Lecture 01
Digital Modulation

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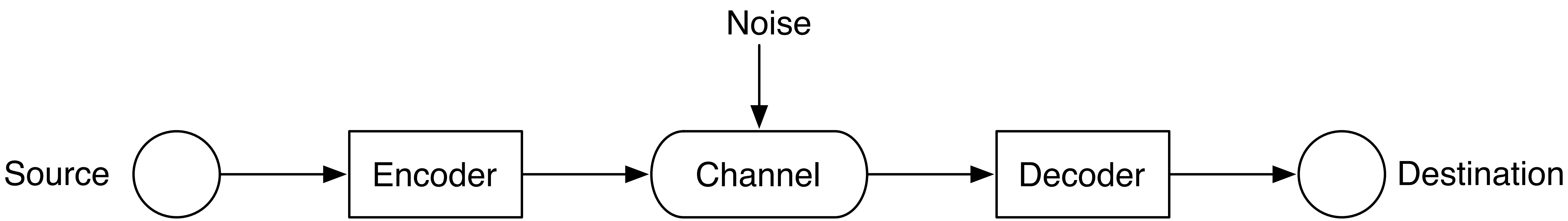
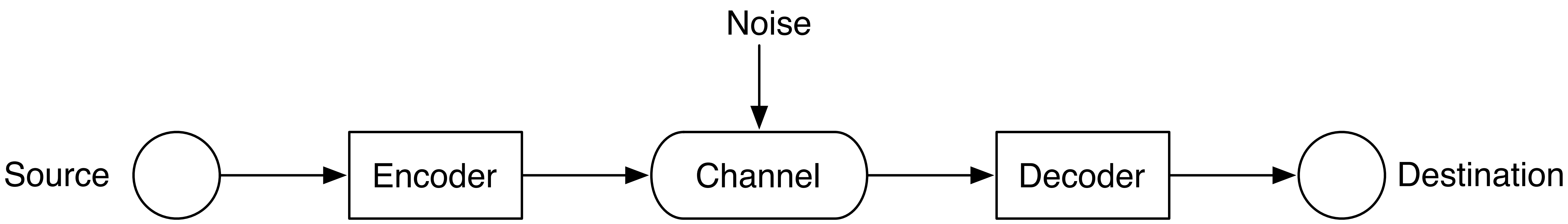
A communication system



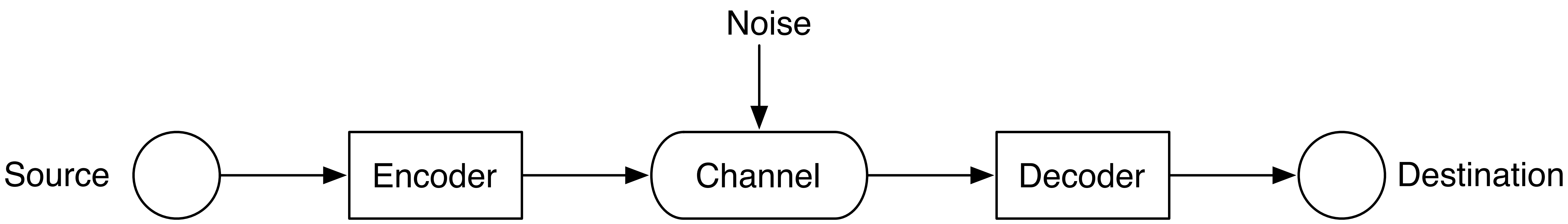
- The simplest point-to-point abstraction
- Given by nature/applications: source, channel, destination
- Engineers' design: encoder, decoder

Source and channel



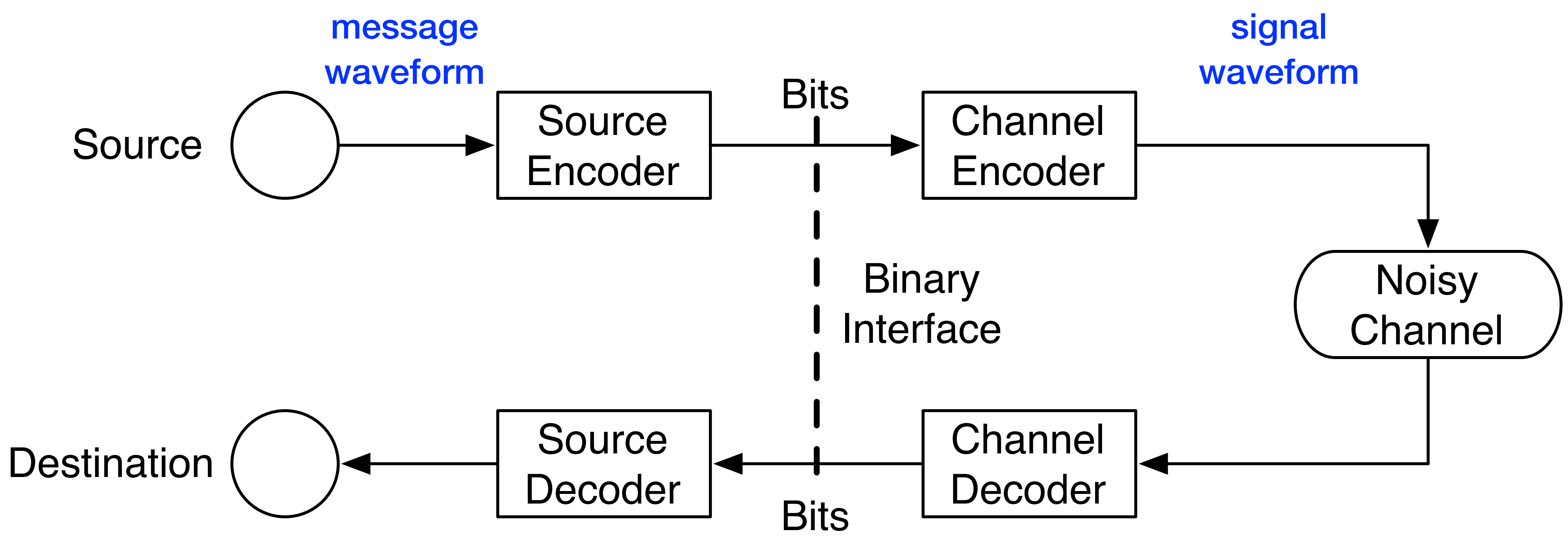
- Source
  - generates the **message** to be delivered
  - message: text, audio, image, video, etc.. → can be abstracted into **signals**
  - in general, signals can be continuous-time and continuous-valued → **waveforms**!
- Channel
  - abstraction of the physical medium: light, sound, wire, optical fiber, EM radiation, etc.
  - distorts the transmitted signal in some way → **noisy** channel, using stochastic models

Encoder and decoder



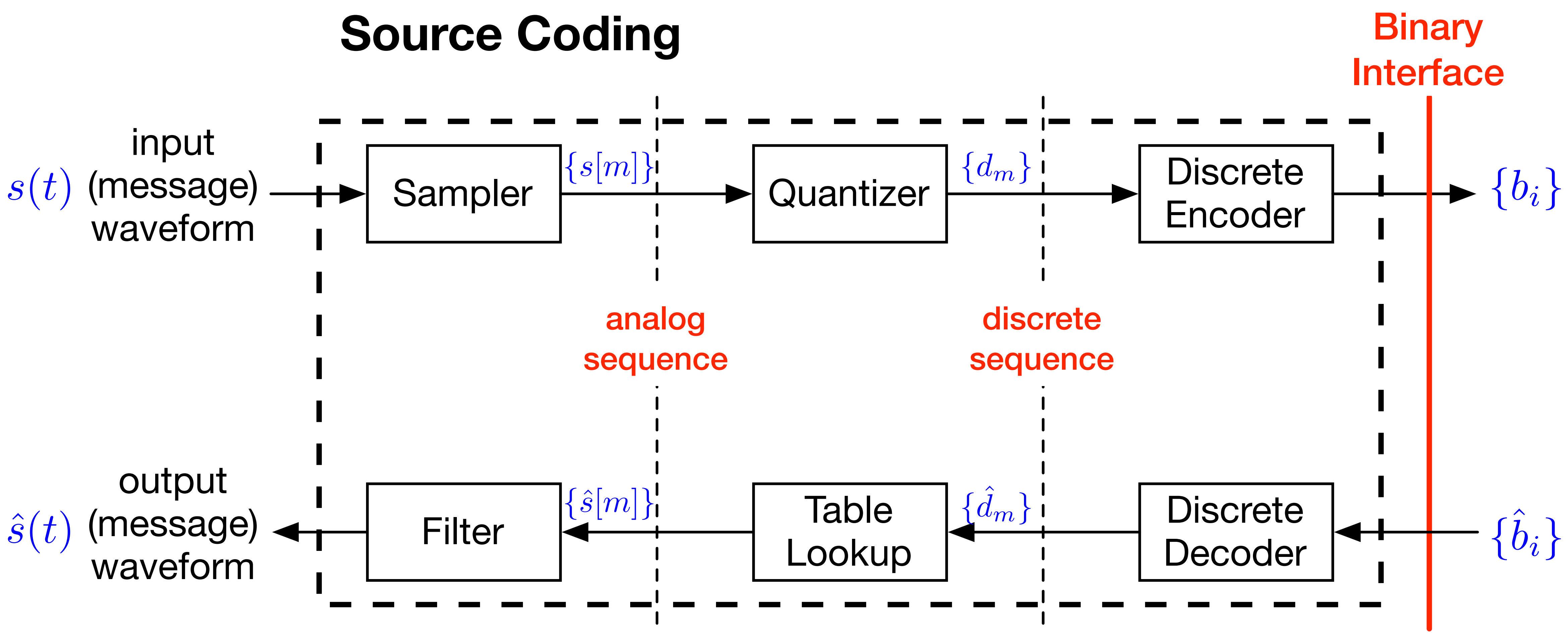
- Encoder
  - converts the message into physical signals that is ready for transmission
  - functions include sampling, quantization, compression, error-correction coding, modulation, etc.
- Decoder
  - reconstructs the message from the received signals
  - depending on the goal of the destination, can also carry out other tasks such as computing a function of the messages.

