Lecture 04
Wideband Communication

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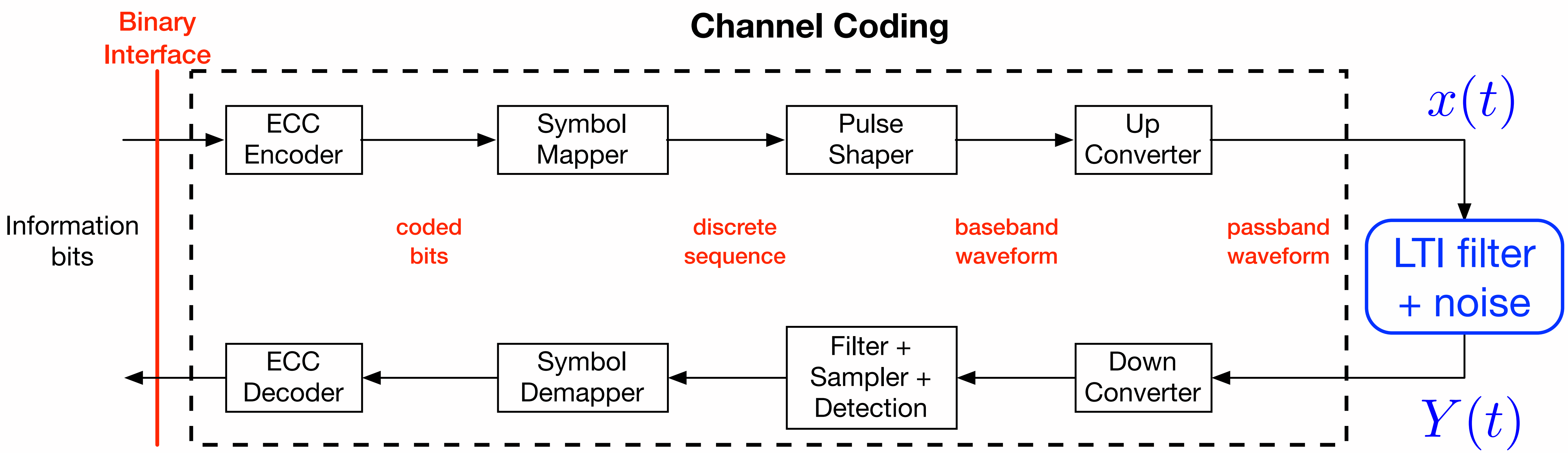
This Lecture

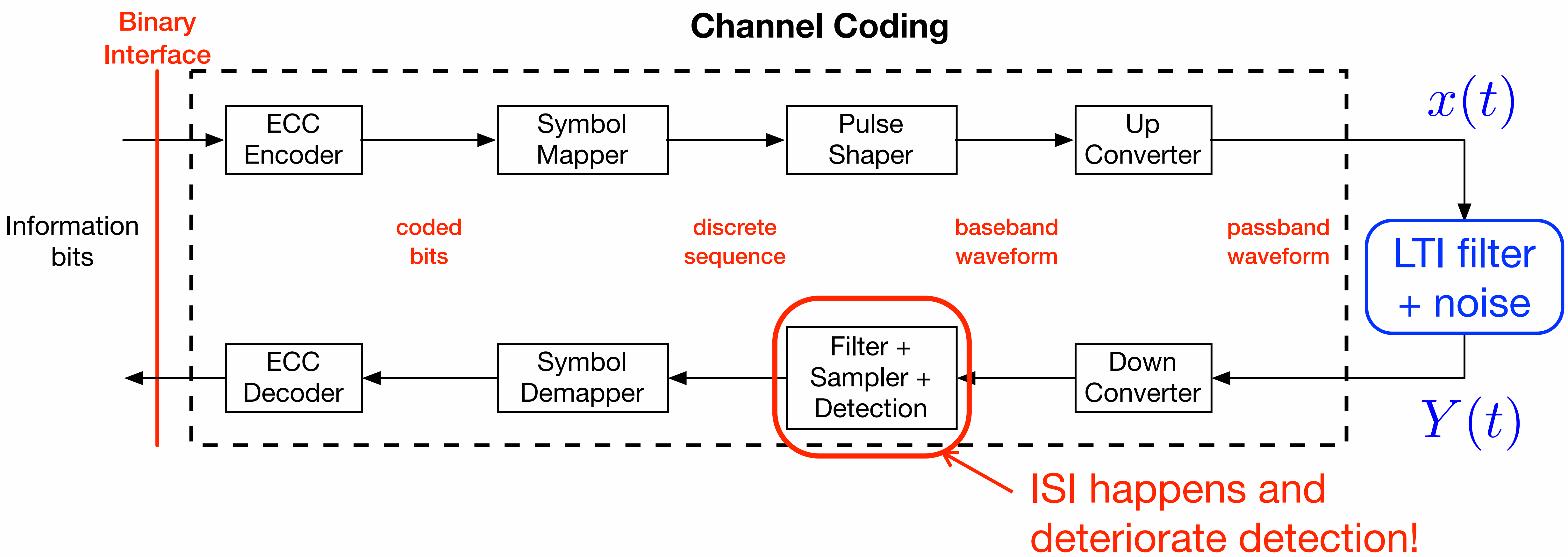


- Physical channel model for wideband communication

$$Y(t) = (h * x)(t) + Z(t),\ \mathsf{S}_Z(f) = \frac{N_0}{2}$$

  - Intuition: when the band is wide, signals in difference band will experience different frequency response of the channel
  - Use an LTI filter to model the channel

This Lecture



- New challenge: **inter-symbol interference** (ISI)
  - Detect each symbol individually is no longer optimal
- Our focus: mitigate ISI in the digital world (after sampling)
  - HW1 tells us that dealing with ISI in the analog world is a pretty bad idea
  - Receiver-side solution, transmitter-side solution, and Tx-Rx solution

Outline

- LTI filter channel and inter-symbol interference (ISI)
- Optimal Rx-side solution: MLSD
- Rx-side solution: linear equalizations
- Tx-Rx-side solution: OFDM

Part I. LTI Filter Channel and Inter-Symbol Interference

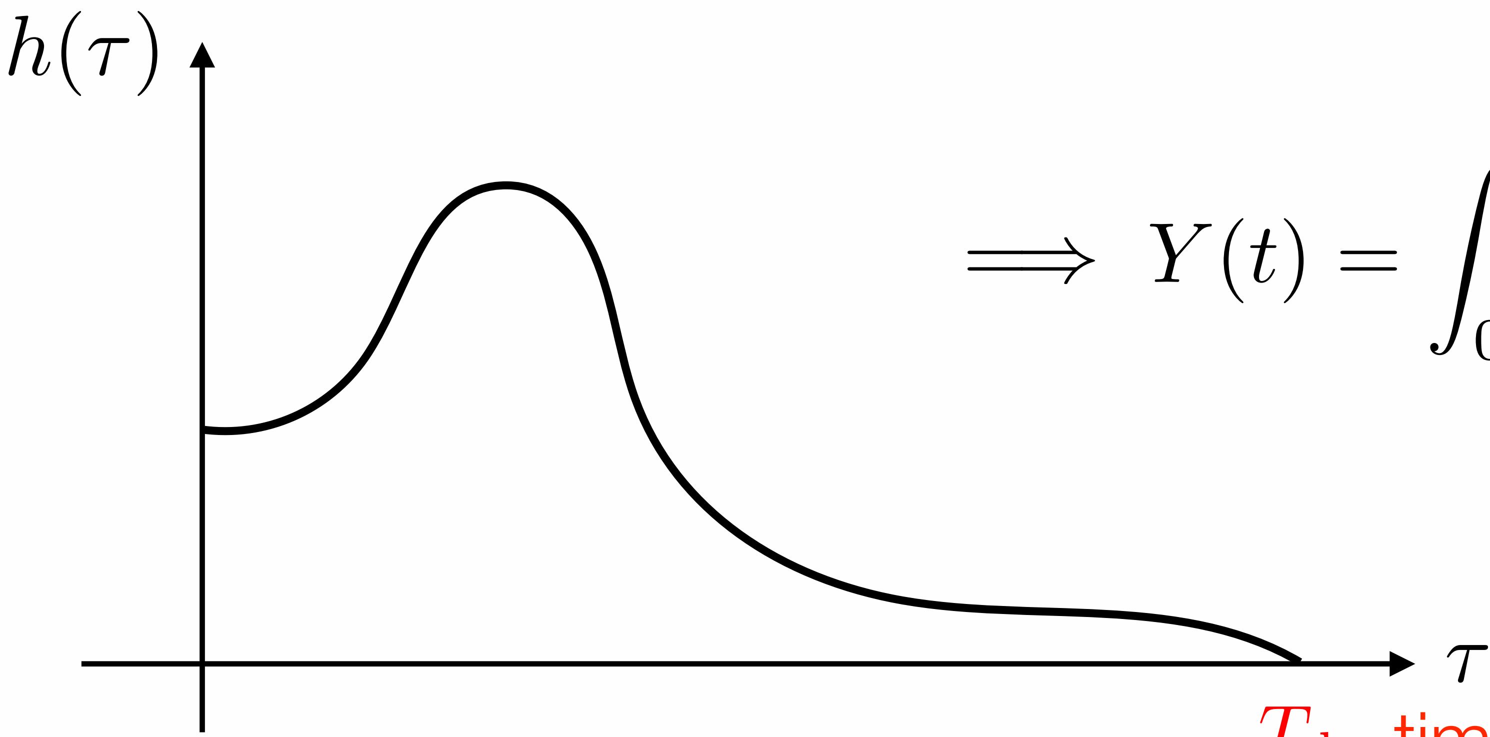
Equivalent Discrete-Time Baseband Channel;
Inter-Symbol Interference; MLSD

Physical Channel Model

$$Y(t) = (h * x)(t) + Z(t),\ \mathsf{S}_Z(f) = \frac{N_0}{2}$$
$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau + Z(t)$$

- Use LTI filter to model wireline channels
  - Examples: telephone lines, Ethernet cables, cable TV wires, optical fibers
  - Operating bandwidth range from 1~2MHz to 250~500 MHz.
- Why use LTI filter to model wireline channels?
  - Frequency responses are no longer flat
  - Channel is rather stationary compared to wireless channels
  - Within the interest of time, can be assumed to be time-invariant

Features of the LTI Filter Channel



$$\Longrightarrow \; Y(t) = \int_0^{T_\mathrm{d}} h(\tau)x(t-\tau)\,\mathrm{d}\tau + Z(t)$$

T_d time dispersion (delay spread)

- **Causal**: naturally, impulse response should be causal.

$$h(\tau) = 0, \quad \forall\, \tau < 0$$

- **Dispersive**: naturally, input signal cannot “stay” in the channel for too long, and hence most energy of the impulse response of the channel should be contained in an interval $[0, T_\mathrm{d}]$

$$h(\tau) = 0, \quad \forall\, \tau > T_\mathrm{d}$$

Derivation of the Discrete-Time Model

- Step 1: real passband complex baseband (ignore noise)

Pulse shaping: $x_b(t) \triangleq \sum_m u_m \, p(t - mT)$

Up conversion: $x(t) \triangleq \mathrm{Re}\left\{x_b(t)\sqrt{2}\exp(\mathrm{j}2\pi f_c t)\right\}$

check!

