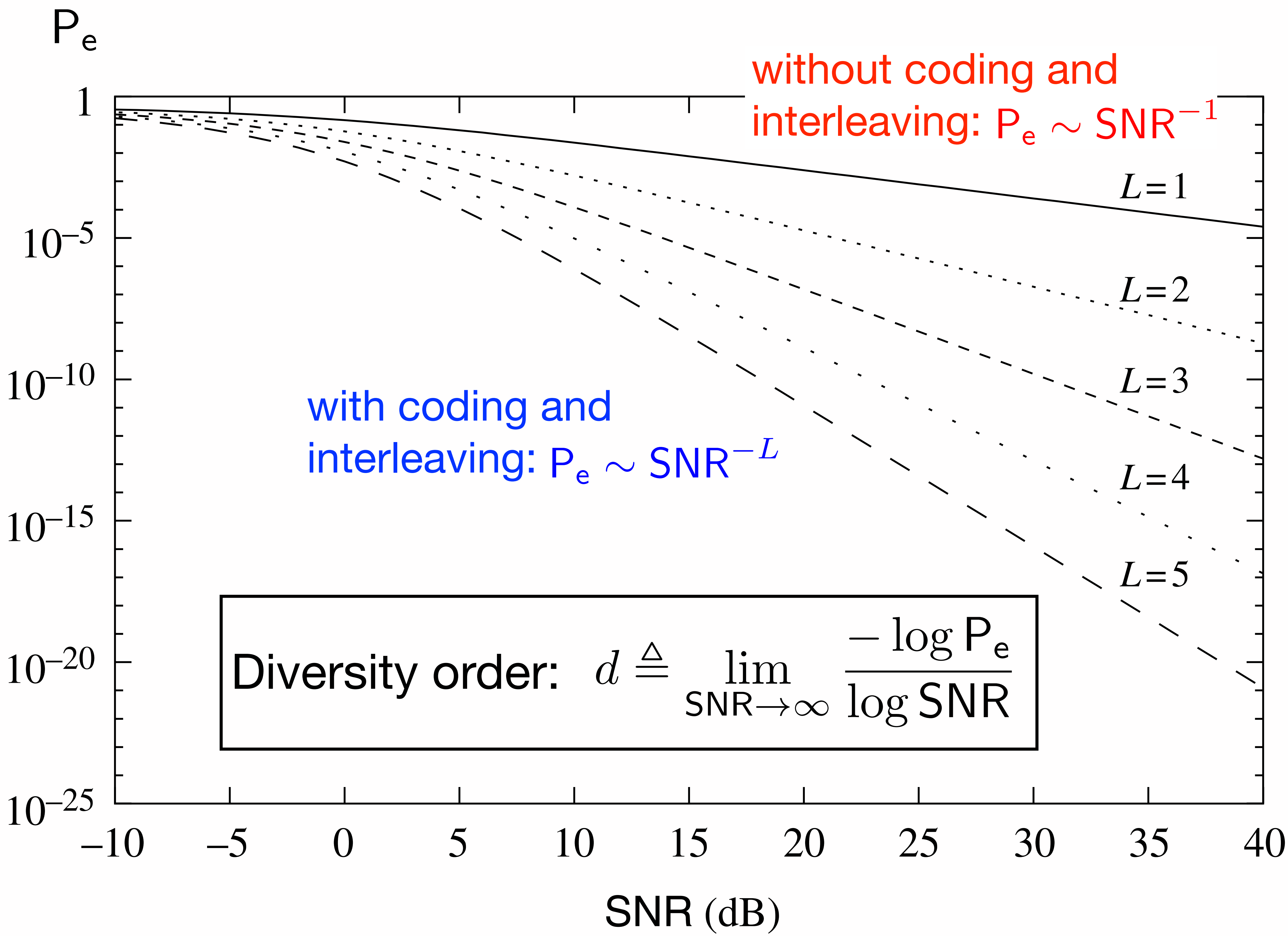
  - Approximation at high SNR:

$$\mathsf{P}\{\|\boldsymbol{H}\|^2 < \mathsf{SNR}^{-1}\} \approx \int_0^{\mathsf{SNR}^{-1}} \frac{1}{(L-1)!} x^{L-1} \, \mathrm{d}x = \frac{1}{L!} \mathsf{SNR}^{-L}$$

order: SNR^{-L}


χ^2_{2L}
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$\mathsf{P}\{\|\boldsymbol{H}\|^2 < \mathsf{SNR}^{-1}\} \approx \frac{1}{L!}\mathsf{SNR}^{-L}$
$L = 1$
$L = 2$
$L = 3$
$L = 4$
$L = 5$
0
2.5
5
7.5
10
SNR^{-1}