Digital communication system



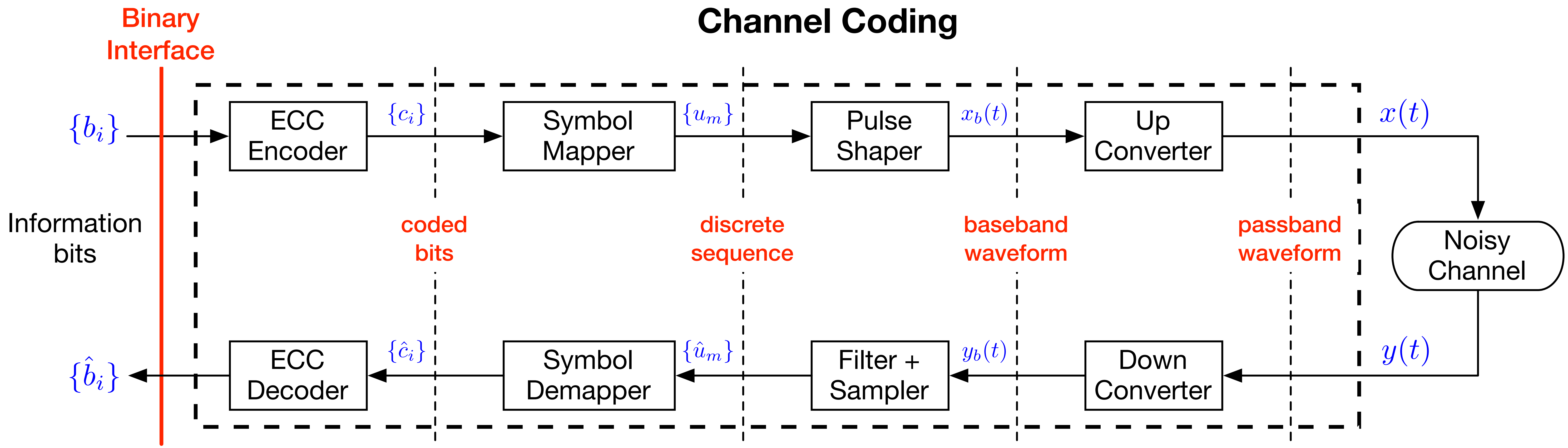
- A digital communication system uses **digital sequence** (**bits**) as an interface between **source coding** and **channel coding**
- Why separation?
  - Digital hardware is cheap, reliable, and scalable.
  - Source coding and channel coding can be independently designed.
  - Source-channel separation attains optimal transmission efficiency (Shannon).
  - Bits as universal currency of the digital world → a channel can be extended to a **network**

Source coding: main building blocks

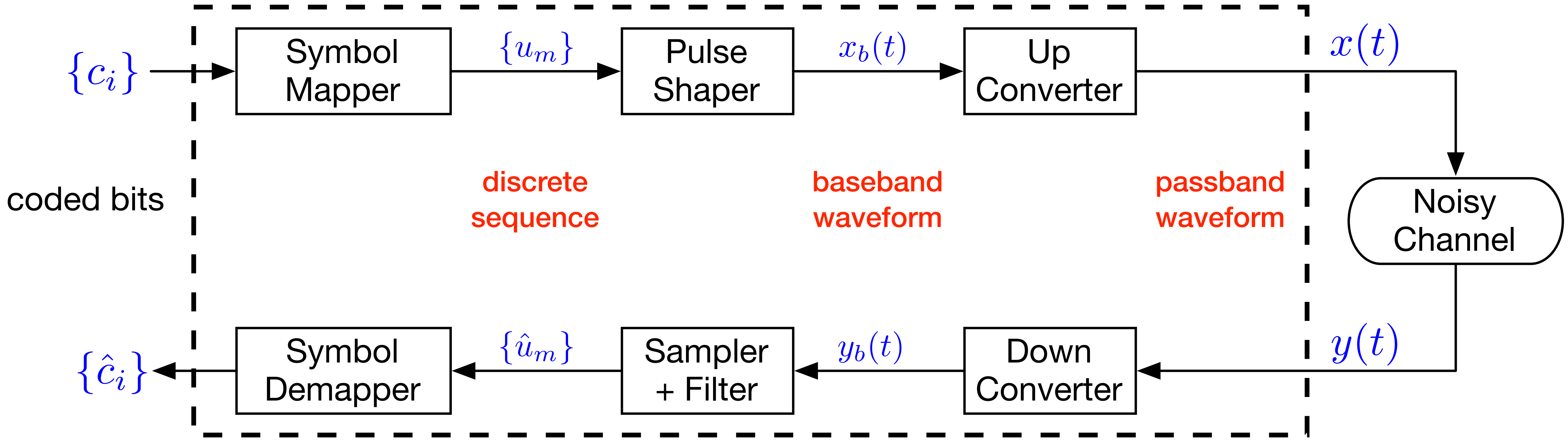


- Sampling: (continuous-time) waveform → (discrete-time) sequence
    - (Review: Sampling Theorem, Nyquist Rate)
- Quantization: continuous-valued sequence → discrete-valued sequence
- Compression: discrete-valued sequence → bit sequence, remove redundancy

Channel coding: main building blocks

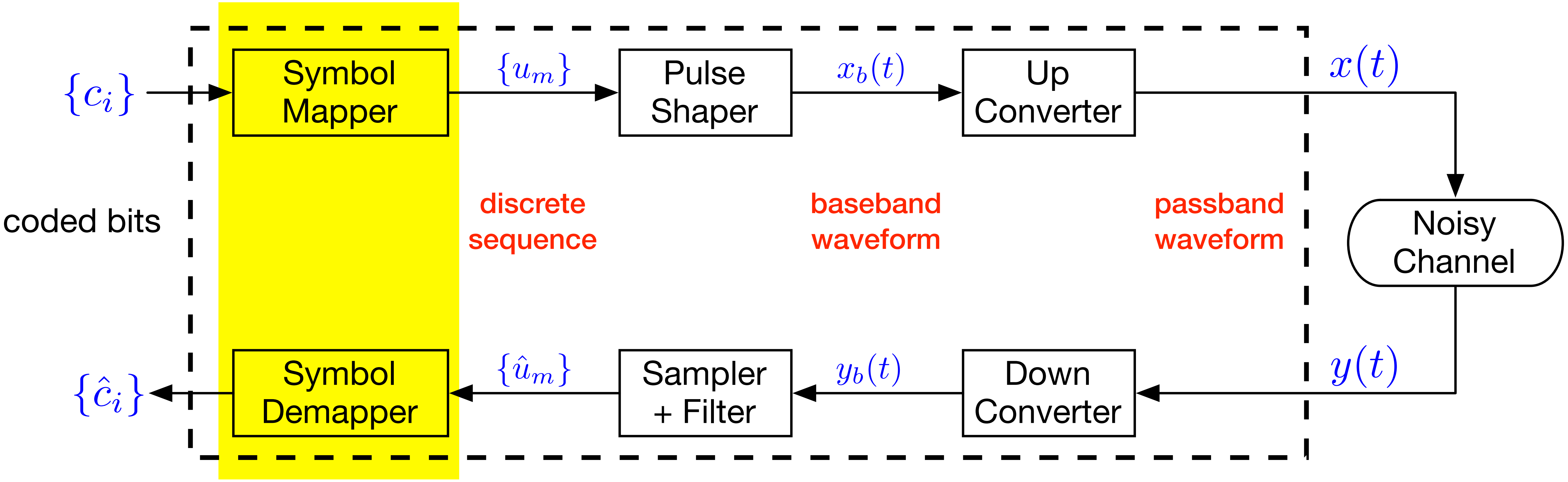


System architecture of digital modulation



- Three major components (for the Tx):
  - Symbol mapping: bit sequence → symbol sequence
  - Pulse shaping: symbol sequence → (baseband) waveform
  - Up conversion: baseband waveform → passband waveform

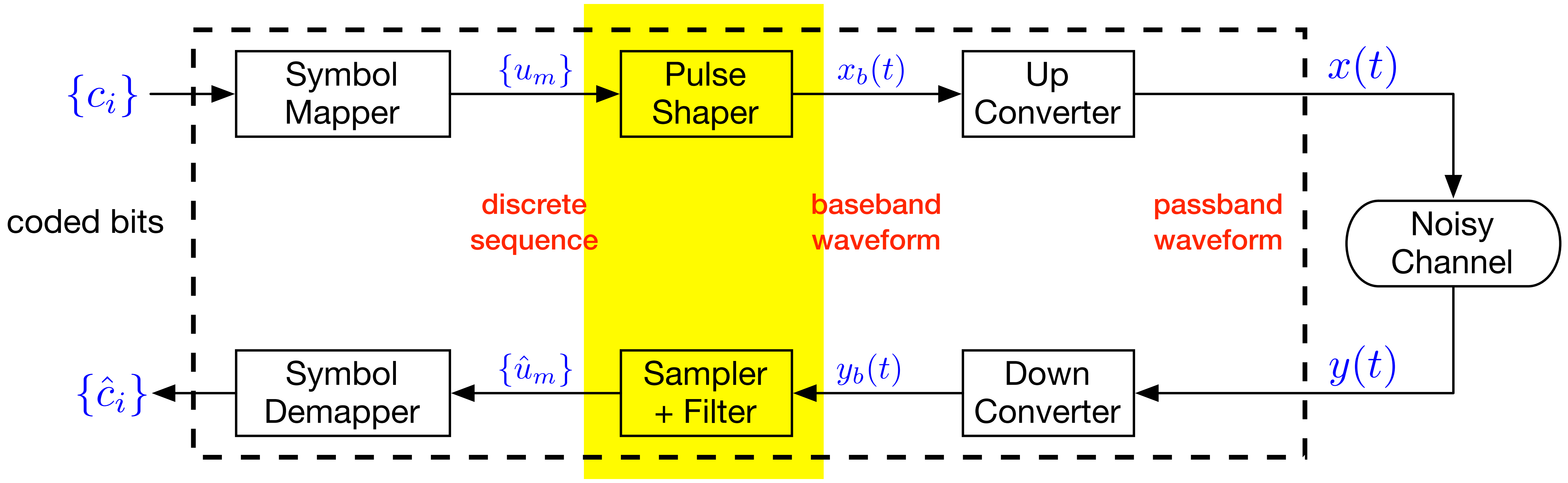
Bits $\longleftrightarrow$ Symbols



- To be designed: the constellation set and how to map bits to symbols

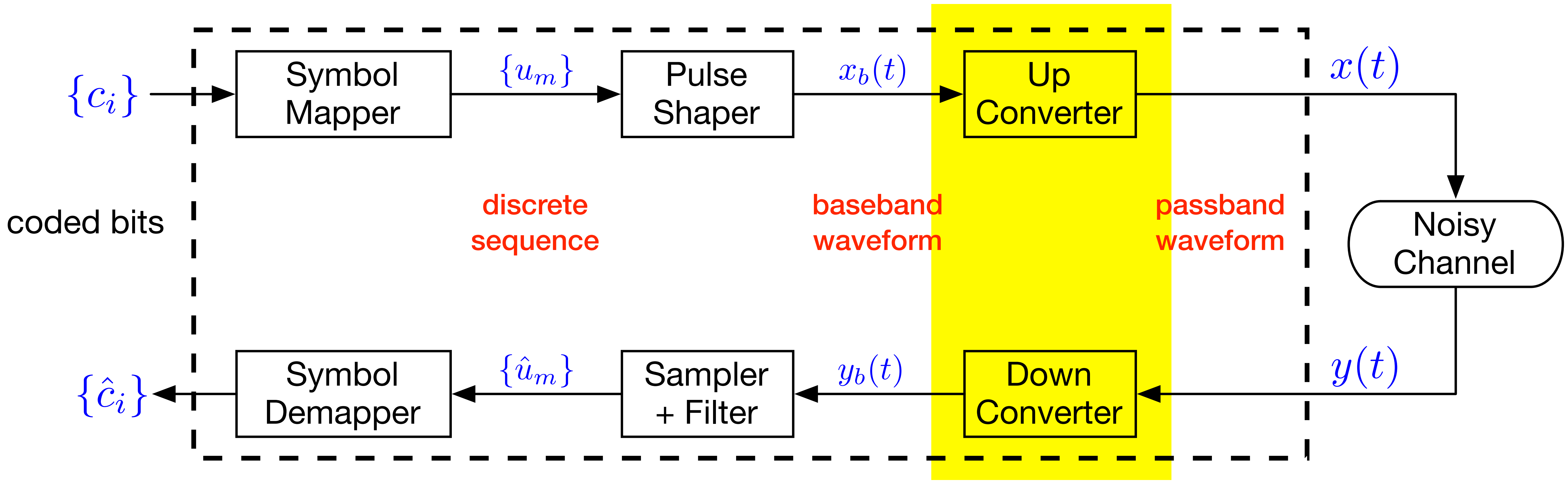
- Constellation sets to be covered: standard PSK, standard PAM, standard QAM
- Mapping: Gray mapping