LTI channel: $y(t) = (h * x)(t) = \mathrm{Re}\left\{(h_b * x_b)(t)\sqrt{2}\exp(\mathrm{j}2\pi f_c t)\right\}$

$h_b(\tau) \triangleq h(\tau)\exp(-\mathrm{j}2\pi f_c \tau)$

Down conversion: $y_b(t) = (h_b * x_b)(t)$

- Step 2: continuous-time discrete-time

$x_b(t) \triangleq \sum_k u_k\, p(t-kT)$

Demodulation: $\hat{u}_m = (y_b * q)(mT) = (x_b * h_b * q)(mT)$

$g(t) \triangleq (p * q)(t)$

$$= \sum_k u_k \int_0^{T_\mathrm{d}} h_b(\tau) g(mT - kT - \tau)\, \mathrm{d}\tau$$

$$= \sum_k u_k h_{m-k} = \boxed{(u * h_d)_m}$$

$$h_d[\ell] \triangleq (h_b * g)(\ell T) = (p * h_b * q)(\ell T)$$

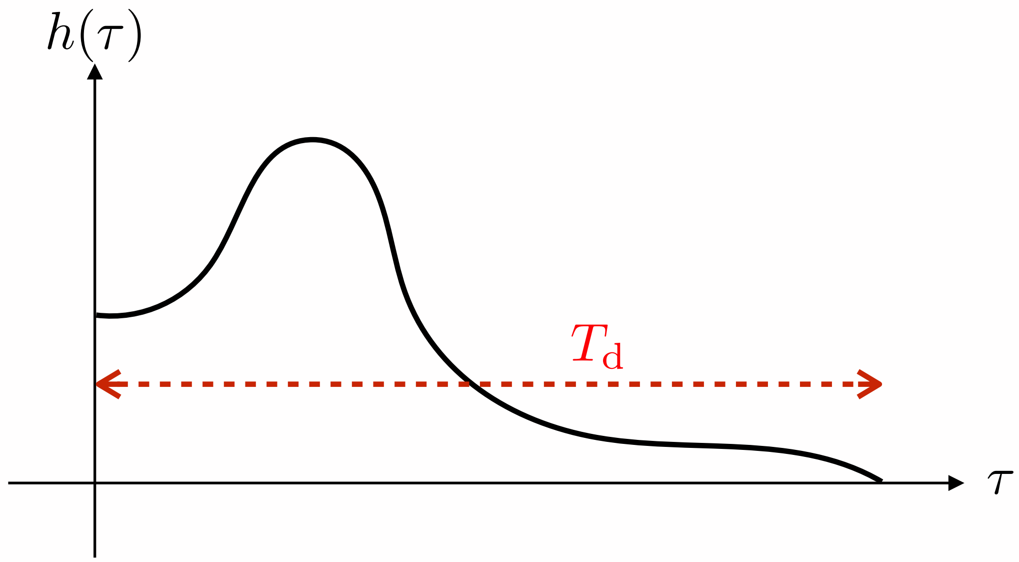
- Step 3: adding noise back

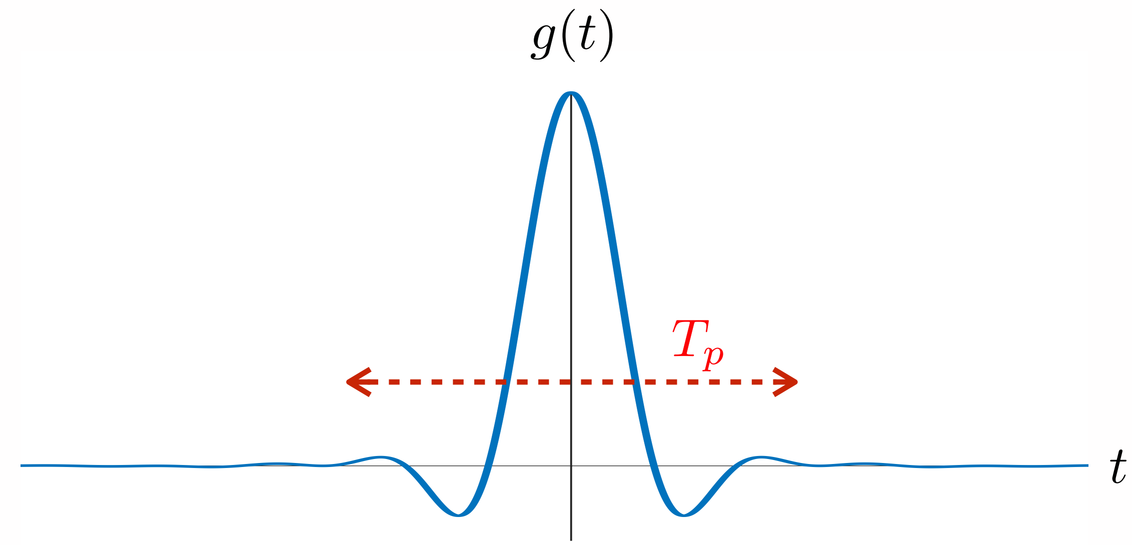
$$\boxed{V_m = \sum_\ell h_d[\ell] u_{m-\ell} + Z_m, \quad Z_m \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_0)}$$

Number of Taps

$$V_m = \sum_\ell h_d[\ell] u_{m-\ell} + Z_m, \quad Z_m \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_0)$$

- What is the range of ℓ in the summation of the discrete-time convolution in the equivalent discrete-time model?
  - Recall: $h_d[\ell] \triangleq (h_b * g)(\ell T) \quad h_b(\tau) \triangleq h(\tau)\exp(-\mathrm{j}2\pi f_c \tau)$





  - The overall "spread" of the digital filter is hence $\dfrac{T_p + T_\mathrm{d}}{T}$
- The equivalent discrete-time filter has **finite impulse response**, that is, the number of taps $L \approx \frac{T_p + T_\mathrm{d}}{T}$ is finite:

$$V_m = \sum_{\ell=0}^{L-1} h_d[\ell] u_{m-\ell} + Z_m, \quad Z_m \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_0)$$

Discrete-Time Complex Baseband Model

- With a little abuse of notation, identifying $h_d[\ell] \equiv h_\ell$, the equivalent discrete-time baseband channel model is given as

$$V_m = \sum_{\ell=0}^{L-1} h_\ell \, u_{m-\ell} + Z_m,$$

$$Z_m \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_0)$$

$$\{u_m\} \to \boxed{\text{FIR } L\text{-tap LTI filter}} \longrightarrow \oplus \longrightarrow \{V_m\} \quad (\{Z_m\} \downarrow \oplus)$$

$$\boldsymbol{h} \triangleq [h_0\, h_1 \ldots h_{L-1}] \in \mathbb{C}^L$$

- The filter tap coefficients $\{h_\ell\}$ depends on
  - one-sided bandwidth W (or symbol time $T = \frac{1}{2W}$)
  - carrier frequency f_c
  - modulation pulse $g(t)$
  - channel impulse response $h(\tau)$
- In practice, these taps are measured via training: sending known pilot symbols to estimate the tap coefficients.
- Total # of taps is proportional to bandwidth: $L \approx \frac{T_p + T_\mathrm{d}}{T} \propto W$

Inter-Symbol Interference

- Narrowband channel (no ISI)

$$V_m = h_0 \, u_m + Z_m, \quad Z_m \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_0)$$

- Wideband channel (with ISI)

$$V_m = \sum_{\ell=0}^{L-1} h_\ell \, u_{m-\ell} + Z_m, \quad Z_m \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_0)$$

$$= h_0 \, u_m + \boxed{(h_1 u_{m-1} + \ldots + h_{L-1} u_{m-L+1})} + Z_m$$

I_m inter-symbol interference

- With ISI, it is no longer optimal to detect each symbol u_m from the single observed V_m only.
- ISI introduces memory, and hence one needs to detect the entire sequence jointly **Maximum Likelihood Sequence Detection**

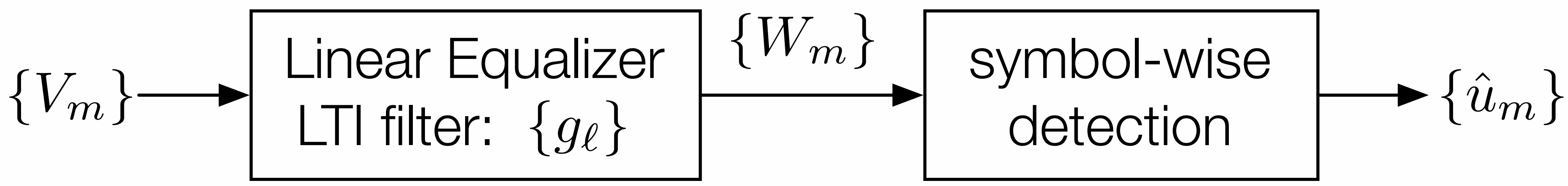
Optimal Receiver: MLSD

- ISI channel is the same as the convolutional encoder, except that the arithmetic is in the complex field, not the finite (binary) field
- Hence, each possible sequence in MLSD can be represented by a path on a trellis
- Procedure of MLSD: received a length-n sequence $(V_1, \ldots, V_n)$
  - Define the state as the past interfering symbols:
  $$s_m \triangleq (u_{m-1}, \ldots, u_{m-L+1})$$
  - Each transition u_m has $|\mathcal{A}|$ possible outgoing arrows. $\mathcal{A}$: constellation set
  - Each transition outputs a symbol:
  $$\hat{u}_m = h_0 u_m + \sum_{\ell=1}^{L-1} h_\ell \, u_{m-\ell}$$
  - Goal of MLSD: find a path on the trellis such that $\sum_{m=1}^{n} |V_m - \hat{u}_m|^2$ is minimized $\Longrightarrow$ Viterbi algorithm!
- However, the complexity of Viterbi algorithm is $\Theta(n\,|\mathcal{A}|^L)$, while the # of taps is quite large (100~200) in wideband systems (DSL).
- MLSD is optimal but infeasible in practice for wideband systems.

Part II. Linear Equalizations

Matched-Filter, Zero-Forcing,
MMSE Equalization

Mitigate ISI with Linear Filters



- ISI is caused by a (discrete-time) LTI filter due to the frequency selectivity of the channel
- Why not use another discrete-time LTI filter at the receiver to mitigate ISI, and do symbol-wise detection at the filtered output?
- Design of the filter $\{g_\ell\}$ requires some **objectives** for optimization:
  - Probability of error? hard to analyze
  - Energy will be easier to handle
- Since the ISI is treated as noise in the symbol-wise detection, we should try to maximize the signal-to-interference-and-noise ratio (SINR) at the filtered output $\{W_m\}$

Linear Equalizers to be Introduced

- Use Z transform to represent the discrete-time LTI filter

$$g_\ell \longleftrightarrow \check{g}(\zeta) \triangleq \sum_\ell g_\ell \zeta^{-\ell}$$

  - Recall its relation with DTFT: $\breve{g}(f) = \check{g}(e^{j2\pi f})$

- Three kinds of linear equalizers:
  - Matched filter (MF): $\check{g}^{(\mathrm{MF})}(\zeta) = \check{h}^*(1/\zeta^*)$.
  - Zero forcing (ZF): $\check{g}^{(\mathrm{ZF})}(\zeta) = (\check{h}(\zeta))^{-1}$.
  - Minimum mean squared error (MMSE): maximize SINR

$$\check{g}^{(\mathrm{MMSE})}(\zeta) = \frac{E_s \check{h}^*(1/\zeta^*)}{N_0 + E_s \check{h}^*(1/\zeta^*)\check{h}(\zeta)}$$

- Low SNR regime ($E_s \ll N_0$): $\check{g}^{(\mathrm{MMSE})}(\zeta) \approx \frac{E_s}{N_0}\check{g}^{(\mathrm{MF})}(\zeta)$
- High SNR regime ($E_s \gg N_0$): $\check{g}^{(\mathrm{MMSE})}(\zeta) \approx \check{g}^{(\mathrm{ZF})}(\zeta)$

Matrix Representation of ISI Channel

$$
\begin{array}{lclclclclclcl}
V_1 & = & h_0 u_1 & & & & & & & + & Z_1 \\
V_2 & = & h_0 u_2 & + & h_1 u_1 & & & & & + & Z_2 \\
 & \vdots & & & & & & & & & \\
V_L & = & h_0 u_L & + & h_1 u_{L-1} & + & \cdots & + & h_{L-1} u_1 & + & Z_L \\
V_{L+1} & = & h_0 u_{L+1} & + & h_1 u_L & + & \cdots & + & h_{L-1} u_2 & + & Z_{L+1} \\
 & \vdots & & & & & & & & & \\
V_n & = & h_0 u_n & + & h_1 u_{n-1} & + & \cdots & + & h_{L-1} u_{n-L+1} & + & Z_n \\
V_{n+1} & = & & & h_1 u_n & + & \cdots & + & h_{L-1} u_{n-L+2} & + & Z_{n+1} \\
 & \vdots & & & & & & & & & \\
V_{n+L-1} & = & & & & & & & h_{L-1} u_n & + & Z_{n+L-1}
\end{array}
$$