Diversity Order: $1 \to L$

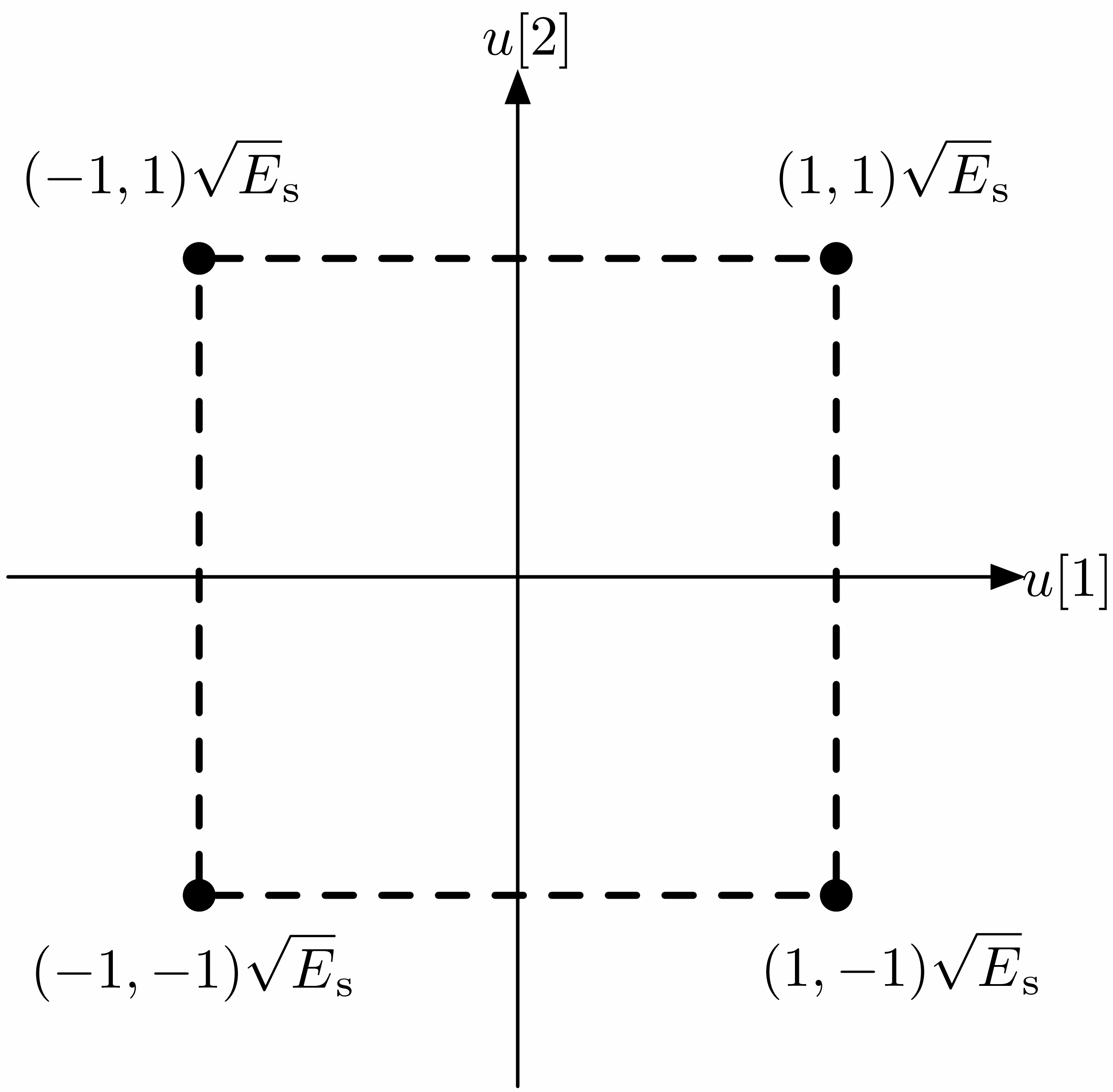


Time-Diversity Code

- Full diversity order:
  - Total L independent looks (interleave over L coherence time intervals)
  - The scheme can achieve full diversity order if its diversity order is L.
- Repetition coding
  - achieves full diversity order
  - suffers loss in transmission rate
- Is it possible to achieve full diversity order without compromising the transmission rate?
- The answer is yes, with Time-Diversity Code.