Sequence ⟷ Waveform



- A pragmatic approach: **Pulse Amplitude Modulation (PAM)**
- To be designed: the modulating pulse
- System parameter: **bandwidth**
- **Nyquist criterion**: a sufficient condition for the pulse to satisfy in order to avoid aliasing effect

Baseband ⟷ Passband



- A pragmatic approach: **Quadrature Amplitude Modulation (QAM)**
- Essentially speaking, PAM with two branches:
  - one mixed with cosine, the other with sine
  - system parameter: **carrier frequency**
- Equivalent complex baseband representation

Part I. Signal Space

A Linear Algebraic Point to View for the
Conversion between Sequences and Waveforms

Fourier series for time-limited signals

Analysis (waveform → sequence)

$$x(t) \to x[m]$$

$$x[m] = \frac{1}{T}\int_{\mathcal{T}} x(t)e^{-\frac{\mathrm{j}2\pi m}{T}t}\,\mathrm{d}t$$

$$x[m] = \int_{-\infty}^{\infty} x(t)\underbrace{\boxed{\frac{1}{\sqrt{T}}e^{-\frac{\mathrm{j}2\pi m}{T}t}}}_{\phi_m^*(t)}\mathrm{d}t$$

Synthesis (sequence → waveform)

$$x[m] \to x(t)$$

$$x(t) = \sum_{m=-\infty}^{\infty} x[m]e^{\mathrm{j}\frac{2\pi m}{T}t}$$

$$x(t) = \sum_{m=-\infty}^{\infty} x[m]\underbrace{\boxed{\frac{1}{\sqrt{T}}e^{\mathrm{j}\frac{2\pi m}{T}t}}}_{\phi_m(t)}$$

Fourier Basis: $\boldsymbol{\phi}_m \equiv \phi_m(t) \triangleq \frac{1}{\sqrt{T}}\exp(\mathrm{j}\frac{2\pi}{T}mt), \quad m \in \mathbb{Z},$

Sampling theorem for band-limited signals

Analysis (waveform → sequence)

$$x(t) \to x[m]$$

$$x[m] = x(t)|_{t=\frac{m}{2W}} = x\left(\frac{m}{2W}\right)$$

$$x[m] = \frac{1}{\sqrt{2W}} x(m/2W)$$

check! ← $$= \int_{-\infty}^{\infty} x(t) \underbrace{\boxed{\sqrt{2W}\operatorname{sinc}(2Wt - m)}}_{\phi_m^*(t)} \mathrm{dt}$$

Synthesis (sequence → waveform)

$$x[m] \to x(t)$$

$$x(t) = \sum_{m=-\infty}^{\infty} x[m] \operatorname{sinc}(2Wt - m)$$

$$x(t) = \sum_{m=-\infty}^{\infty} x[m] \underbrace{\boxed{\sqrt{2W}\operatorname{sinc}(2Wt - m)}}_{\phi_m(t)}$$

Sinc Basis: $\phi_m \equiv \phi_m(t) \triangleq \sqrt{2W}\operatorname{sinc}(2Wt - m), \quad m \in \mathbb{Z}$

Signal space interpretation

waveform → sequence

$$x(t) \to x[m]$$

$$x(t) \to \boxed{\phi_m(t)} \to x[m]$$

$$= \int_{-\infty}^{\infty} x(t)\phi_m^*(t)\,\mathrm{d}t$$

sequence → waveform

$$x[m] \to x(t)$$

$$\{x[m]\} \to \boxed{\{\phi_m(t)\}} \to x(t)$$

$$= \sum_{m=-\infty}^{\infty} x[m]\phi_m(t)$$

Signal space intrepretation

waveform → sequence

$$x(t) \to x[m]$$

$$\boldsymbol{x} \to \boxed{\phi_m} \to x[m] = \langle \boldsymbol{x}, \boldsymbol{\phi}_m \rangle$$

projection onto an orthonormal basis

sequence → waveform

$$x[m] \to x(t)$$

$$\{x[m]\} \to \boxed{\{\phi_m\}} \to \boldsymbol{x} = \sum_{m=-\infty}^{\infty} x[m]\boldsymbol{\phi}_m$$

expansion over an orthonormal basis

$x(t)$ waveform ⟷ vectors $\boldsymbol{x}$

$\int_{-\infty}^{\infty} u(t)v^*(t)\,\mathrm{d}t$ integration ⟷ inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

Part II. Pulse Amplitude Modulation

A pragmatic approach to convert symbols to baseband waveforms and back

Modulation basis as time-shifted pulses

$$\{u_m\} \to \boxed{\{\phi_m(t)\}} \to x_b(t)$$

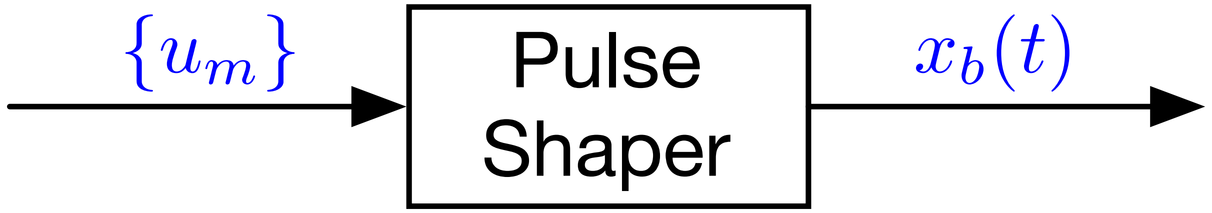
$$\{u_m\} \longrightarrow \boxed{\text{Pulse Shaper}} \longrightarrow x_b(t)$$

$$\phi_m(t) = p(t - mT), \; T = \frac{1}{2W}$$

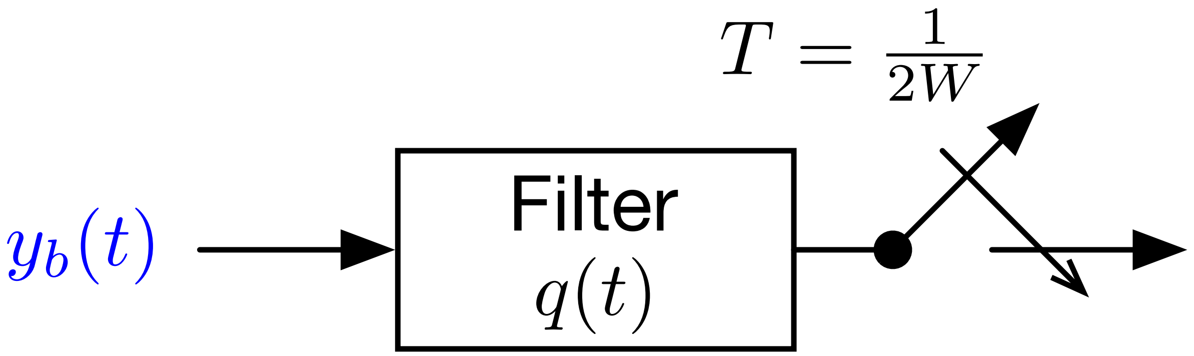
$$x_b(t) = \sum_{m=-\infty}^{\infty} u_m \, p(t - mT).$$

- $T = 1/2\,W$: transmission interval
- W : operational bandwidth
- Desired properties of the pulse function $p(t)$:
  - Time-limited (approximately)
  - Band-limited

PAM modulation and demodulation



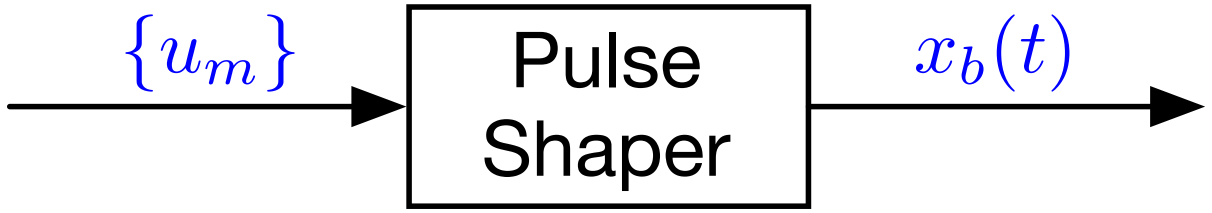
$$x_b(t) = \sum_{m=-\infty}^{\infty} u_m \, p(t - mT).$$



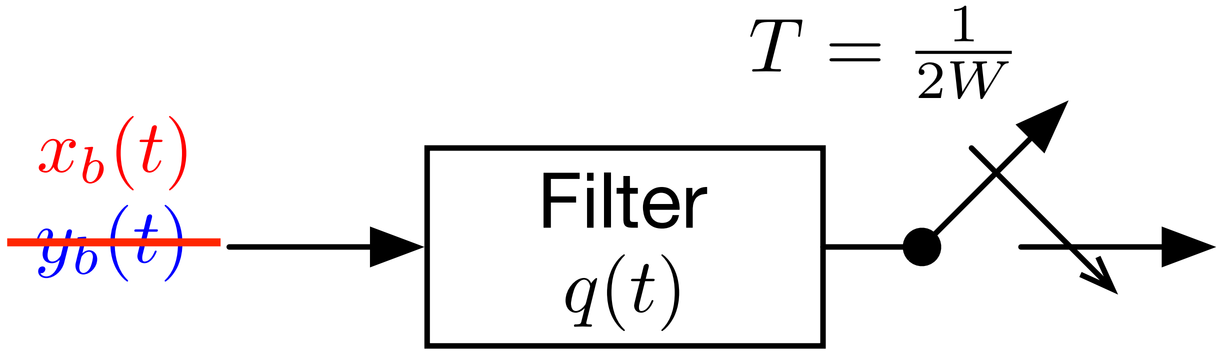
$$\hat{u}_m = \int_{-\infty}^{\infty} y_b(\tau) q(mT - \tau) \, \mathrm{d}\tau$$

- Key question: how to design the pulse $p(t)$ and the filter $q(t)$?