Matrix Representation of ISI Channel

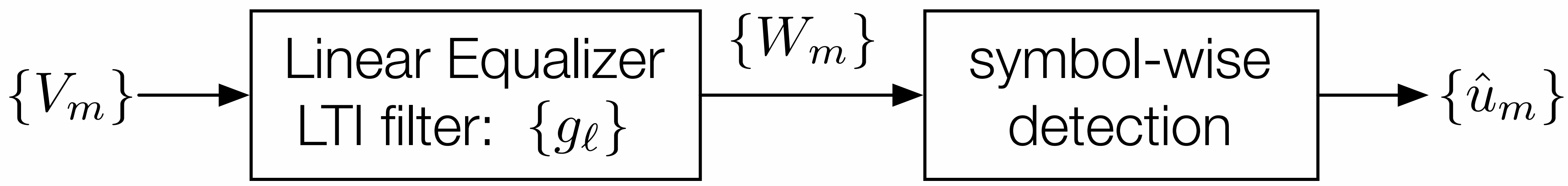
$$\boldsymbol{V} = \mathbf{h}\boldsymbol{u} + \boldsymbol{Z} = u_m[\mathbf{h}]_m + \sum_{i \neq m} u_i[\mathbf{h}]_i + \boldsymbol{Z}$$

$$\mathbf{h} \triangleq \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & & \vdots \\ \vdots & h_1 & & \\ h_{L-1} & \vdots & & 0 \\ 0 & h_{L-1} & & h_0 \\ \vdots & 0 & & h_1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & h_{L-1} \end{bmatrix}$$

$m \sim (m+L-1)$-th elements are $h_0, h_1, ...h_{L-1}$

$[\mathbf{h}]_m$

Matrix Representation of Equalizer



$$W_m = \langle \boldsymbol{V}, [\mathbf{g}]_m \rangle = [\mathbf{g}]_m^{\mathtt{H}} \boldsymbol{V}$$

$$\tilde{Z}_m \triangleq [\mathbf{g}]_m^{\mathtt{H}} \boldsymbol{Z}$$

$$= ([\mathbf{g}]_m^{\mathtt{H}} [\mathbf{h}]_m) u_m + \sum_{i \neq m} ([\mathbf{g}]_m^{\mathtt{H}} [\mathbf{h}]_i) u_i + \tilde{Z}_m$$

signal ISI noise

Goal: maximize $\text{SINR} = \dfrac{|\langle [\mathbf{h}]_m, [\mathbf{g}]_m \rangle|^2 E_s}{\sum_{i \neq m} |\langle [\mathbf{h}]_i, [\mathbf{g}]_m \rangle|^2 E_s + \|[\mathbf{g}]_m\|^2 N_0}$

Low SNR Regime

$$E_s \ll N_0 \implies \text{neglect ISI.}$$

$$W_m = ([\mathbf{g}]_m^{\mathtt{H}} [\mathbf{h}]_m) u_m + \cancel{\sum_{i \neq m} ([\mathbf{g}]_m^{\mathtt{H}} [\mathbf{h}]_i) u_i} + \tilde{Z}_m$$

$$\mathsf{SINR} = \frac{|\langle [\mathbf{h}]_m, [\mathbf{g}]_m \rangle|^2 E_s}{\cancel{\sum_{i \neq m} |\langle [\mathbf{h}]_i, [\mathbf{g}]_m \rangle|^2 E_s} + \|[\mathbf{g}]_m\|^2 N_0}$$

$$= \left(\frac{|\langle [\mathbf{h}]_m, [\mathbf{g}]_m \rangle|}{\|[\mathbf{g}]_m\|} \right)^2 \frac{E_s}{N_0}$$

$$\implies [\mathbf{g}^{(\mathrm{MF})}]_m = [\mathbf{h}]_m$$

Matched Filter

$$W_m = h_0^* V_m + h_1^* V_{m+1} + \ldots + h_{L-1}^* V_{m+L-1}$$

$$= \sum_{\ell=0}^{L-1} h_\ell^* V_{m+\ell} = \sum_{\ell=-(L-1)}^{0} h_{-\ell}^* V_{m-\ell} = \sum_{\ell=-(L-1)}^{0} g_\ell^{(\mathrm{MF})} V_{m-\ell},$$

$$\Longrightarrow \; g_\ell^{(\mathrm{MF})} = h_{-\ell}^* \qquad\qquad \breve{g}^{(\mathrm{MF})}(\zeta) = \breve{h}^*(1/\zeta^*)$$

$$\breve{g}^{(\mathrm{MF})}(f) = \breve{h}^*(f)$$

project the signal onto the signal direction,
so that the signal energy is maximized.

High SNR Regime

$$E_s \gg N_0 \implies \text{neglect noise.}$$

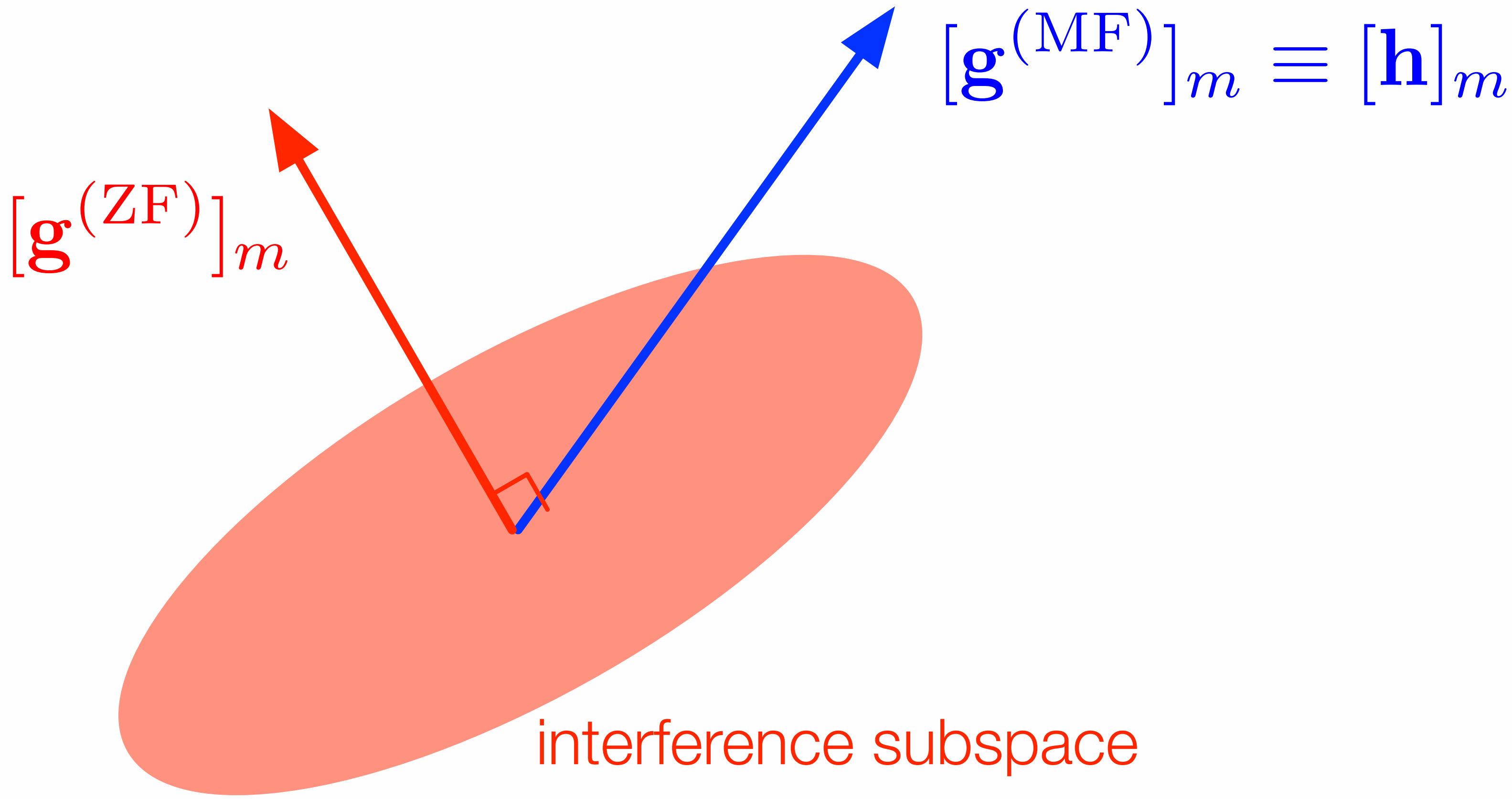
$$W_m = ([\mathbf{g}]_m^{\mathtt{H}}[\mathbf{h}]_m)u_m + \sum_{i \neq m}([\mathbf{g}]_m^{\mathtt{H}}[\mathbf{h}]_i)u_i + \sout{\tilde{Z}_m}$$

$$\text{SINR} = \frac{|\langle[\mathbf{h}]_m, [\mathbf{g}]_m\rangle|^2 E_s}{\sum_{i \neq m}|\langle[\mathbf{h}]_i, [\mathbf{g}]_m\rangle|^2 E_s + \sout{\|[\mathbf{g}]_m\|^2 N_0}}$$

$$\implies [\mathbf{g}^{(\mathrm{ZF})}]_m \perp [\mathbf{h}]_i, \ \forall i \neq m$$

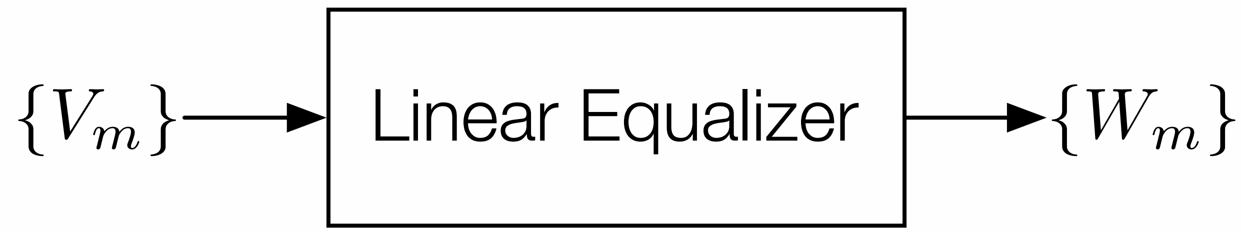
one choice: $[\mathbf{g}^{(\mathrm{ZF})}]_m = (\mathbf{h}^{\dagger})^{\mathtt{H}}\boldsymbol{e}_m = \mathbf{h}(\mathbf{h}^{\mathtt{H}}\mathbf{h})^{-1}\boldsymbol{e}_m$

Geometric Interpretation



subspace spanned by the $n-1$ columns of $\mathbf{h}$ (except $[\mathbf{h}]_m$)

Max. SINR $\equiv$ Min. MSE



$$W_m = \sum_k g_k V_{m-k} = \sum_k \sum_{\ell=0}^{L-1} g_k h_\ell u_{m-k-\ell} + \sum_k g_k Z_{m-k}$$

$$= \left(\sum_{\ell=0}^{L-1} g_{-\ell} h_\ell \right) u_m + \tilde{I}_m + \tilde{Z}_m$$

the same for all m ⇒ WLOG assume it is 1

$$= u_m + \tilde{I}_m + \tilde{Z}_m$$

Ξ_m : kind of estimation error

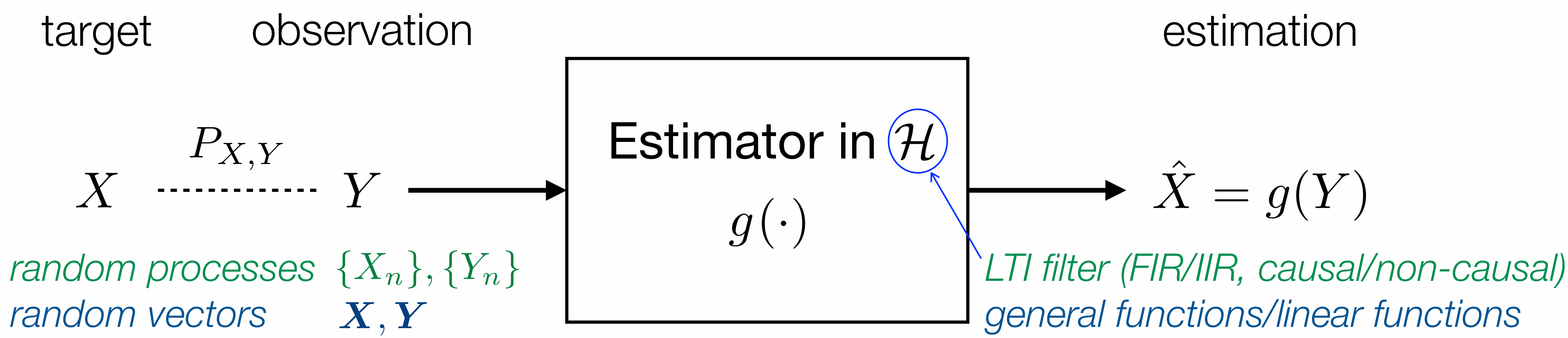
$$\text{SINR} = \frac{\mathsf{E}\left[|U_m|^2\right]}{\mathsf{E}\left[|\Xi_m|^2\right]} = \frac{E_s}{\mathsf{E}\left[|\Xi_m|^2\right]}$$

$$\max \text{SINR} \equiv \min \mathsf{E}\left[|\Xi_m|^2\right]$$

mean squared error (MSE)