Sending 2 BPSK Symbols for $L = 2$ "Looks"

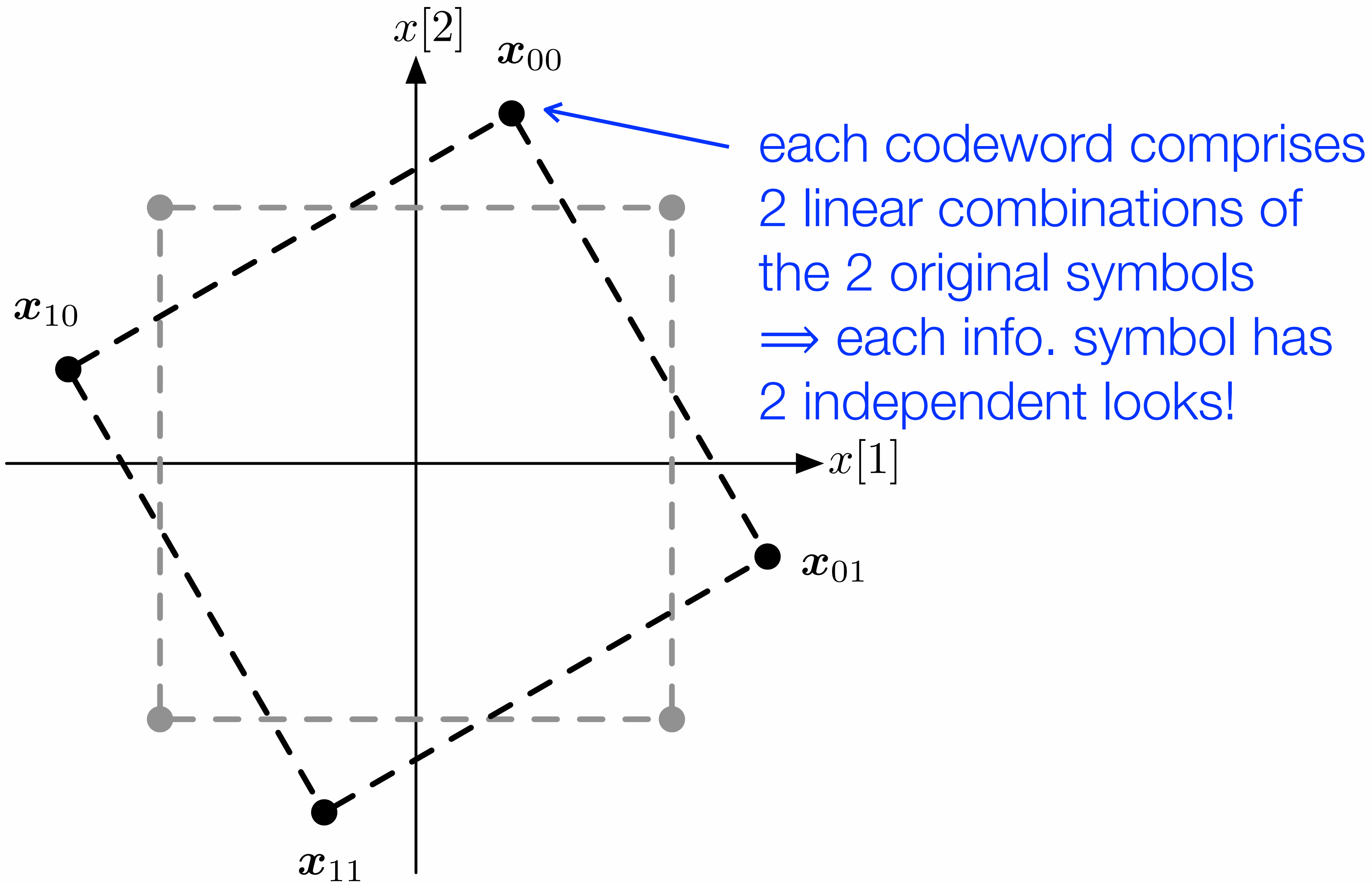
- Consider sending 2 independent BPSK symbols ($u[1]$, $u[2]$) over two (interleaved) time slots ($L = 2$)
  - Diversity order = 1 because each BPSK symbol has only one "look"