ISI-free condition when the channel is perfect



$$x_b(t) = \sum_{m=-\infty}^{\infty} u_m\, p(t - mT).$$



$$\hat{u}_m = \int_{-\infty}^{\infty} \overset{x_b(\tau)}{\cancel{y_b(\tau)}}\, q(mT - \tau)\, \mathrm{d}\tau$$

Want: $\hat{u}_m = u_m, \forall\, m.$

$$\hat{u}_m = (x_b * q)(mT) = \sum_{k=-\infty}^{\infty} u_k\, g(mT - kT) \qquad g(t) \triangleq (p * q)(t)$$

$$= \sum_{k=-\infty}^{\infty} u_k\, g((m-k)T))$$

A sufficient condition:

$$\boxed{g(\hat{k}T) = \begin{cases} 0 & \text{if } \hat{k} \neq 0 \\ 1 & \text{if } \hat{k} = 0 \end{cases}}$$

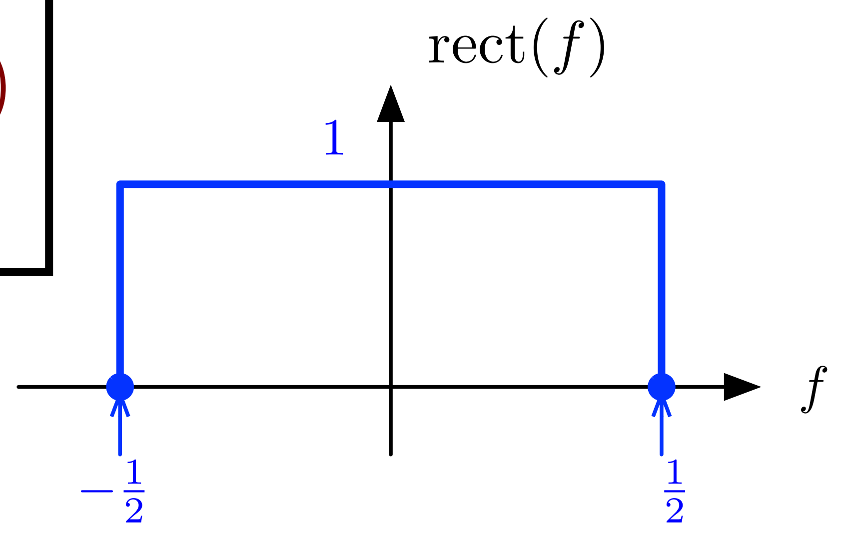
Ideal Nyquist and the Nyquist criterion

A sufficient condition (in time domain)

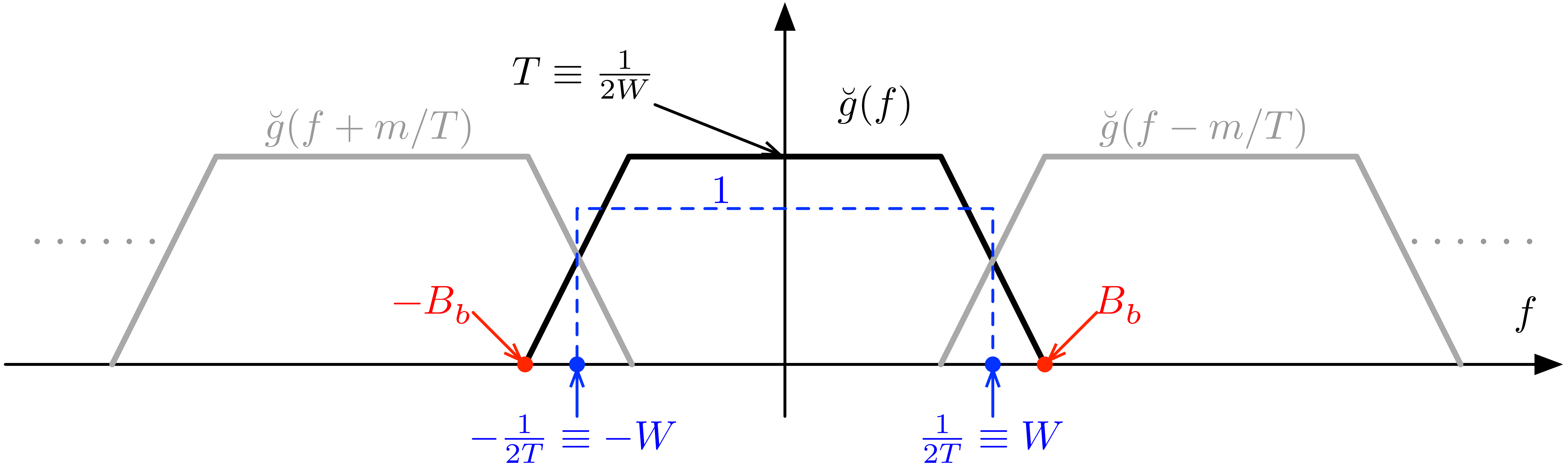
$$g(\hat{k}T) = \begin{cases} 0 & \text{if } \hat{k} \neq 0 \\ 1 & \text{if } \hat{k} = 0 \end{cases}$$

An equivalent condition (in frequency domain)

$$\boxed{T\,\mathrm{rect}(Tf) = \sum_m \breve{g}\left(f - \frac{m}{T}\right)\mathrm{rect}(Tf)}$$



Nyquist criterion



$$T\,\mathrm{rect}(Tf) = \sum_m \breve{g}\left(f - \frac{m}{T}\right)\mathrm{rect}(Tf)$$

Excessive bandwidth: $B_b - W$ ← this should not be too large

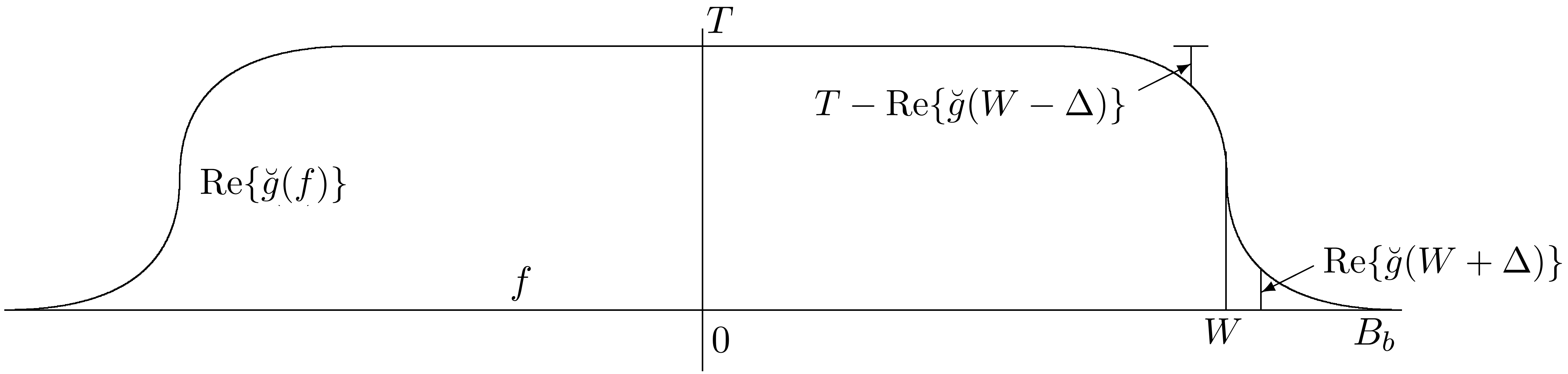
Typical choice: $W \leq B_b \leq 2W$

Band-edge symmetry

When taking the typical choice: $W \leq B_b \leq 2W$

the Nyquist criterion can be simplified to the following **band-edge symmetry**:

$$\boxed{\breve{g}^*(W-\Delta) + \breve{g}(W+\Delta) = T, \quad \forall\,\Delta \in [0, W]}$$

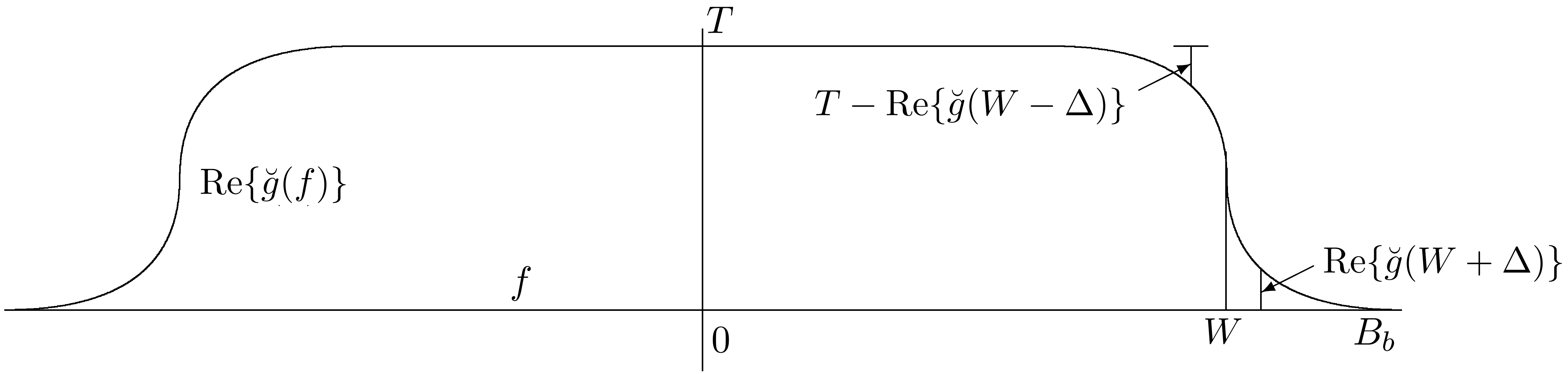


$$\Longleftrightarrow \begin{cases} \mathrm{Re}\left\{\breve{g}(W-\Delta)\right\} + \mathrm{Re}\left\{\breve{g}(W+\Delta)\right\} = T \\ \mathrm{Im}\left\{\breve{g}(W-\Delta)\right\} = \mathrm{Im}\left\{\breve{g}(W+\Delta)\right\} \end{cases}, \quad \forall\,\Delta \in [0, W]$$