Minimum MSE Estimation

- In general, one can consider the following estimation problem:
  - Given a random observation, estimate a target s.t. the MSE is minimized



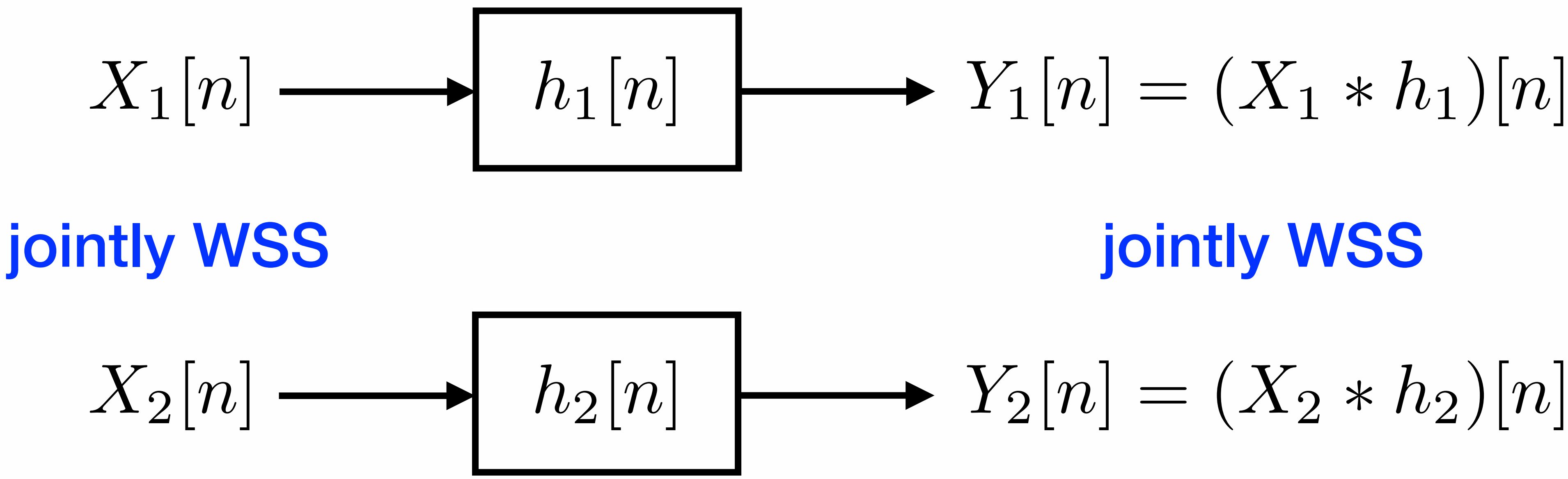
$$g^{(\mathrm{MMSE})}(\cdot) = \underset{g \in \mathcal{H}}{\mathrm{argmin}}\, \mathsf{MSE}(X, g(Y)) \quad \mathsf{MSE}(X, \hat{X}) \triangleq \mathsf{E}\left[\left\| X - \hat{X} \right\|^2\right]$$

  - You might be familiar with the general case: $g^{(\mathrm{MMSE})}(\boldsymbol{Y}) = \mathsf{E}[\boldsymbol{X} | \boldsymbol{Y}]$
- Here, we focus on the random process case and linear estimators without any causality and finite-tap constraints.
  - After deriving the optimal filter for MMSE estimation, we apply it back to the original problem

Recap: Discrete-Time Random Process

	General	(joint) WSS
First moment	$\mu_X[n] \triangleq \mathsf{E}[X_n]$	$\mu_X[n] \equiv \mu_X$
Second moment	$\mathsf{R}_X[n_1, n_2] \triangleq \mathsf{E}\left[X_{n_1} X_{n_2}^*\right]$ (auto-correlation)	$\mathsf{R}_X[n+k, n] \equiv \mathsf{R}_X[k]$ $\mathsf{R}_X[-k] = \mathsf{R}_X^*[k]$
	$\mathsf{R}_{XY}[n_1, n_2] \triangleq \mathsf{E}\left[X_{n_1} Y_{n_2}^*\right]$ (cross-correlation)	$\mathsf{R}_{XY}[n+k, n] \equiv \mathsf{R}_{XY}[k]$ $\mathsf{R}_{YX}[k] = \mathsf{R}_{XY}^*[-k]$
PSD		$\mathsf{R}_X[k] \longleftrightarrow \mathsf{S}_X(\zeta)$ $\mathsf{R}_{XY}[k] \longleftrightarrow \mathsf{S}_{XY}(\zeta)$ $\mathsf{S}_{YX}(\zeta) = \mathsf{S}_{XY}^*(1/\zeta^*)$

Recap: Filtering Random Processes



Cross-correlation: $\mathsf{R}_{Y_1,Y_2}[k] = \left(h_1 * \mathsf{R}_{X_1,X_2} * h_{2,\mathrm{rv}}\right)[k]$

Cross PSD: $\mathsf{S}_{Y_1,Y_2}(\zeta) = \check{h}_1(\zeta)\mathsf{S}_{X_1,X_2}(\zeta)\check{h}_2^*(1/\zeta^*)$

Derivation of the Optimal Filter

jointly WSS

$\{X_n\}$ ········· $\{Y_n\}$ → **Estimation via Linear Filter** $\{g_k\} \longleftrightarrow \check{g}(\zeta)$ → $\{\hat{X}_n\} = \{(g * Y)_n\}$

$$\mathsf{MSE} \triangleq \mathsf{E}\left[\left|\underbrace{X_n - \hat{X}_n}_{\Xi_n}\right|^2\right]$$

Ξ_n also WSS!

Goal: $\{g_k^{(\mathrm{MMSE})}\} = \underset{\{g_k\}}{\operatorname{argmin}} \, \mathsf{MSE}$

Note: $\mathsf{MSE} = \mathsf{E}\left[(X_n - \hat{X}_n)(X_n - \hat{X}_n)^*\right] = \mathsf{E}\left[\Xi_n \left(X_n - \sum_k g_k Y_{n-k}\right)^*\right]$

$$\forall\, k,\ 0 = \frac{\partial}{\partial g_k^*}\mathsf{MSE} = -\mathsf{E}\left[\Xi_n Y_{n-k}^*\right] = \mathsf{E}\left[(g * Y)_n Y_{n-k}^*\right] - \mathsf{E}\left[X_n Y_{n-k}^*\right]$$

$$\iff \forall\, k,\ (g * \mathsf{R}_Y)[k] = \mathsf{R}_{XY}[k] \iff \check{g}(\zeta)\mathsf{S}_Y(\zeta) = \mathsf{S}_{XY}(\zeta)$$

Solution: $\check{g}^{(\mathrm{MMSE})}(\zeta) = (\mathsf{S}_Y(\zeta))^{-1}\mathsf{S}_{XY}(\zeta)$ (non-causal IIR Wiener filter)

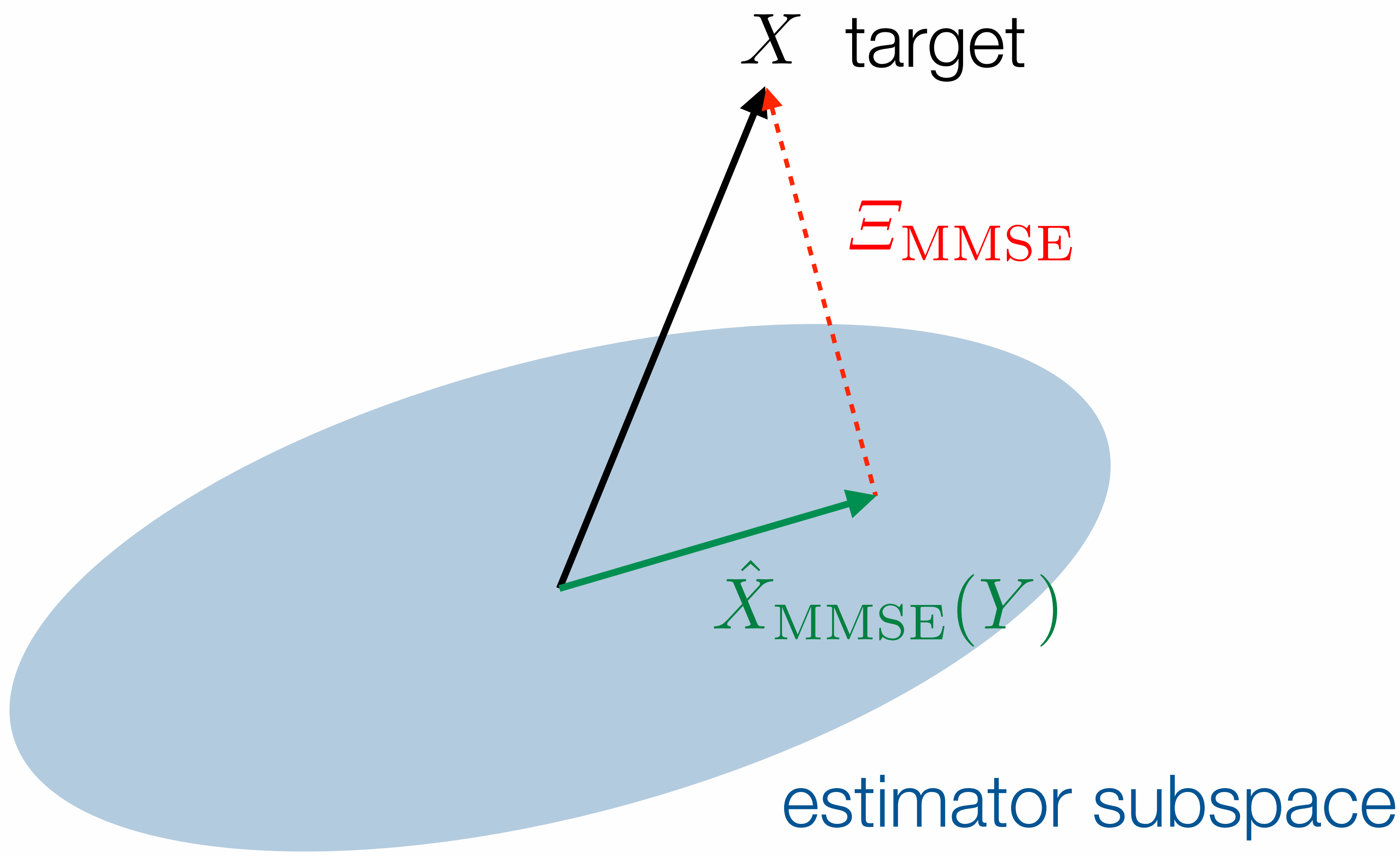
Orthogonality Principle

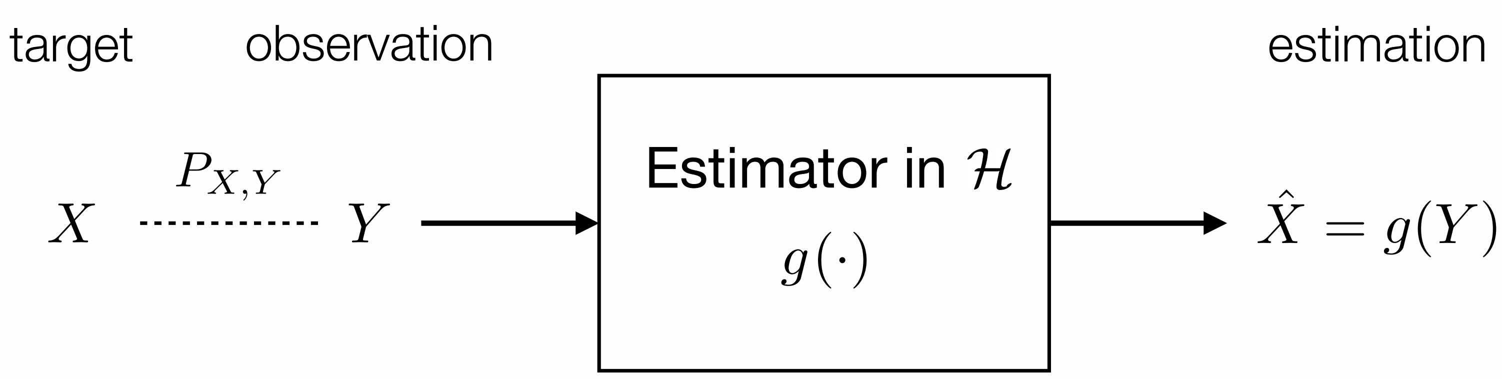
- A key equation in deriving the optimal estimator is

$$\mathsf{E}\left[\Xi_n Y_{n-k}^*\right] = 0,\ \forall k \iff \langle \Xi_n, (f * Y)_n \rangle = 0,\ \forall \, \text{LTI filter } \{f_\ell\}$$