Rotation Code for $L = 2$

- How about rotating the equivalent constellation set?

$$\boldsymbol{x} = \mathbf{r}_\theta \boldsymbol{u}, \quad \mathbf{r}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Performance Analysis of Rotation Code

- Equivalent vector (2-dim) channel:

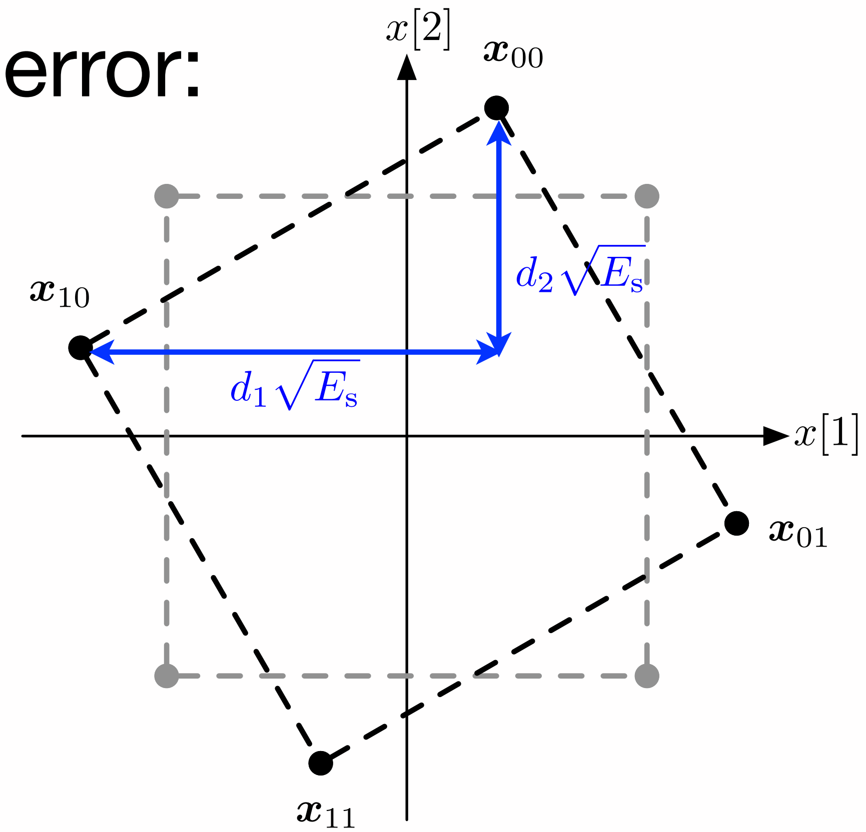
$$\boldsymbol{V} = \begin{bmatrix} H[1] & 0 \\ 0 & H[2] \end{bmatrix} \boldsymbol{x} + \boldsymbol{Z} = \tilde{\boldsymbol{x}} + \boldsymbol{Z}$$

- Union bound via pairwise probability of error:

$$\begin{aligned} \mathsf{P_e}(\phi_{\mathrm{ML}}; \boldsymbol{H} = \boldsymbol{h}) \le\ & \mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{01} | \boldsymbol{H} = \boldsymbol{h}\} \\ & + \mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{11} | \boldsymbol{H} = \boldsymbol{h}\} \\ & + \mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{10} | \boldsymbol{H} = \boldsymbol{h}\} \end{aligned}$$

$$\mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{10} | \boldsymbol{H} = \boldsymbol{h}\} = \mathrm{Q}\left(\frac{\|\tilde{\boldsymbol{x}}_{00} - \tilde{\boldsymbol{x}}_{10}\|}{\sqrt{2N_0}}\right)$$

$$= \mathrm{Q}\left(\sqrt{\frac{|h[1]|^2|d_1|^2 + |h[2]|^2|d_2|^2}{2}\mathsf{SNR}}\right)$$