Excessive bandwidth and rolloff factor

Excessive bandwidth: $B_b - W$

Rolloff factor: $\dfrac{B_b}{W} - 1$



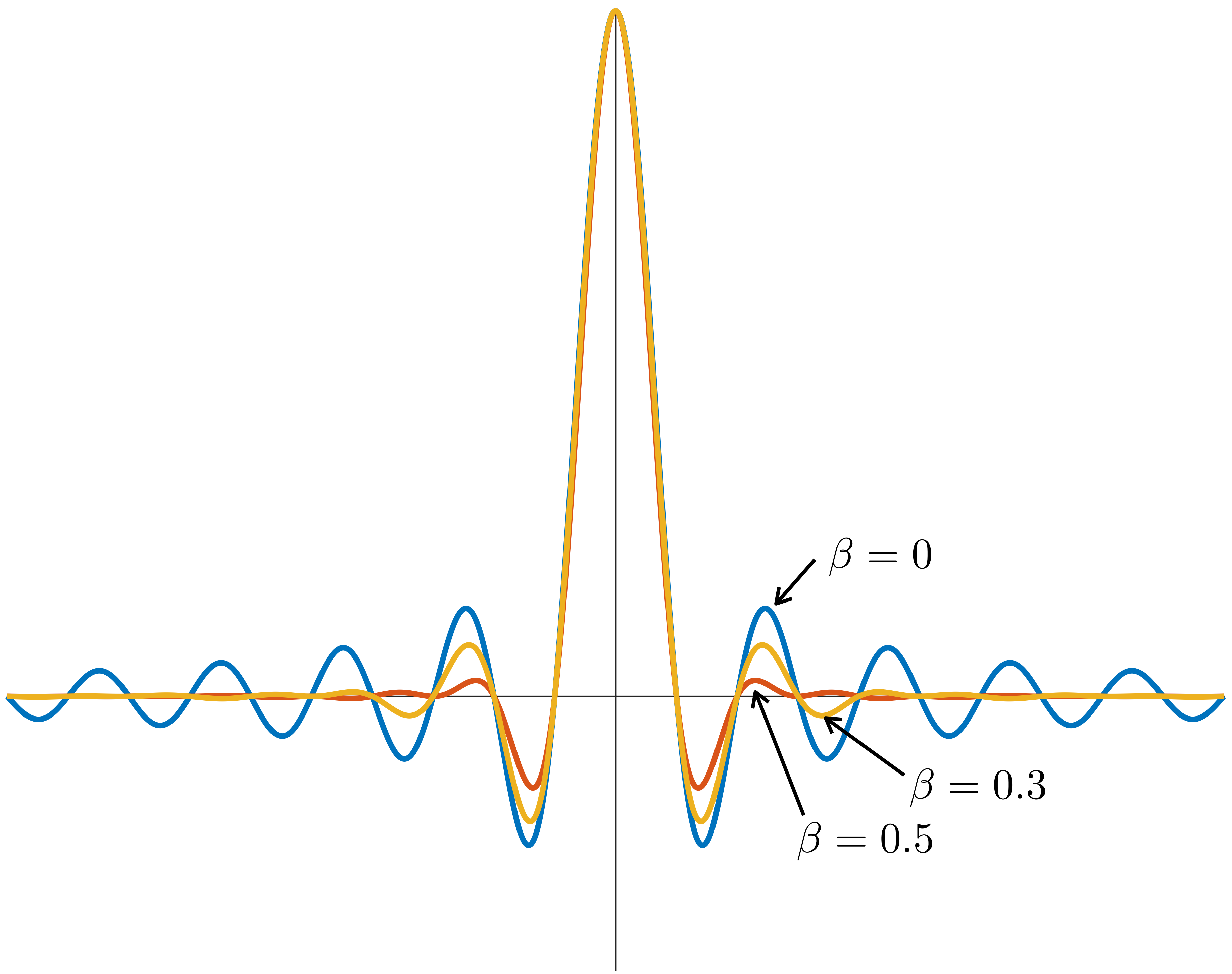
Raised cosine pulse

time domain

$$g_\beta(t) = \begin{cases} \frac{\pi}{4}\mathrm{sinc}\left(\frac{1}{2\beta}\right), & \text{if } |t| = \frac{T}{2\beta} \\ \mathrm{sinc}\left(\frac{t}{T}\right)\frac{\cos\left(\frac{\pi\beta t}{T}\right)}{1-4\frac{\beta^2 t^2}{T^2}}, & \text{otherwise} \end{cases}$$

rolloff factor = β

Decay to zero with speed $\sim \dfrac{1}{t^3}$ as $t \to \infty$ when $\beta > 0$

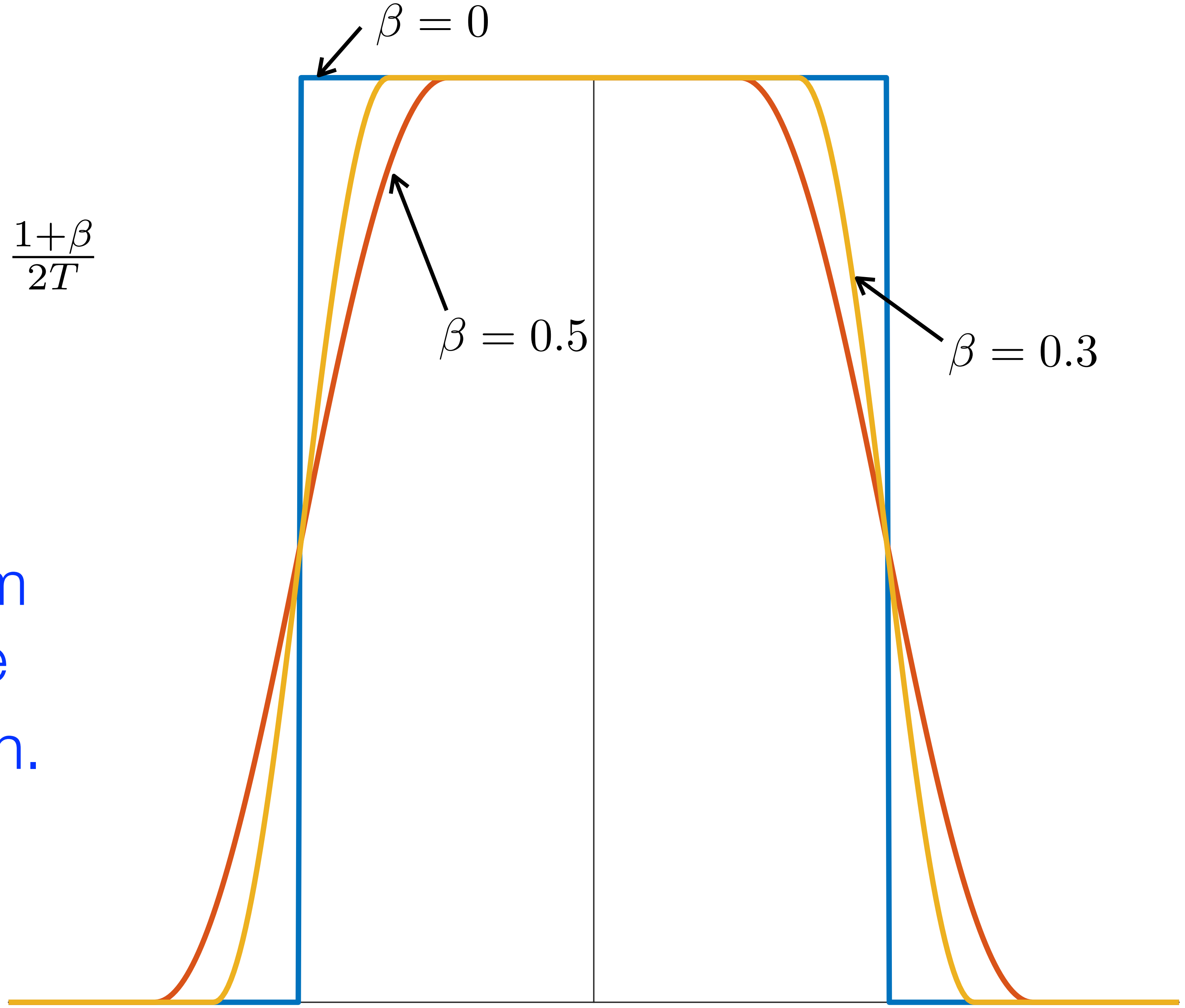


Raised cosine pulse

frequency domain

$$\breve{g}_\beta(f) = \begin{cases} T & \text{if } |f| \le \frac{1-\beta}{2T} \\ 0 & \text{if } |f| > \frac{1+\beta}{2T} \\ T\cos^2\left(\frac{\pi T}{2\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right) & \text{if } \frac{1-\beta}{2T} < |f| \le \frac{1+\beta}{2T} \end{cases}$$

rolloff factor = β



The larger it is, the smoother it transits from T to 0 in the frequency domain, and hence converges to zero faster in the time domain.

Choosing the shifted pulses as an orthonormal set

- A theorem:

  $\{p(t-mT): m \in \mathbb{Z}\}$ form an orthonormal set $\iff$ $|\breve{p}(f)|^2$ satisfies the Nyquist Criterion

- The principle of designing $p(t)$ and $q(t)$
  - Choose $\breve{p}(f)$ such that $|\hat{p}(f)|^2$ satisfies the Nyquist Criterion
  - Choose $\breve{q}(f) = \breve{p}^*(f)$
  - If $p(t) \in \mathbb{R}$ (which is normally the case), then $\breve{q}(f) = \breve{p}^*(f) = \breve{p}(-f)$ and hence $q(t) = p(-t)$.
  - For faster decay in the time-domain (less approximation error) in $t \implies$ need "larger room" for smoother transition from T to 0 in the frequency domain.

Part III. Quadrature Amplitude Modulation

A pragmatic approach to convert baseband to passband waveforms and back

How to shift the frequency response?

- We want to shift baseband signals to passband with center frequency f_c:
- Recall the frequency-shift property of Fourier Transform:

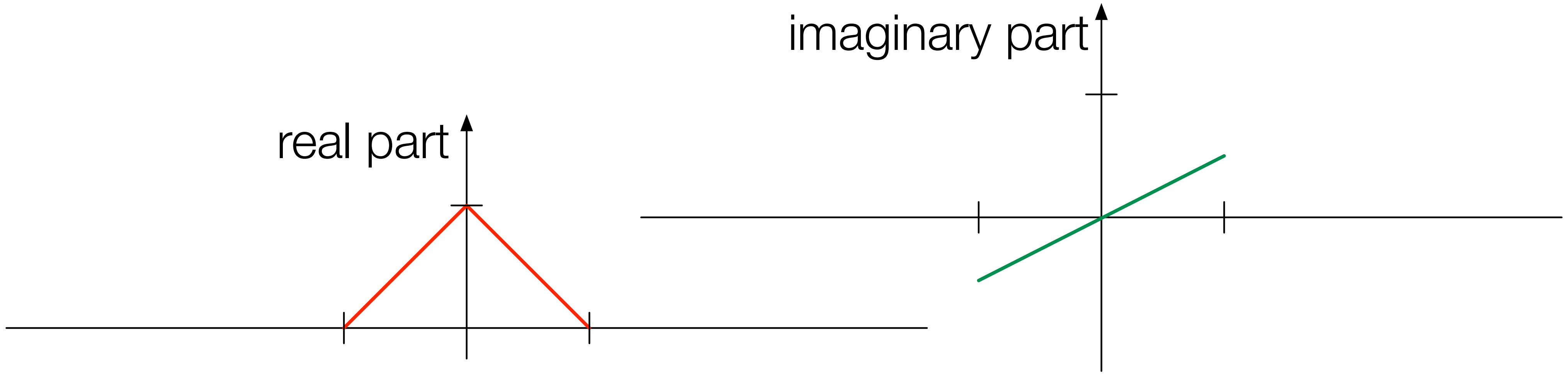
$$\exp(\mathrm{j}2\pi f_0 t)s(t) \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad \breve{s}(f - f_0)$$

- So, a naive way is to multiply the signal by a complex sinusoid
- But, at this point we don't know how to implement a **complex** signal in real world
- We can take the real part after multiplying with the complex sinusoid:

$$\mathrm{Re}\left\{\exp(\mathrm{j}2\pi f_0 t)s(t)\right\} = s(t)\cos(2\pi f_c t) \qquad\qquad s(t) \in \mathbb{R}$$

- But this is a waste of spectrum.

$$\begin{cases} \mathrm{Re}\{\breve{s}(f)\} = \mathrm{Re}\{\breve{s}(-f)\} \\ \mathrm{Im}\{\breve{s}(f)\} = -\mathrm{Im}\{\breve{s}(-f)\} \\ |\breve{s}(f)| = |\breve{s}(-f)| \\ \angle\breve{s}(f) = -\angle\breve{s}(-f) \bmod 2\pi \end{cases}$$