  - For two r.v.'s (X, Y), we define the "inner product" as $\langle X, Y \rangle \triangleq \mathsf{E}[XY^*]$
  - (you can check the axioms of inner product space …)

- A geometric interpretation: for an estimator that minimizes MSE, its estimation error should be "orthogonal" to the any estimators that one can choose
  - Caveat: the family of estimators (which are also r.v.'s) should form a "subspace" of the r.v. inner product space


X target
Ξ_{MMSE}
$\hat{X}_{\mathrm{MMSE}}(Y)$
estimator subspace


target
observation
estimation
X
$P_{X,Y}$
Y
Estimator in $\mathcal{H}$
$g(\cdot)$
$\hat{X} = g(Y)$

The Minimum MSE

$$
\begin{aligned}
\min \ \mathsf{MSE} &= \mathsf{E}\left[\Xi_n \Xi_n^*\right] = \mathsf{E}\left[\Xi_n X_n^*\right] \\
&= \mathsf{E}\left[X_n X_n^*\right] - \mathsf{E}\left[\left(g^{(\mathrm{MMSE})} * Y\right)_n X_n^*\right] \\
&= \mathsf{R}_X[0] - \sum_k g_k^{(\mathrm{MMSE})} \mathsf{R}_{YX}[-k] \\
&= \mathsf{R}_X[0] - \left(g^{(\mathrm{MMSE})} * \mathsf{R}_{YX}\right)[0] \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathsf{S}_X(f) - \breve{g}^{(\mathrm{MMSE})}(f)\mathsf{S}_{YX}(f)\right) \mathrm{d}f \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathsf{S}_X(f) - \frac{\left|\mathsf{S}_{XY}(f)\right|^2}{\mathsf{S}_Y(f)}\right) \mathrm{d}f
\end{aligned}
$$

Other kinds of Wiener Filter

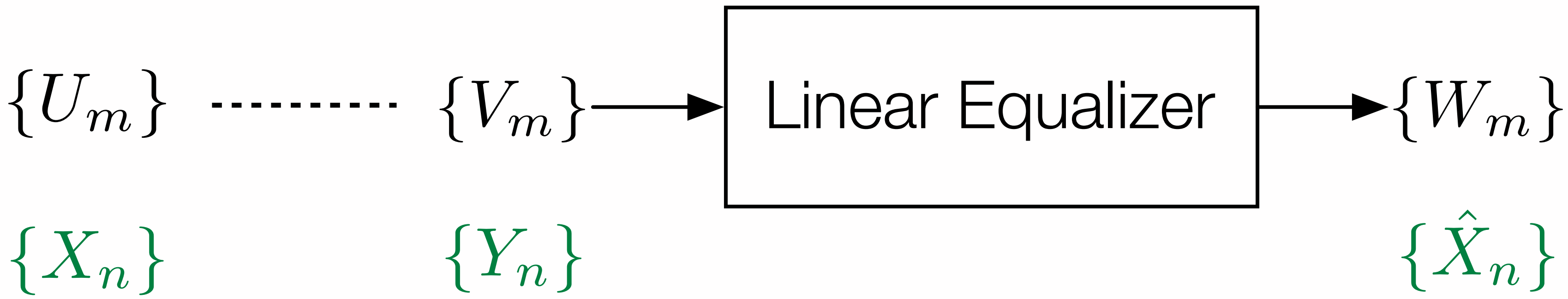
FIR Wiener Filter

IIR Causal Wiener Filter

Optimal Linear Equalizer

Back to our problem of linear equalization

$$V_m = (h * U)_m + Z_m$$



$$\mathsf{S}_U(\zeta) = E_s$$
$$\mathsf{S}_Z(\zeta) = N_0$$

$$V_m = (h * U)_m + Z_m \implies \mathsf{S}_V(\zeta) = \check{h}(\zeta)\mathsf{S}_U(\zeta)\check{h}^*(1/\zeta^*) + \mathsf{S}_Z(\zeta)$$

$$\mathsf{S}_{UV}(\zeta) = \mathsf{S}_U(\zeta)\check{h}^*(1/\zeta^*)$$

Optimal linear equalizer:

$$\check{g}^{(\mathrm{MMSE})}(\zeta) = \frac{\mathsf{S}_{UV}(\zeta)}{\mathsf{S}_V(\zeta)} = \frac{E_s\check{h}^*(1/\zeta^*)}{E_s\check{h}^*(1/\zeta^*)\check{h}(\zeta) + N_0}$$

The Maximum SINR

$$\max \mathsf{SINR} = \frac{E_s}{\min \mathsf{MSE}} = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left| \breve{h}(f) \right|^2 \frac{E_s}{N_0} + 1 \right)^{-1} \mathrm{d}f \right)^{-1}$$

$$\min \mathsf{MSE} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathsf{S}_U(f) - \frac{|\mathsf{S}_{UV}(f)|^2}{\mathsf{S}_V(f)} \right) \mathrm{d}f$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(E_s - \frac{\left| \breve{h}(f) \right|^2 E_s^2}{\left| \breve{h}(f) \right|^2 E_s + N_0} \right) \mathrm{d}f$$

$$= E_s \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}f}{|\breve{h}(f)|^2 \frac{E_s}{N_0} + 1}$$

Part III. OFDM

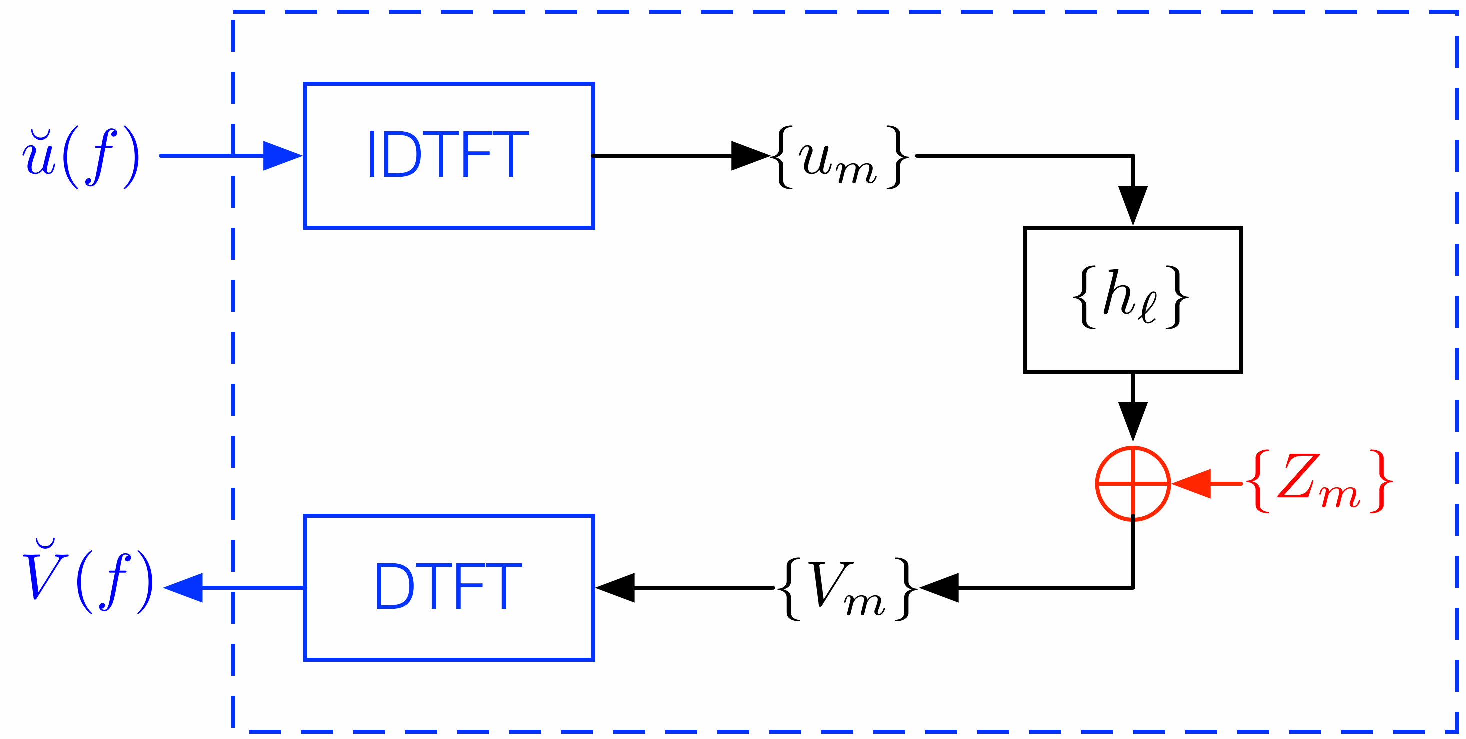
Discrete Fourier Transform; Circular Convolution;
Eigen Decomposition of Circulant Matrices

Motivation

- Previous parts: only receiver-centric methods
  - MLSD with Viterbi algorithm: optimal but computationally infeasible
  - Linear equalizations: simple but suboptimal.
- Is it possible to "pre-process" $\{u_m\}$ at Tx and "post-process" $\{V_m\}$ at Rx, so that the end-to-end channel is ISI-free?
  - Note: ZF can already remove ISI completely, but the noises after ZF are not independent anymore
  - Post processing should preserve mutual independence of the noises
- Observation: IDTFT and DTFT will work, "if" we are willing to roll back to analog communication

IDTFT: $u_m = \int_{-1/2}^{1/2} \breve{u}(f) e^{\mathrm{j}2\pi m f} \, \mathrm{d}f$

DTFT: $\breve{V}(f) = \sum_m V_m e^{-\mathrm{j}2\pi m f}$



$$V_m = (h * u)_m + Z_m \longleftrightarrow \boxed{\breve{V}(f) = \breve{h}(f)\breve{u}(f) + \breve{Z}(f)}$$

Why it works: because $e^{\mathrm{j}2\pi m f}$ is an **eigenfunction** to any LTI filter.

Using these eigenfunctions as a new basis to carry data renders infinite # of ISI-free channels in the frequency domain.

In frequency domain, the outcome at a frequency only depends on the input at that frequency:

$\Longrightarrow$ no ISI!

Caveat: analog communication in the frequency domain

Discretized DTFT: Discrete Fourier Transform

- Idea: use the discretized version of DTFT/IDTFT

IDTFT: $u_m = \int_{-1/2}^{1/2} \breve{u}(f) e^{j2\pi m f}\, \mathrm{d}f$

DTFT: $\breve{V}(f) = \sum_m V_m e^{-j2\pi m f}$

$f = \frac{k}{N}$ ⟷

$k = 0, ..., N-1$
$m = 0, ..., N-1$

N-pt. IDFT: $u_m = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \breve{u}[k] e^{j2\pi \frac{mk}{N}}$

N-pt. DFT: $\breve{V}[k] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} V_m e^{-j2\pi \frac{mk}{N}}$

- Note: N-point DFT/IDFT are transforms between two length-N sequences, indexed from 0 to N–1.