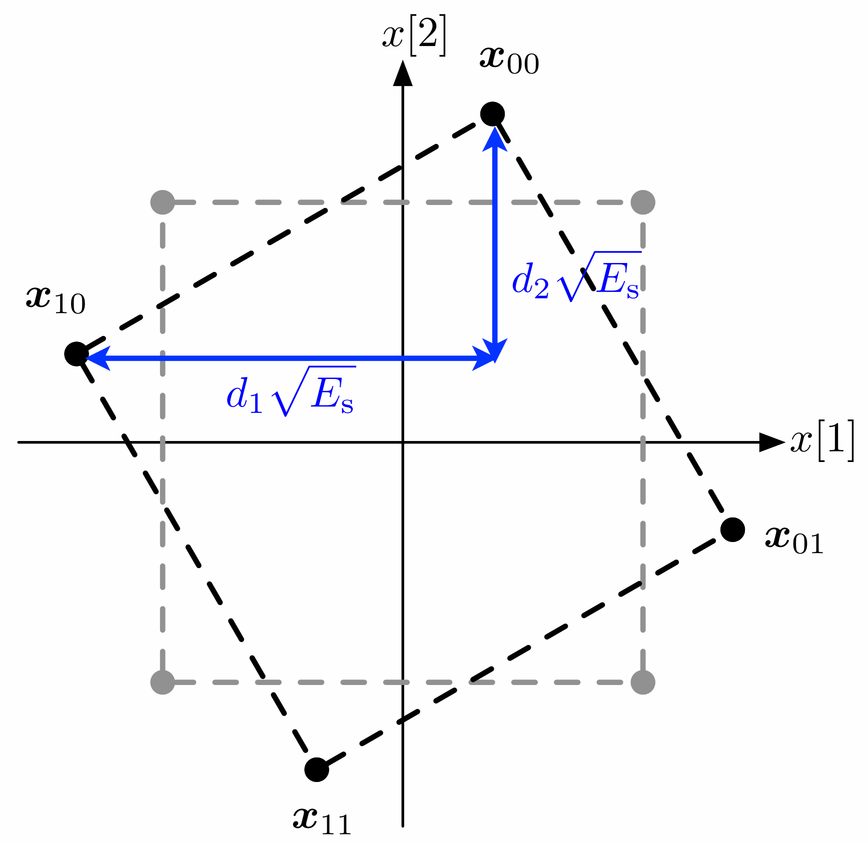


$$\|\tilde{\boldsymbol{x}}_{00} - \tilde{\boldsymbol{x}}_{10}\|^2 = E_\mathrm{s}(|h[1]|^2|d_1|^2 + |h[2]|^2|d_2|^2)$$

$$d_1 = 2\cos\theta,\ d_2 = 2\sin\theta$$

$$\mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{10} | \boldsymbol{H} = \boldsymbol{h}\} = \mathrm{Q}\left(\frac{\|\tilde{\boldsymbol{x}}_{00} - \tilde{\boldsymbol{x}}_{10}\|}{\sqrt{2N_0}}\right)$$

$$= \mathrm{Q}\left(\sqrt{\frac{|h[1]|^2|d_1|^2 + |h[2]|^2|d_2|^2}{2}\mathsf{SNR}}\right)$$



$$\mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{10}\}$$

$$\leq \mathsf{E}_{H[1],H[2]}\left[\frac{1}{2}e^{-\frac{1}{4}(|H[1]|^2|d_1|^2+|H[2]|^2|d_2|^2)\mathsf{SNR}}\right]$$

$$= \frac{1}{2}\frac{1}{1+\frac{|d_1|^2}{4}\mathsf{SNR}}\frac{1}{1+\frac{|d_2|^2}{4}\mathsf{SNR}}$$

$$\approx \frac{8}{|d_1 d_2|^2}\mathsf{SNR}^{-2} = \frac{8}{\delta_{00\to 10}}\mathsf{SNR}^{-2}$$

$$\|\tilde{\boldsymbol{x}}_{00} - \tilde{\boldsymbol{x}}_{10}\|^2$$
$$= E_s(|h[1]|^2|d_1|^2 + |h[2]|^2|d_2|^2)$$

$$d_1 = 2\cos\theta,\ d_2 = 2\sin\theta$$

squared product distance:

$$\delta_{00\to 10} \triangleq |d_1 d_2|^2 = 4\sin^2(2\theta)$$

Rotation Code Achieves Full Diversity

- Total probability of error is upper bounded by

$$\mathsf{P_e}(\phi_{\mathrm{ML}}) \le \mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{01}\} + \mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{11}\} + \mathsf{P}\{\boldsymbol{x}_{00} \to \boldsymbol{x}_{10}\}$$

$$\lesssim 8\left(\frac{1}{\delta_{00\to10}} + \frac{1}{\delta_{00\to11}} + \frac{1}{\delta_{00\to01}}\right)\mathsf{SNR}^{-2}$$

$$\le \frac{24}{\delta_{\min}}\mathsf{SNR}^{-2}$$

- Diversity order = 2
- Coding gain: maximize the minimum squared product distance
  - Compute $\delta_{00\to10} = \delta_{00\to01} = 4\sin^2(2\theta),\ \delta_{00\to11} = 16\cos^2(2\theta)$
  - The best rotation angle that maximize min. squared product distance:

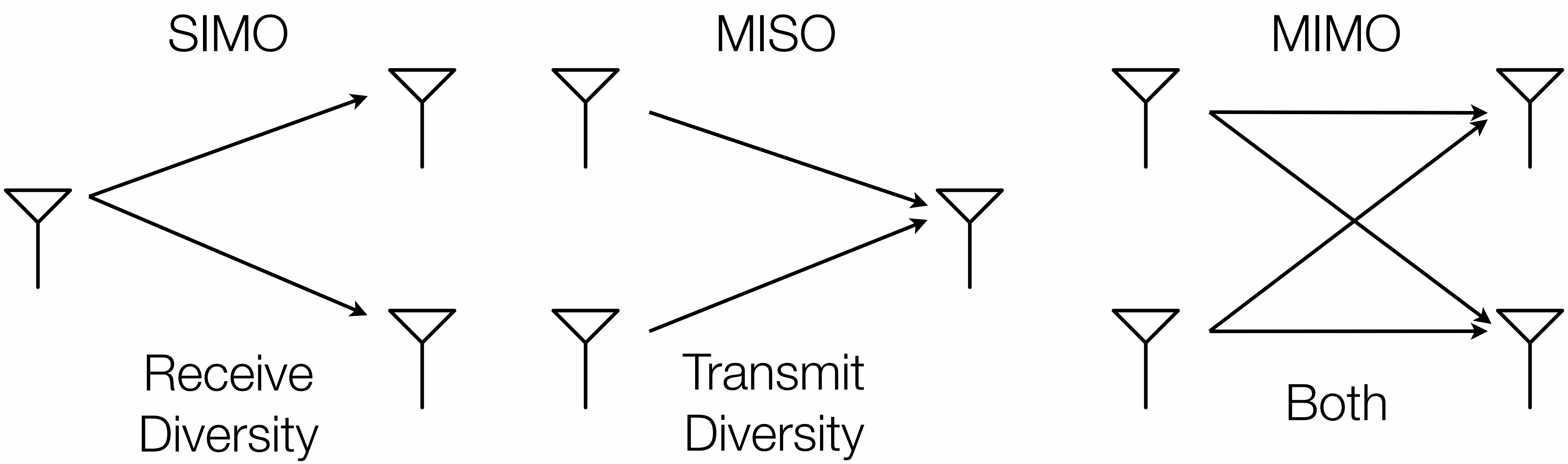
$$4\sin^2(2\theta^*) = 16\cos^2(2\theta^*) \implies \theta^* = \tfrac{1}{2}\tan^{-1}(2)$$

General Time Diversity Code

- The above idea can be generalized to arbitrary L
- Diversity order and coding gain can be analyzed with union bound
- Time diversity code can be used at the bit-level (merged into ECC) or used at the symbol level.