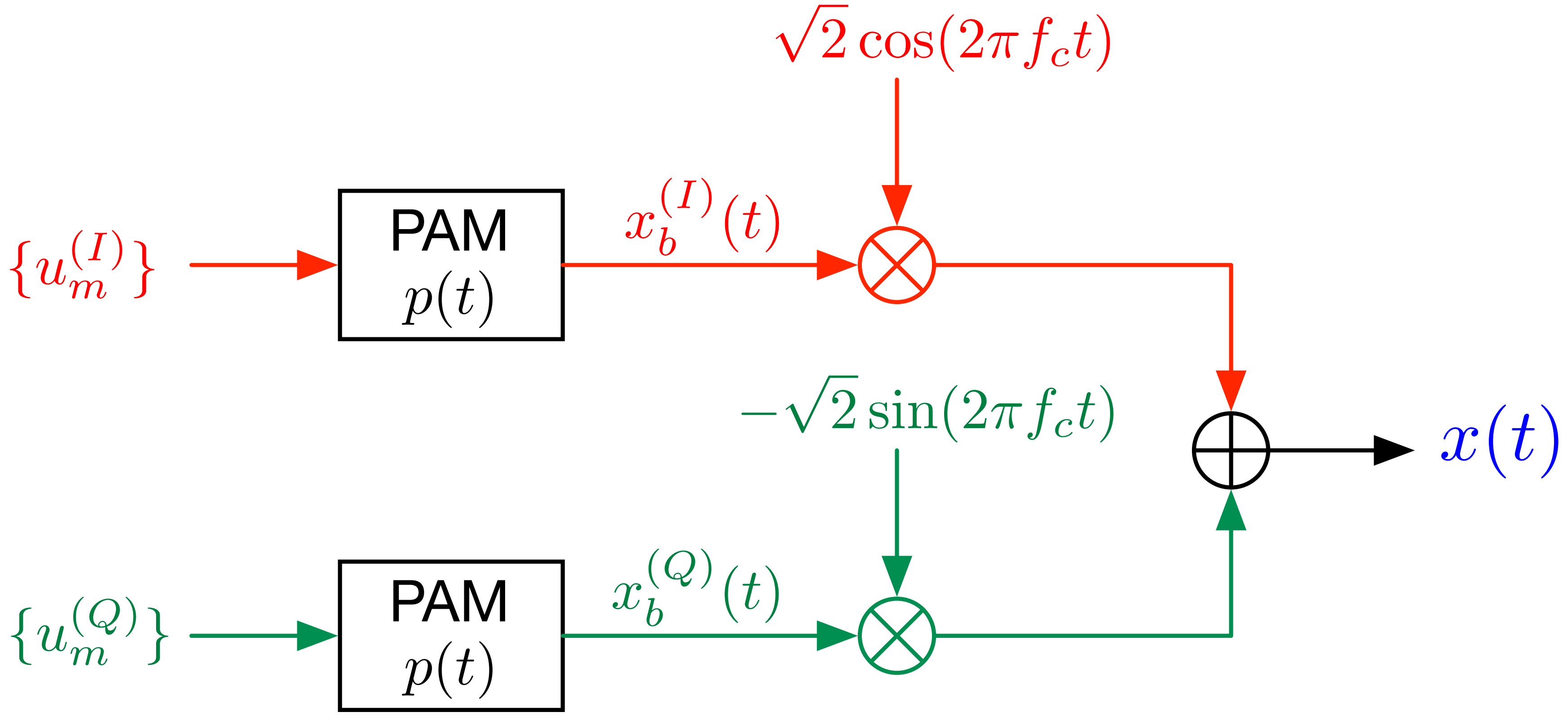


Two degrees of freedom for complex signal

- Why not multiplex two individual baseband waveforms?

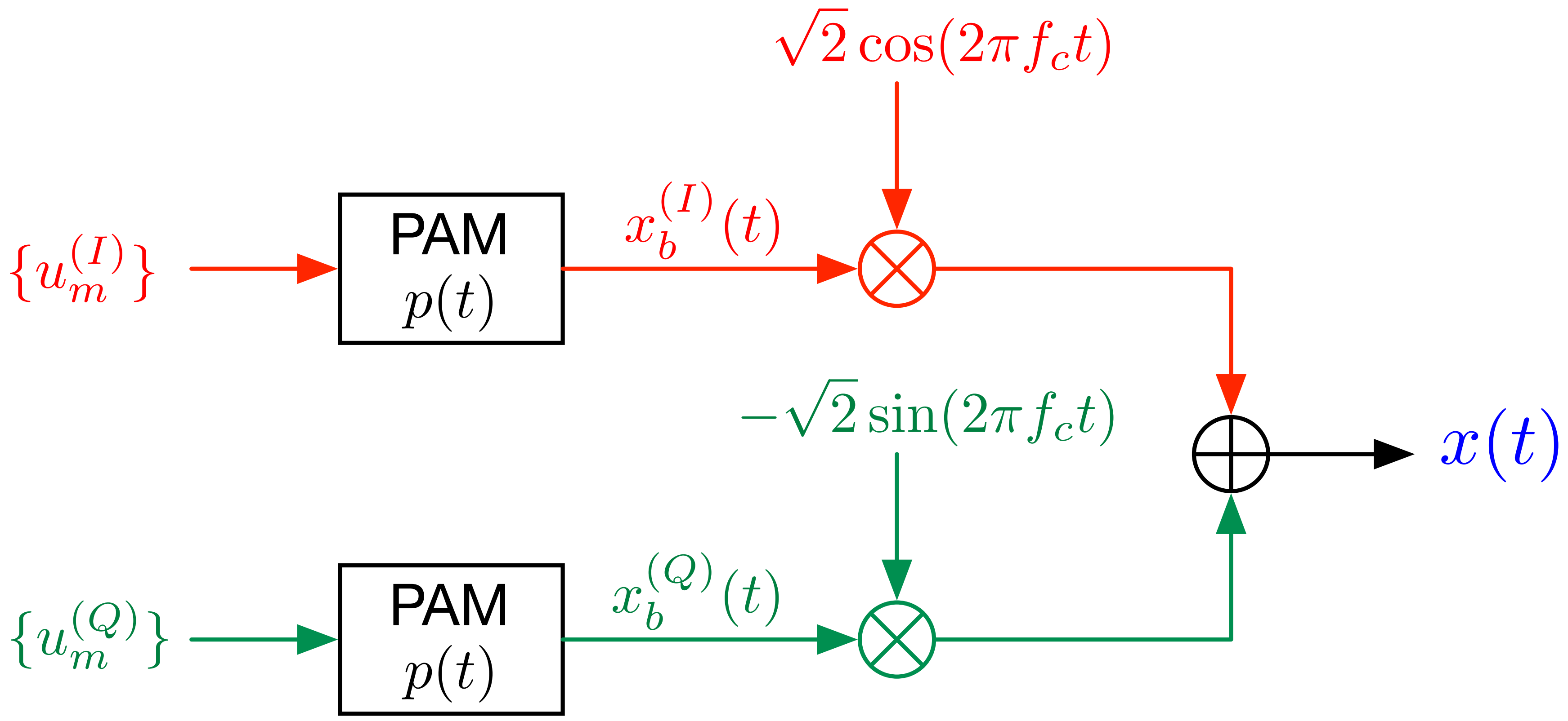
$$x(t) = x_b^{(I)}(t)\sqrt{2}\cos(2\pi f_c t) - x_b^{(Q)}(t)\sqrt{2}\sin(2\pi f_c t)$$

- Quadrature amplitude modulation (QAM):



QAM modulation: real-domain implementation

$$x(t) = \underbrace{x_b^{(I)}(t)}_{\text{in-phase component}}\sqrt{2}\cos(2\pi f_c t) - \underbrace{x_b^{(Q)}(t)}_{\text{quadrature component}}\sqrt{2}\sin(2\pi f_c t)$$

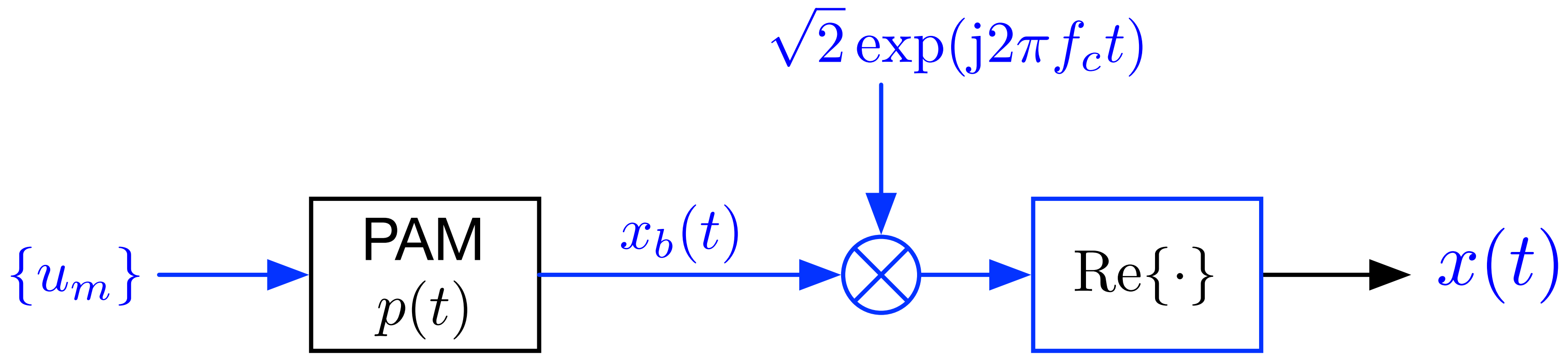


QAM modulation: equivalent complex

$$x(t) = x_b^{(I)}(t)\sqrt{2}\cos(2\pi f_c t) - x_b^{(Q)}(t)\sqrt{2}\sin(2\pi f_c t)$$

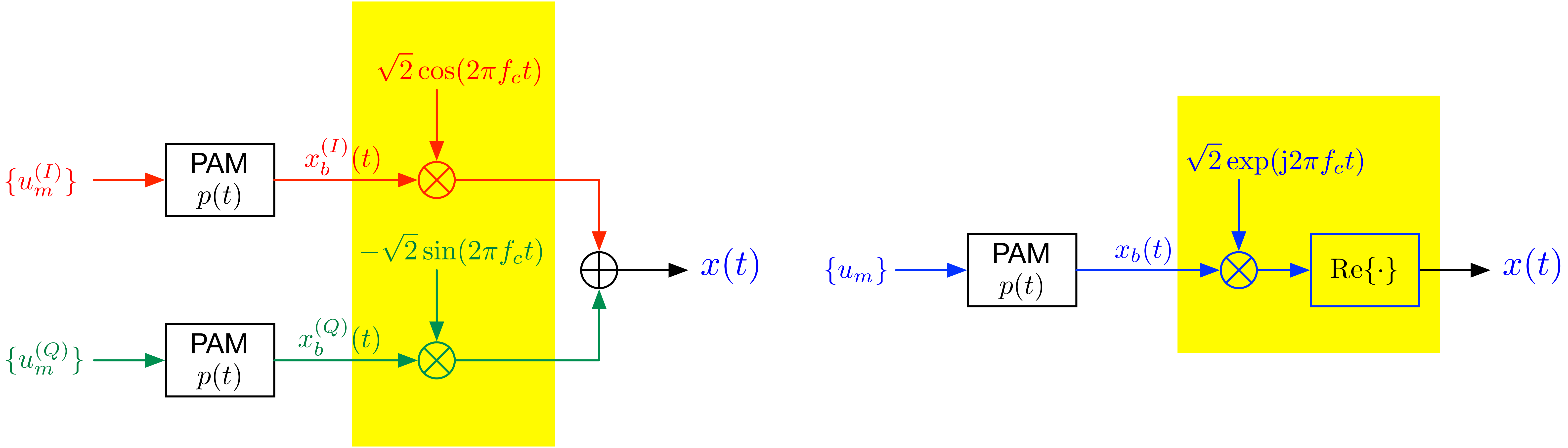
$$= \sqrt{2}\mathrm{Re}\left\{x_b(t)\exp(\mathrm{j}2\pi f_c t)\right\}$$