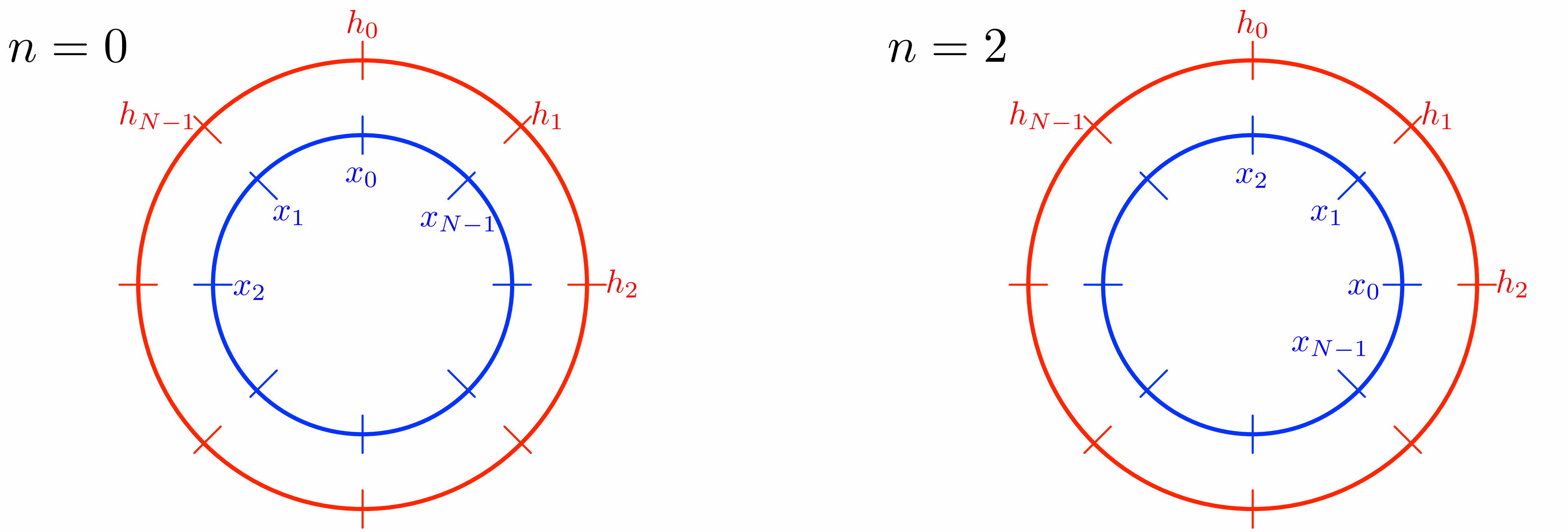
- Unfortunately, the convolution-multiplication property of the DTFT-IDTFT pair no longer holds

- **We need a new kind of convolution for DFT-IDFT pair!**

Circular Convolution

- Definition: for two length-N sequences $\{x_n\}_{n=0}^{N-1}, \{h_n\}_{n=0}^{N-1}$

$$(h \circledast x)_n \triangleq \sum_{\ell=0}^{N-1} h_\ell \, x_{(n-\ell) \bmod N}, \qquad n = 0, 1, \ldots, N-1$$



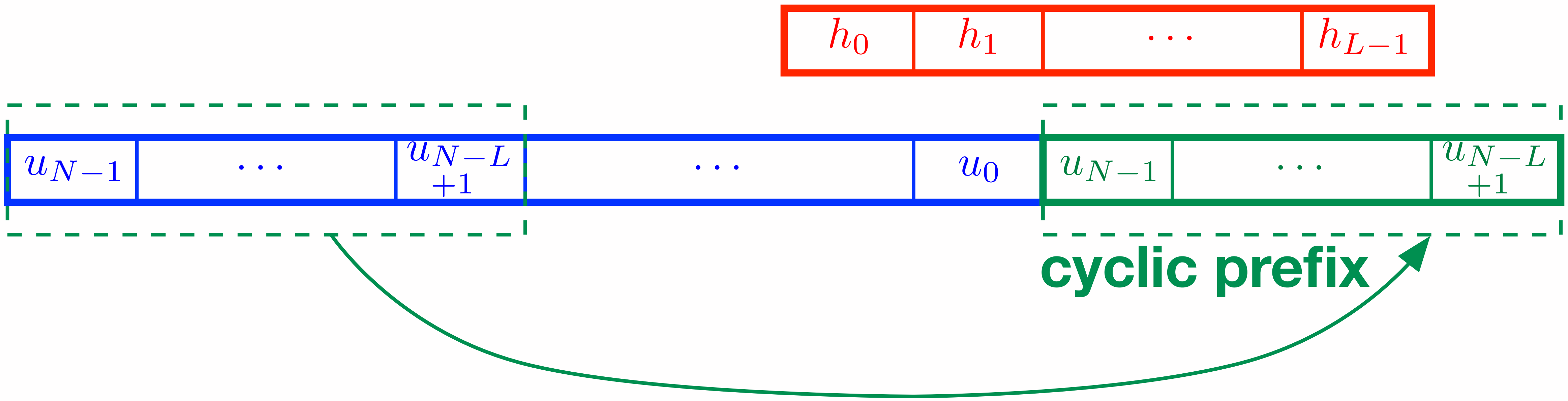
- Convolution-multiplication property: for length-N sequences $\{x_n\}_{n=0}^{N-1}, \{y_n\}_{n=0}^{N-1}, \{h_n\}_{n=0}^{N-1}$ with $y_n = (h \circledast x)_n$,

$$\breve{y}[k] = \sqrt{N}\breve{h}[k]\breve{x}[k], \quad \forall\, k = 0, 1, \ldots, N-1$$

Implement Circular Conv. in LTI Channel

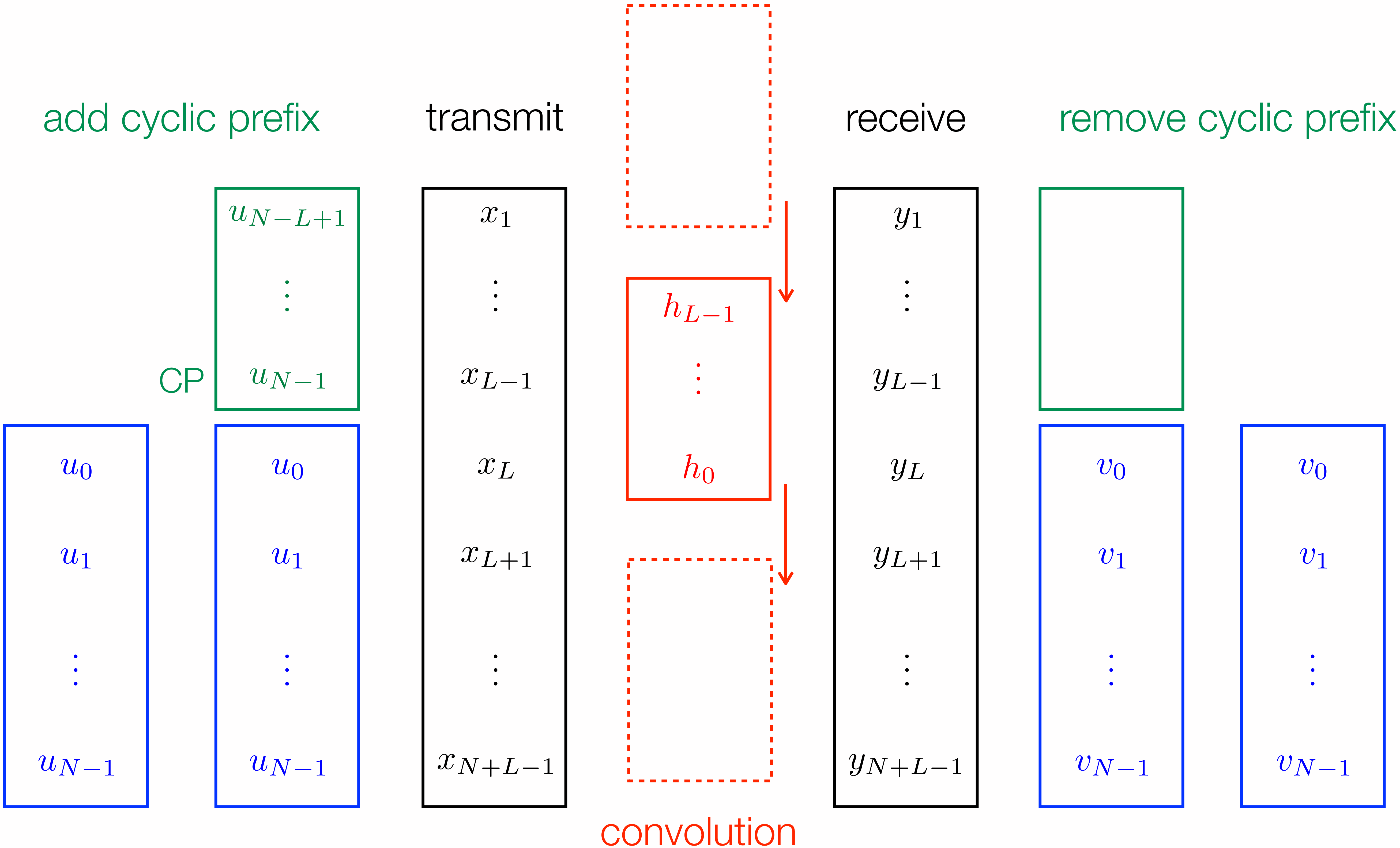
- Original LTI channel (ignore noise): linear convolution

$$v_m = (h * u)_m = \sum_{\ell=0}^{L-1} h_\ell u_{m-\ell}, \; m = 0, \ldots, N-1 \qquad (N \gg L)$$



- Desired channel (ignore noise): circular convolution

$$v_m = (h \circledast u)_m = \sum_{\ell=0}^{L-1} h_\ell u_{(m-\ell) \bmod N}, \; m = 0, \ldots, N-1$$



$$v_m = (h \circledast u)_m, \;\; m = 0, \ldots, N-1$$

Matrix Form of Circular Convolution

$$\boxed{v_m = (h \circledast u)_m, \;\; m = 0, \ldots, N-1}$$

$$\boldsymbol{v} = \mathbf{h}_c \boldsymbol{u} \in \mathbb{C}^N$$

$$\underbrace{\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}}_{\boldsymbol{v}} = \underbrace{\begin{bmatrix} h_0 & 0 & 0 & h_{L-1} & \cdots & h_1 \\ h_1 & h_0 & \vdots & & & \vdots \\ \vdots & h_1 & 0 & & & h_{L-1} \\ h_{L-1} & \vdots & h_0 & & & 0 \\ 0 & h_{L-1} & h_1 & h_0 & & \vdots \\ \vdots & & \vdots & & & 0 \\ 0 & 0 & h_{L-1} & h_{L-2} & \cdots & h_0 \end{bmatrix}}_{\mathbf{h}_c} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{\boldsymbol{u}}$$

with noise: $\boldsymbol{V} = \mathbf{h}_c \boldsymbol{u} + \boldsymbol{Z} \;\; \in \mathbb{C}^N$

Linear Algebraic View

Circulant Matrix

Every row/column is a circular shift of the first row/column

$$
\mathbf{h}_{\mathrm{c}} = \begin{bmatrix}
h_0 & 0 & 0 & h_{L-1} & \cdots & h_1 \\
h_1 & h_0 & \vdots & & & \vdots \\
\vdots & h_1 & 0 & & & h_{L-1} \\
h_{L-1} & \vdots & h_0 & & & 0 \\
0 & h_{L-1} & h_1 & h_0 & & \vdots \\
\vdots & & \vdots & & & 0 \\
0 & 0 & h_{L-1} & h_{L-2} & \cdots & h_0
\end{bmatrix}
$$