Antenna Diversity

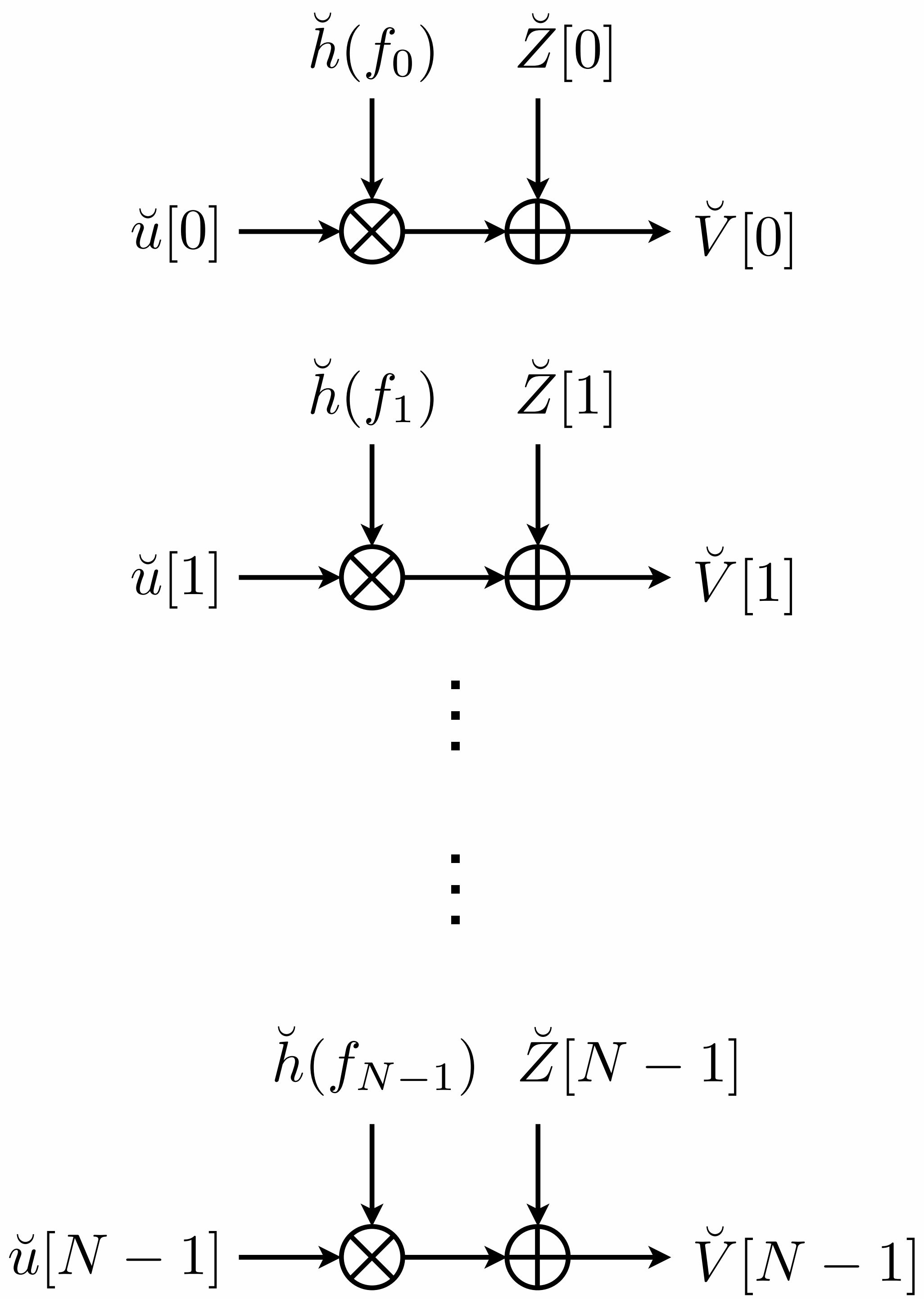
- Typical antenna separation for antenna diversity $\sim \lambda_c = c/f_c$



- Full diversity order: $d = N_{\mathrm{rx}} N_{\mathrm{tx}}$

- <u>**Space-time code**</u> for exploiting diversity and multiplexing capabilities of MIMO systems

Frequency Diversity



- Frequency selectivity can be used to provide diversity
- L-taps channel: each Tx symbol appears in L Rx symbols
- Full diversity order: L
- OFDM extracts full diversity:
  - N parallel channels (subcarriers)
  - coding + interleaving over subcarriers
  - total bandwidth: $2W$
    coherence bandwidth: W_{c}
  - diversity order: $2W/W_{\mathrm{c}} = 2WT_{\mathrm{d}} = L$

Summary

- Fading makes wireless channels unreliable
- Diversity increases reliability and makes the channel more consistent
- Key to increasing diversity: create more independent “looks” of the channel
- Smart codes yields a coding gain in addition to the diversity gain