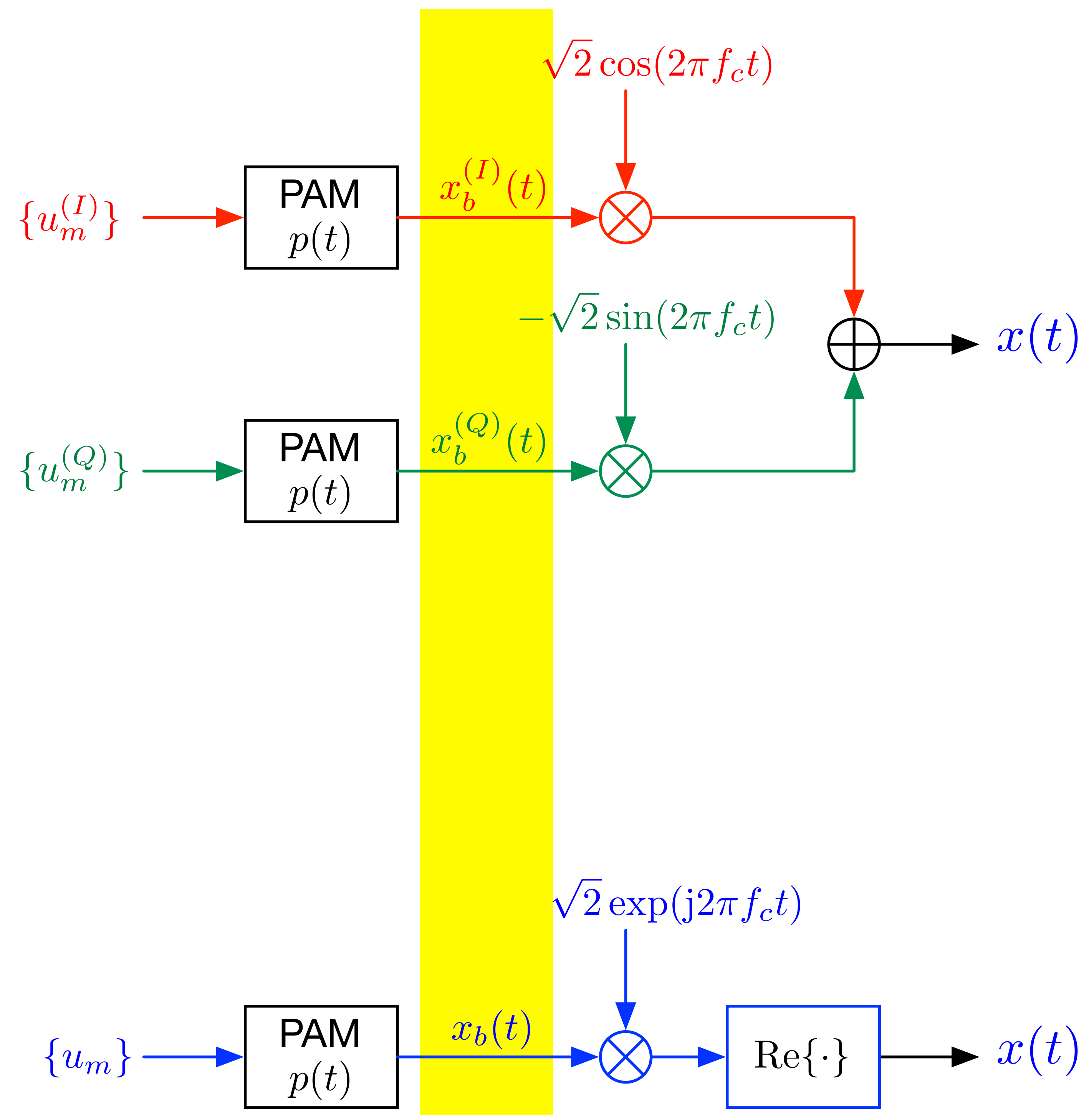
$$x_b(t) \triangleq x_b^{(I)}(t) + \mathrm{j}x_b^{(Q)}(t)$$

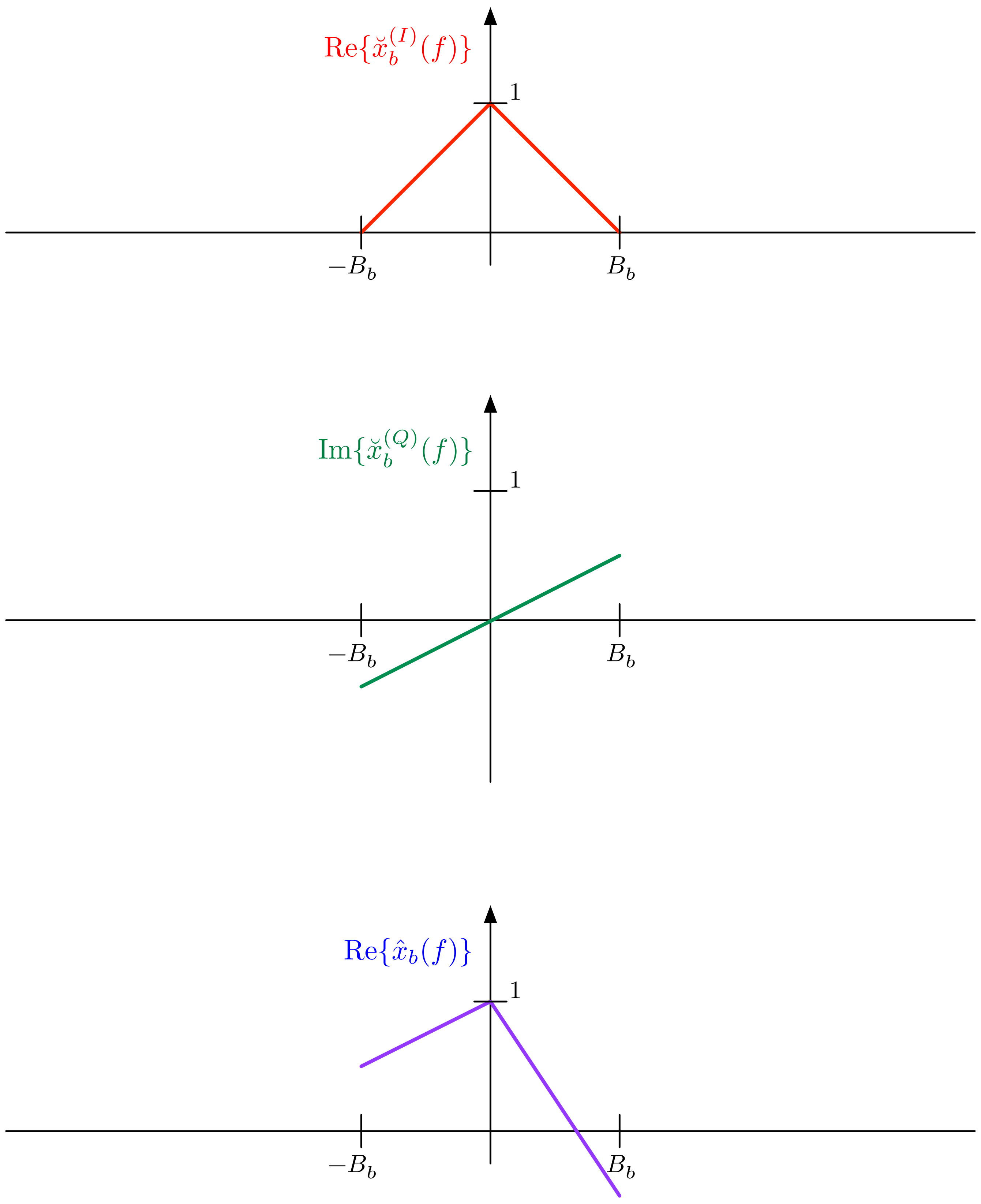


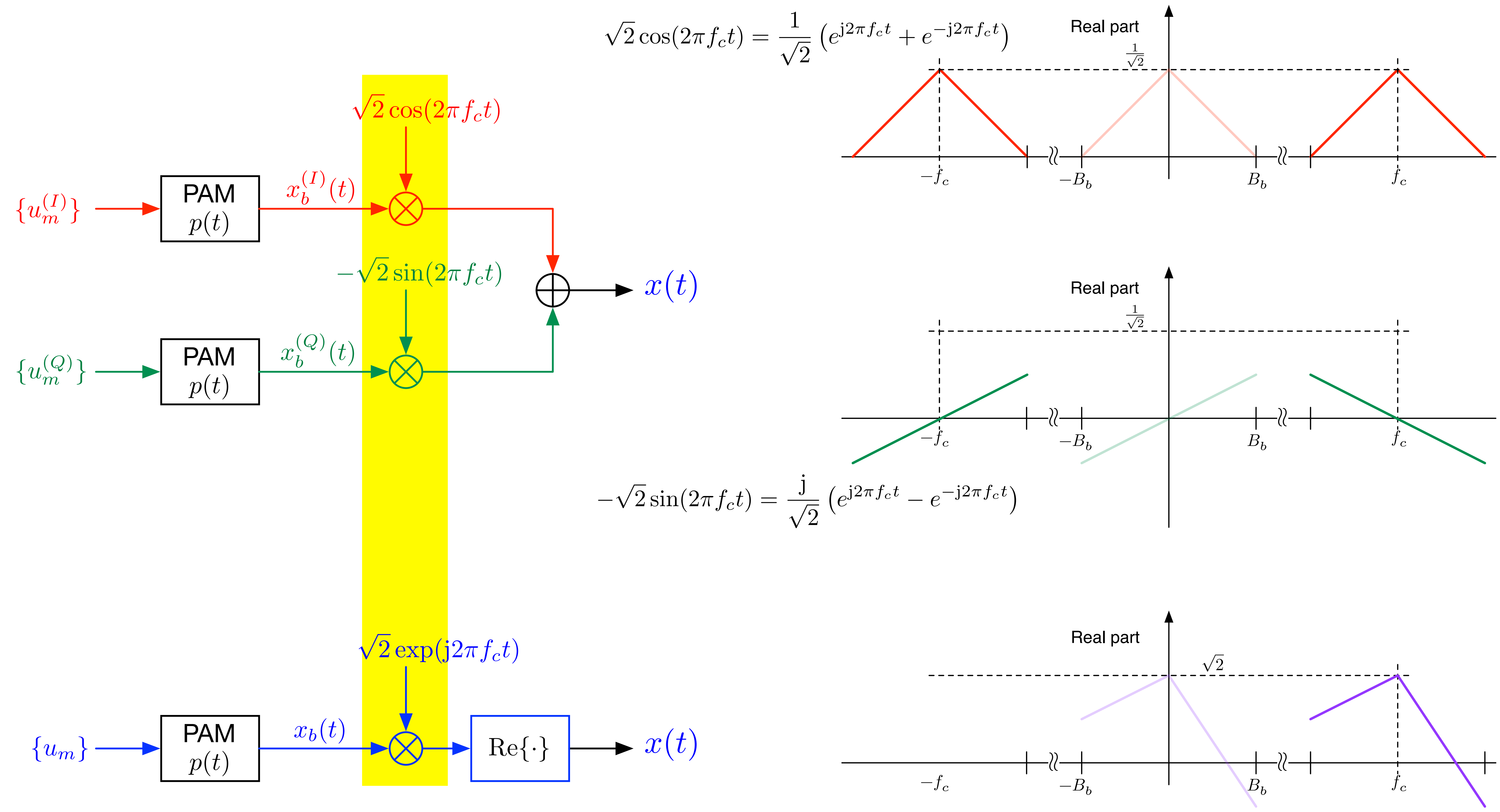
$$u_m \triangleq u_m^{(I)} + \mathrm{j}u_m^{(Q)}$$