- Define $\phi_m^{(k)} \triangleq \frac{1}{\sqrt{N}} e^{\mathrm{j}2\pi \frac{k}{N} m}, \quad m = 0, ..., N-1$
- Can show: for any $\{h_\ell\}_{\ell=0}^{N-1}$, $(h \circledast \phi^{(k)})_m = \sqrt{N}\breve{h}[k]\phi_m^{(k)}$

$$(h \circledast \phi^{(k)})_m = \sqrt{N}\breve{h}[k]\phi_m^{(k)}, \quad m = 0, \dots, N-1$$

$$\implies \boldsymbol{\phi}^{(k)} \text{ is an eigenvector of matrix } \mathbf{h}_{\mathrm{c}}$$
$$\text{with eigenvalue } \sqrt{N}\breve{h}[k], \text{ for all } k = 0, \dots, N-1.$$

$$\underbrace{\sqrt{N}\breve{h}[k]}_{\breve{h}(f)|_{f=\frac{k}{N}}} \; \boldsymbol{\phi}^{(k)} = \underbrace{\begin{bmatrix} h_0 & 0 & 0 & h_{L-1} & \cdots & h_1 \\ h_1 & h_0 & \vdots & & & \vdots \\ \vdots & h_1 & 0 & & & h_{L-1} \\ h_{L-1} & \vdots & h_0 & & & 0 \\ 0 & h_{L-1} & h_1 & h_0 & & \vdots \\ \vdots & & \vdots & & & 0 \\ 0 & 0 & h_{L-1} & h_{L-2} & \cdots & h_0 \end{bmatrix}}_{\mathbf{h}_{\mathrm{c}}} \underbrace{\begin{bmatrix} \phi_0^{(k)} \\ \phi_1^{(k)} \\ \vdots \\ \phi_{N-1}^{(k)} \end{bmatrix}}_{\boldsymbol{\phi}^{(k)}}$$

$$\boxed{\mathbf{h}_{\mathrm{c}}\boldsymbol{\phi}^{(k)} = \sqrt{N}\breve{h}[k]\boldsymbol{\phi}^{(k)}, \quad \forall\, k = 0, \dots, N-1}$$

- Furthermore, can show that $\langle \boldsymbol{\phi}^{(k)}, \boldsymbol{\phi}^{(l)} \rangle = \mathbb{1}\{k = l\}$

  $\implies \{\boldsymbol{\phi}^{(k)} \mid k = 0, \dots, N-1\}$: an orthonormal basis of $\mathbb{C}^N$

- Hence, we can obtain the eigenvalue decomposition of ***any*** **circulant matrix** $\mathbf{h}_\mathrm{c}$

$$\mathbf{h}_\mathrm{c} = \mathbf{\Phi} \mathbf{\Lambda}_{\breve{h}} \mathbf{\Phi}^\mathtt{H}$$

$$\mathbf{\Phi} \triangleq \left[\phi^{(0)} \ \ldots \ \phi^{(N-1)} \right]$$

$$\mathbf{\Lambda}_{\breve{h}} \triangleq \mathrm{diag}(\breve{h}(f_0), \breve{h}(f_1), \ldots, \breve{h}(f_{N-1}))$$

$$f_k \triangleq \tfrac{k}{N}, \quad k = 0, \ldots, N-1$$

$\{h_n \mid n = 0, \ldots, N-1\}$: the first column of $\mathbf{h}_\mathrm{c}$

$\breve{h}(f)$: DTFT of $\{h_n\}$

- Can diagonalize the channel (remove ISI) without knowing it using the DFT basis. Only true for circulant matrix!

- IDFT matrix $\boldsymbol{\Phi}$ and DFT matrix $\boldsymbol{\Phi}^{\mathtt{H}}$:

$$(\boldsymbol{\Phi})_{m,k} = \frac{1}{\sqrt{N}} \exp\left(\mathrm{j}2\pi \frac{mk}{N}\right)$$

$$(\boldsymbol{\Phi}^{\mathtt{H}})_{m,k} = \frac{1}{\sqrt{N}} \exp\left(-\mathrm{j}2\pi \frac{mk}{N}\right)$$

N-pt. IDFT: $u_m = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \breve{u}[k] e^{\mathrm{j}2\pi \frac{mk}{N}}$

N-pt. DFT: $\breve{V}[k] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} V_m e^{-\mathrm{j}2\pi \frac{mk}{N}}$

$$\boldsymbol{u} = \boldsymbol{\Phi}\breve{\boldsymbol{u}}, \quad \breve{\boldsymbol{V}} = \boldsymbol{\Phi}^{\mathtt{H}}\boldsymbol{V}$$

- Pre-processing and post-processing:

$$\boldsymbol{V} = \mathbf{h}_c \boldsymbol{u} + \boldsymbol{Z} = \boldsymbol{\Phi}\boldsymbol{\Lambda}_{\breve{h}}\boldsymbol{\Phi}^{\mathtt{H}}\boldsymbol{u} + \boldsymbol{Z} \qquad \boldsymbol{\Phi}^{\mathtt{H}}\boldsymbol{V} = \boldsymbol{\Lambda}_{\breve{h}}\boldsymbol{\Phi}^{\mathtt{H}}\boldsymbol{u} + \boldsymbol{\Phi}^{\mathtt{H}}\boldsymbol{Z}$$

$$\Longrightarrow \breve{\boldsymbol{V}} = \boldsymbol{\Lambda}_{\breve{h}}\breve{\boldsymbol{u}} + \breve{\boldsymbol{Z}} \qquad \breve{Z}[k] \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_0),\ k = 0, 1, ..., N-1$$

because the DFT matrix $\boldsymbol{\Phi}^{\mathtt{H}}$ is **unitary**

Equivalent Parallel Channels

- OFDM creates N parallel non-interfering sub-channels:

$$\breve{V}[k] = \breve{h}(f_k)\breve{u}[k] + \breve{Z}[k], \quad k = 0, 1, ..., N-1$$

- Channel gain at the k-th branch:

$$\breve{h}(f_k) = \breve{h}(\tfrac{k}{N}) = \sqrt{N}\breve{h}[k]$$

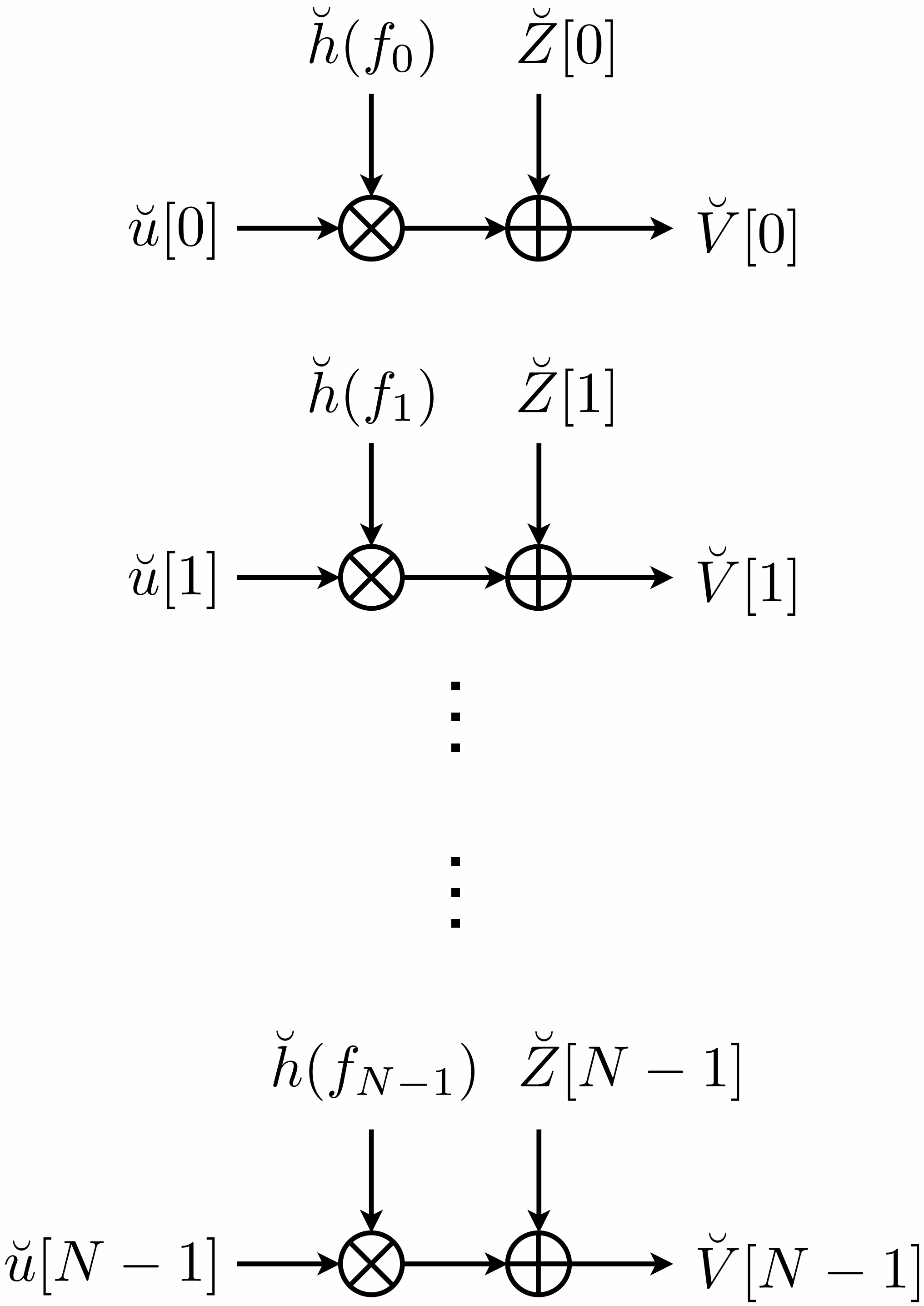
$\breve{h}(f)$: DTFT of $\{h_\ell\}$
= periodic copies of $\breve{h}_a(\frac{f}{T})$, period 1

$h_a(\tau) \triangleq (h_b * g)(\tau)$

- Equivalently, the overall bandwidth $2W$ is partitioned into N narrowbands, and each sub-channel use that narrowband for transmission (centered at $k\frac{2W}{N}, \ k = 0, ..., N-1$)

- Subcarrier spacing: $\frac{2W}{N}$

Capacity of Parallel Channels



- Capacity of N parallel channels is the sum of individual capacities
  coding across subcarriers does not help!
- Since channel gains are different, each branch has different capacity
- Goal: maximize capacity subject to a total power constraint
- Power allocation: maximize rate

P_k: power of branch k $\quad \sum_{k=0}^{N-1} P_k \leq NP$

$$R = \sum_{k=0}^{N-1} \log\left(1 + \frac{\left|\breve{h}(f_k)\right|^2 P_k}{N_0}\right)$$

Water-filling

$$\max_{P_0,\ldots,P_{N-1}} \sum_{k=0}^{N-1} \log\left(1 + \left|\breve{h}(f_k)\right|^2 \frac{P_k}{N_0}\right),$$

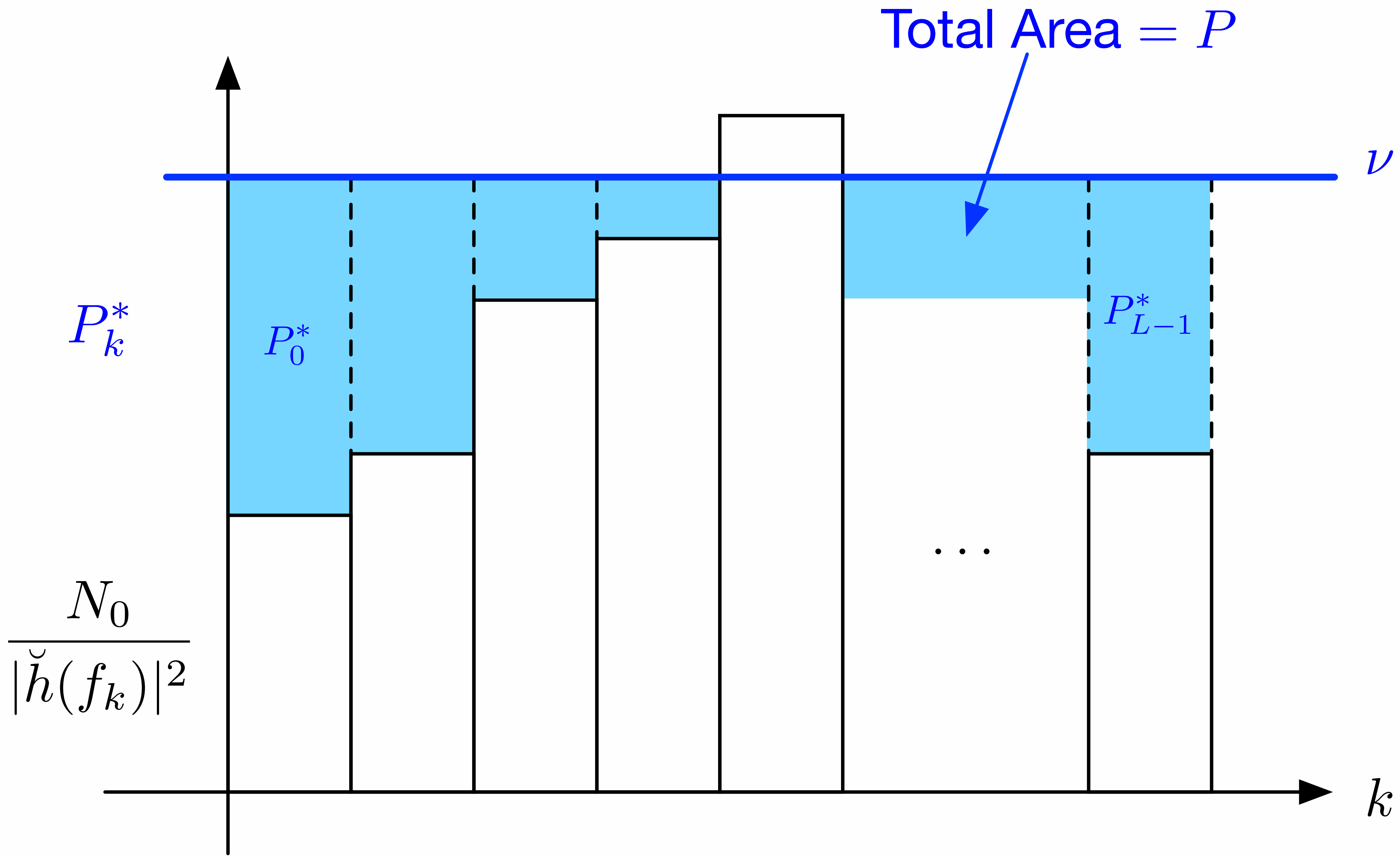
$$\text{subject to} \sum_{n=0}^{N-1} P_k = NP, \quad P_k \geq 0,\ k = 0, \ldots, N-1$$

- Solved by standard techniques in convex optimization (Lagrange multipliers, KKT condition)

- Final solution:

$$P_k^* = \left(\nu - \frac{N_0}{\left|\breve{h}(f_k)\right|^2}\right)^+ \qquad (x)^+ \triangleq \max(0, x)$$

$$\nu \text{ satisfies} \sum_{k=0}^{N-1} \left(\nu - \frac{N_0}{\left|\breve{h}(f_k)\right|^2}\right)^+ = NP$$



$$\breve{h}(f_k) = \breve{h}_b(k\tfrac{2W}{N})\breve{g}(k\tfrac{2W}{N})$$

baseband frequency response at $f = k\frac{2W}{N}$

- Main lesson: one should allocate higher rate when at the branch with better channel condition

Capacity of Frequency Selective Channel

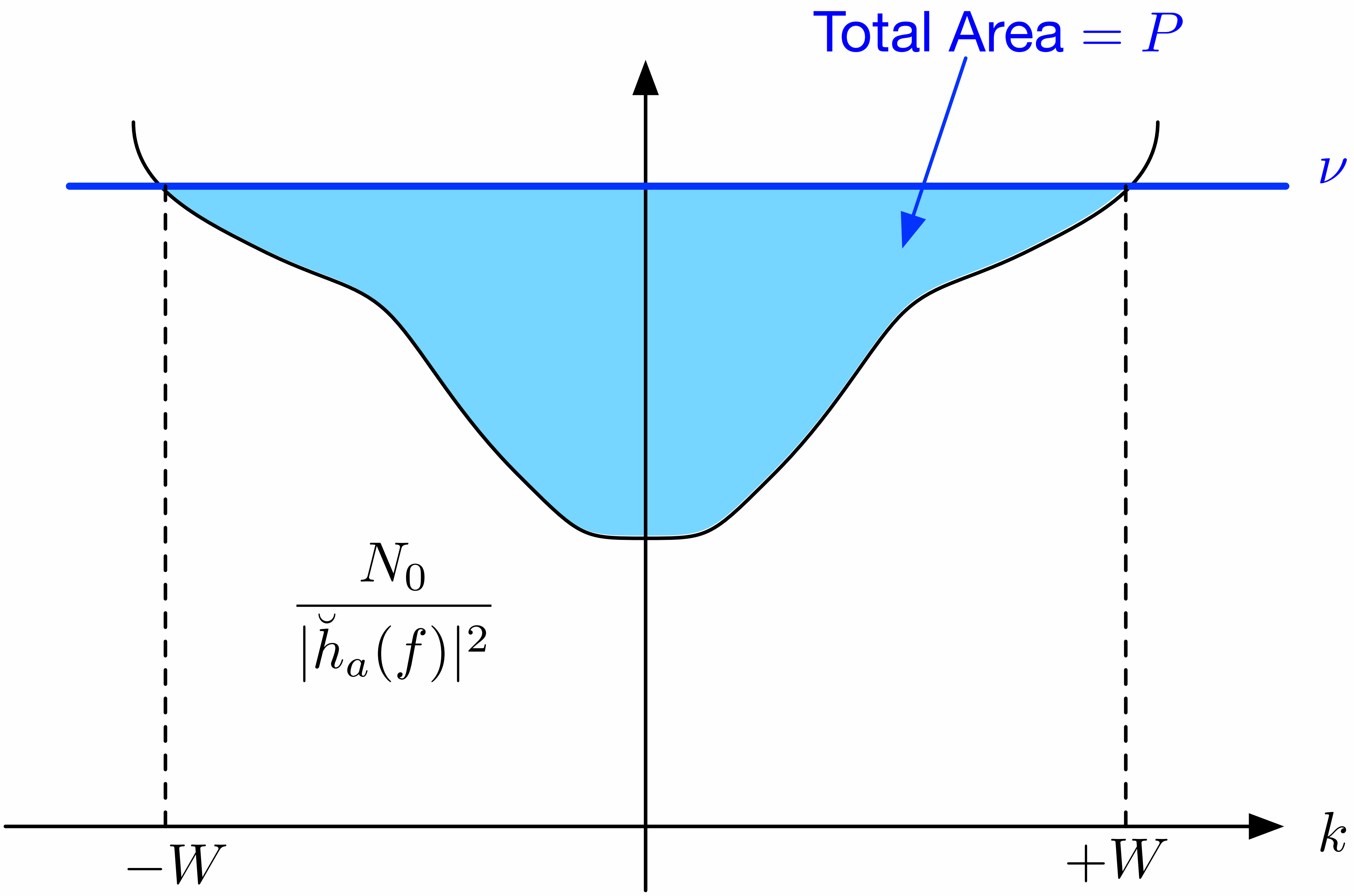
- Pre-processing (IDFT) and post-processing (DFT) are both **invertible** in OFDM systems
- The only loss: length-$(L-1)$ cyclic prefix, negligible when we take $N \to \infty$
- The power allocation problem becomes

$$\max_{P(f)} \int_{-1/2}^{1/2} \log\left(1 + \left|\breve{h}(f)\right|^2 \frac{P(f)}{N_0}\right) \mathrm{d}f,$$

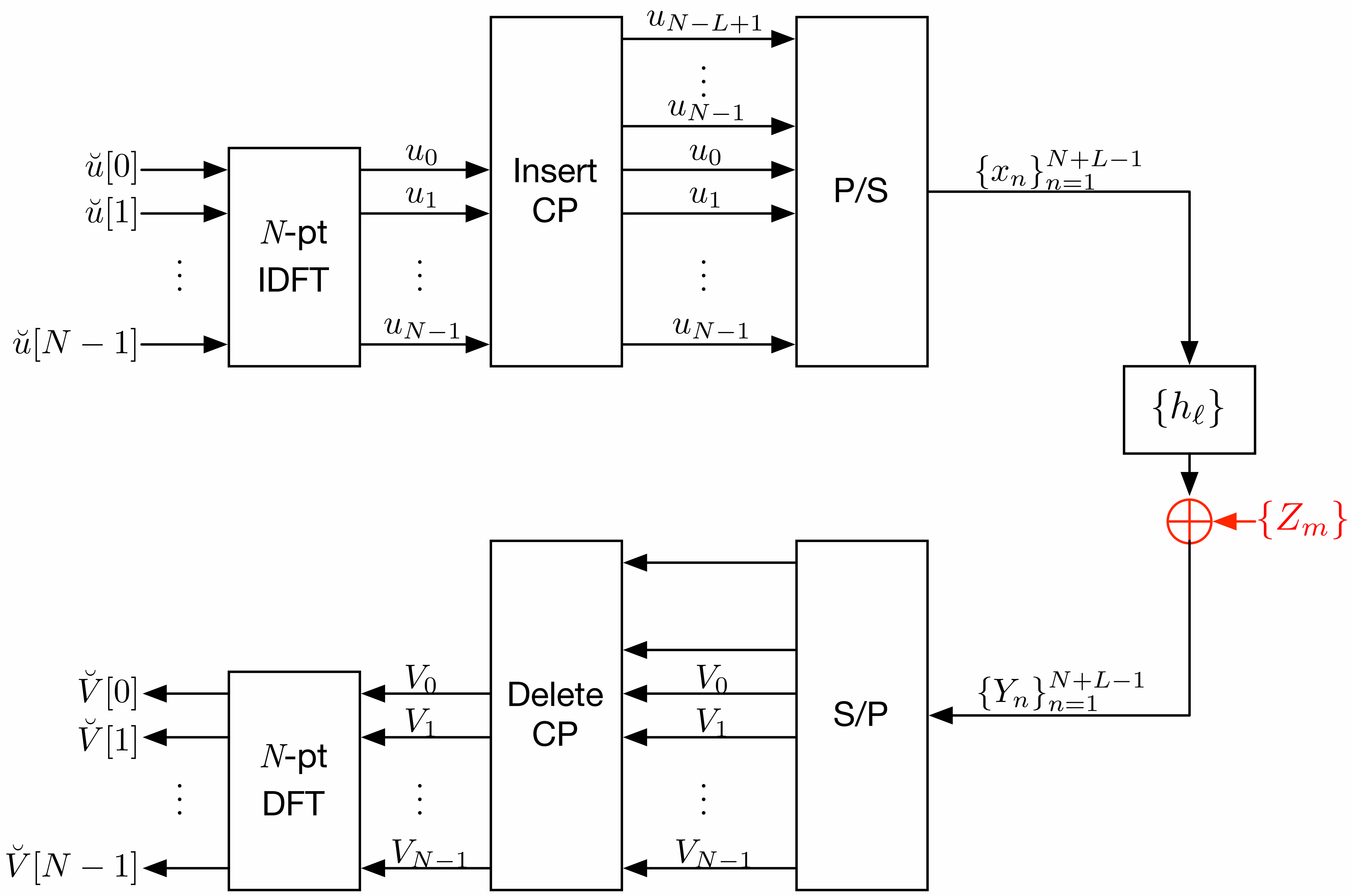
$$\text{subject to} \int_{-1/2}^{1/2} P(f)\,\mathrm{d}f = P, \quad P(f) \geq 0,\ f \in [-1/2, 1/2]$$

- Optimal solution: water-filling on the continuous spectrum

Water-filling in Frequency-Selective Channel



OFDM System Diagram



OFDM System Design

- Cyclic prefix overhead: $\frac{L-1}{N}$ (the smaller the better)
- Subcarrier spacing: $\frac{2W}{N}$ (the larger the better)

  prevent frequency offset/asynchrony

- Subcarriers are basic resource units in OFDM systems

- A critical issue of OFDM in practice: peak-to-average ratio (PAR) is much higher than single-carrier systems.

  It requires a large dynamic range of the linear characteristic of the transmit power amplifier (PA).