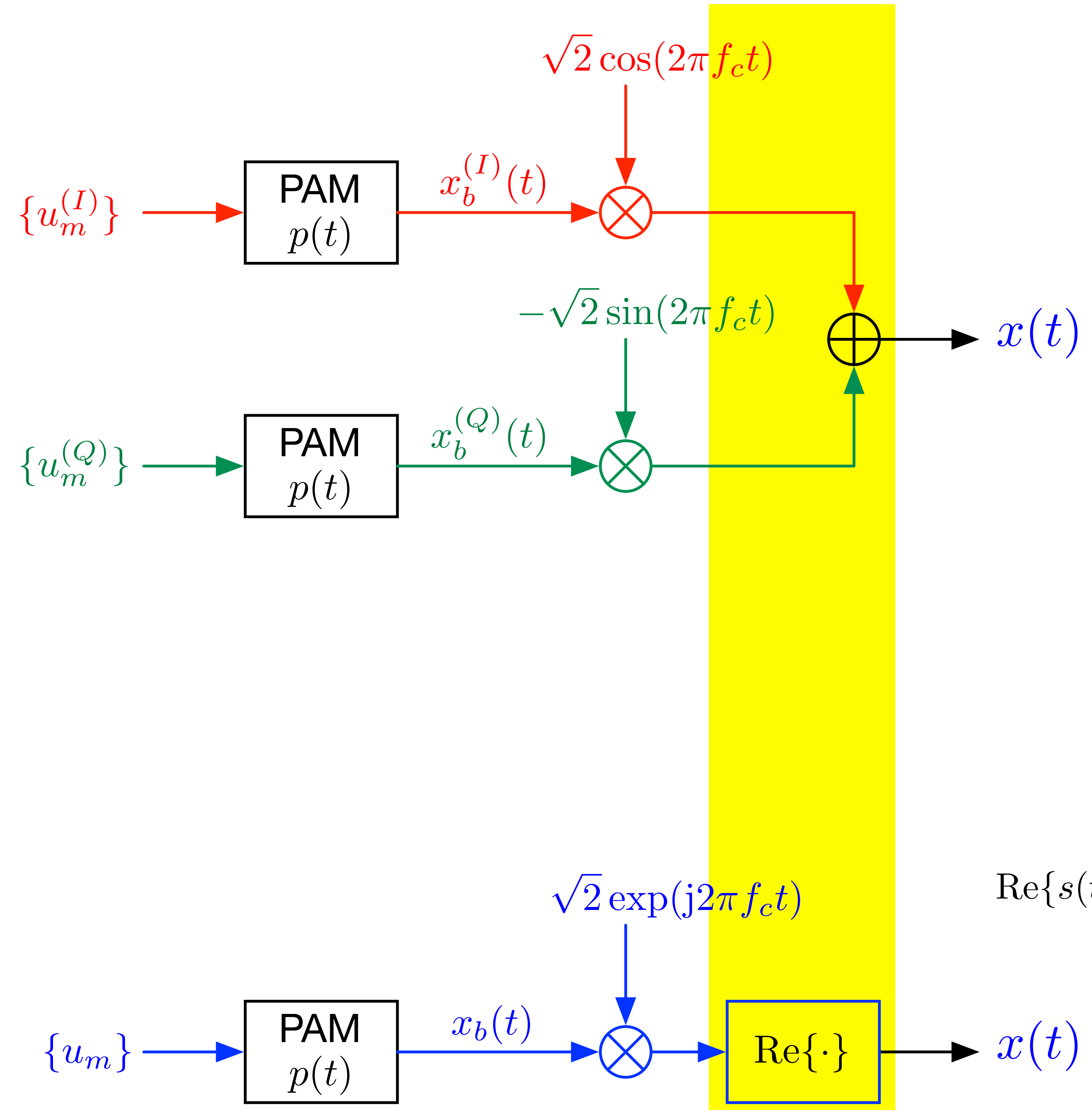
Up conversion

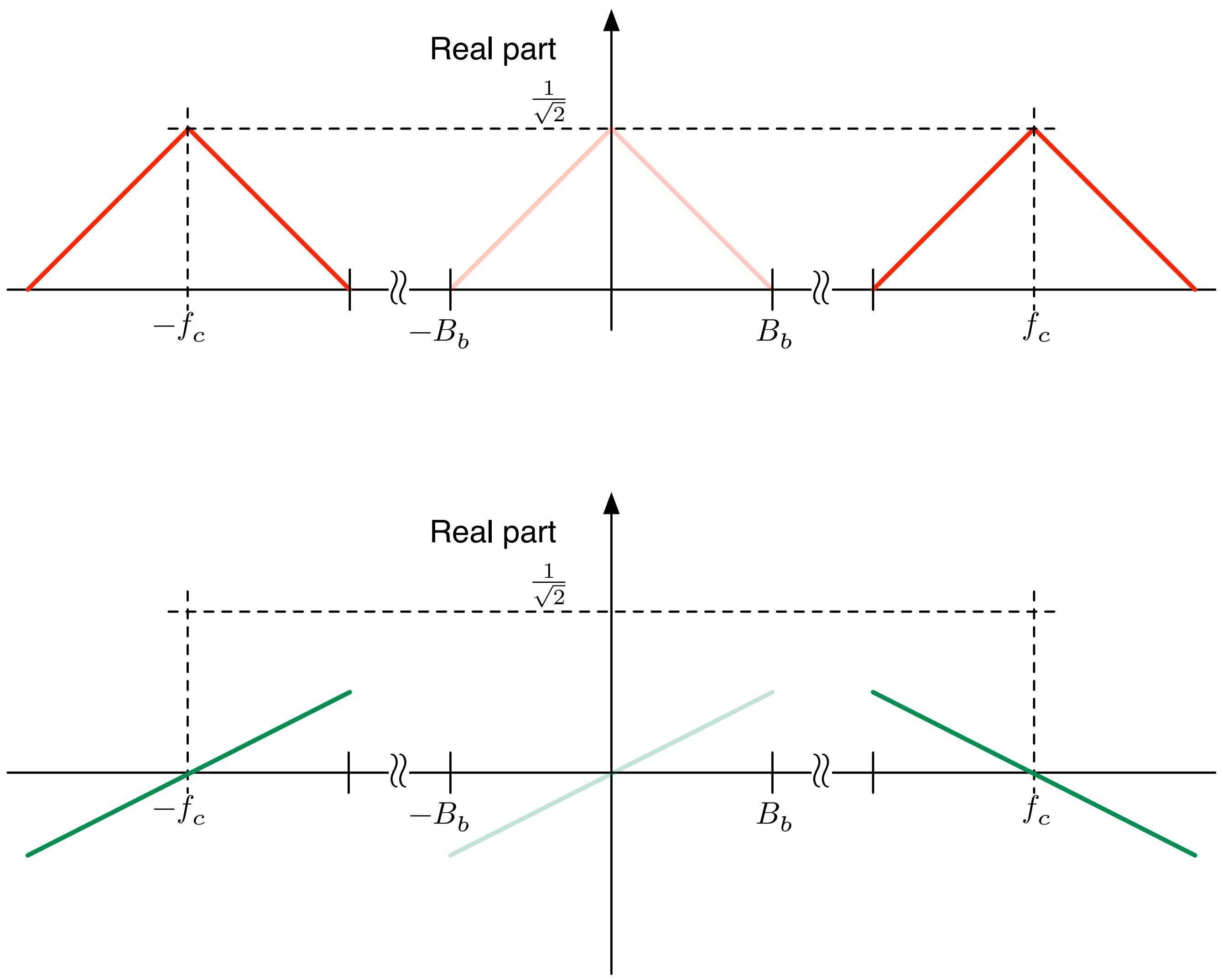



$\{u_m^{(I)}\}$
PAM
$p(t)$
$x_b^{(I)}(t)$
$\sqrt{2}\cos(2\pi f_c t)$
$\{u_m^{(Q)}\}$
PAM
$p(t)$
$x_b^{(Q)}(t)$
$-\sqrt{2}\sin(2\pi f_c t)$
$x(t)$
$\{u_m\}$
PAM
$p(t)$
$x_b(t)$
$\sqrt{2}\exp(\mathrm{j}2\pi f_c t)$
$\mathrm{Re}\{\cdot\}$
$x(t)$

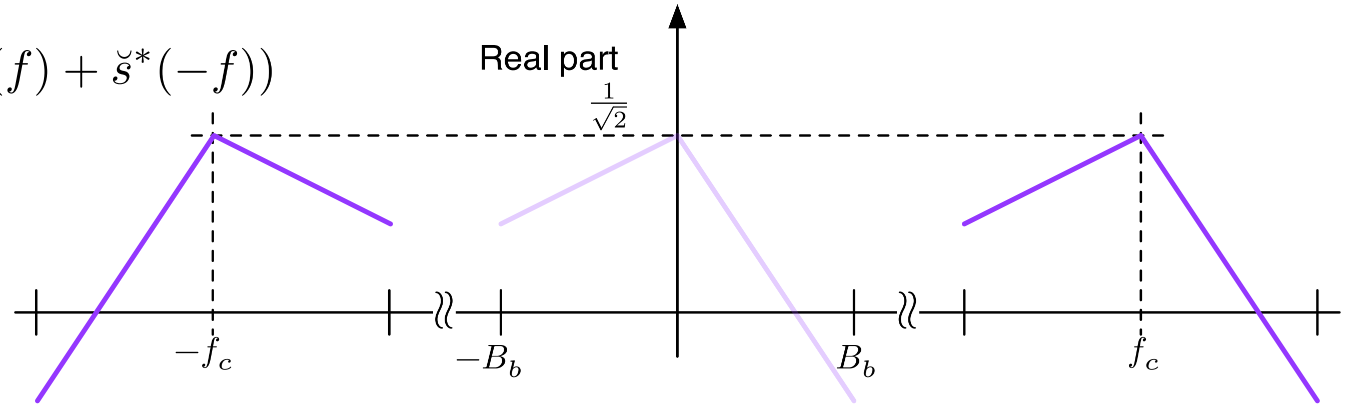

$\mathrm{Re}\{\breve{x}_b^{(I)}(f)\}$
1
$-B_b$
B_b
$\mathrm{Im}\{\breve{x}_b^{(Q)}(f)\}$
1
$-B_b$
B_b
$\mathrm{Re}\{\hat{x}_b(f)\}$
1
$-B_b$
B_b


$\sqrt{2}\cos(2\pi f_c t) = \frac{1}{\sqrt{2}}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right)$
$-\sqrt{2}\sin(2\pi f_c t) = \frac{j}{\sqrt{2}}\left(e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right)$
$\{u_m^{(I)}\}$
PAM
$p(t)$
$x_b^{(I)}(t)$
$\sqrt{2}\cos(2\pi f_c t)$
$\{u_m^{(Q)}\}$
$x_b^{(Q)}(t)$
$-\sqrt{2}\sin(2\pi f_c t)$
$x(t)$
$\{u_m\}$
$x_b(t)$
$\sqrt{2}\exp(j2\pi f_c t)$
$\mathrm{Re}\{\cdot\}$
Real part
$\frac{1}{\sqrt{2}}$
$\sqrt{2}$
$-f_c$
$-B_b$
B_b
f_c

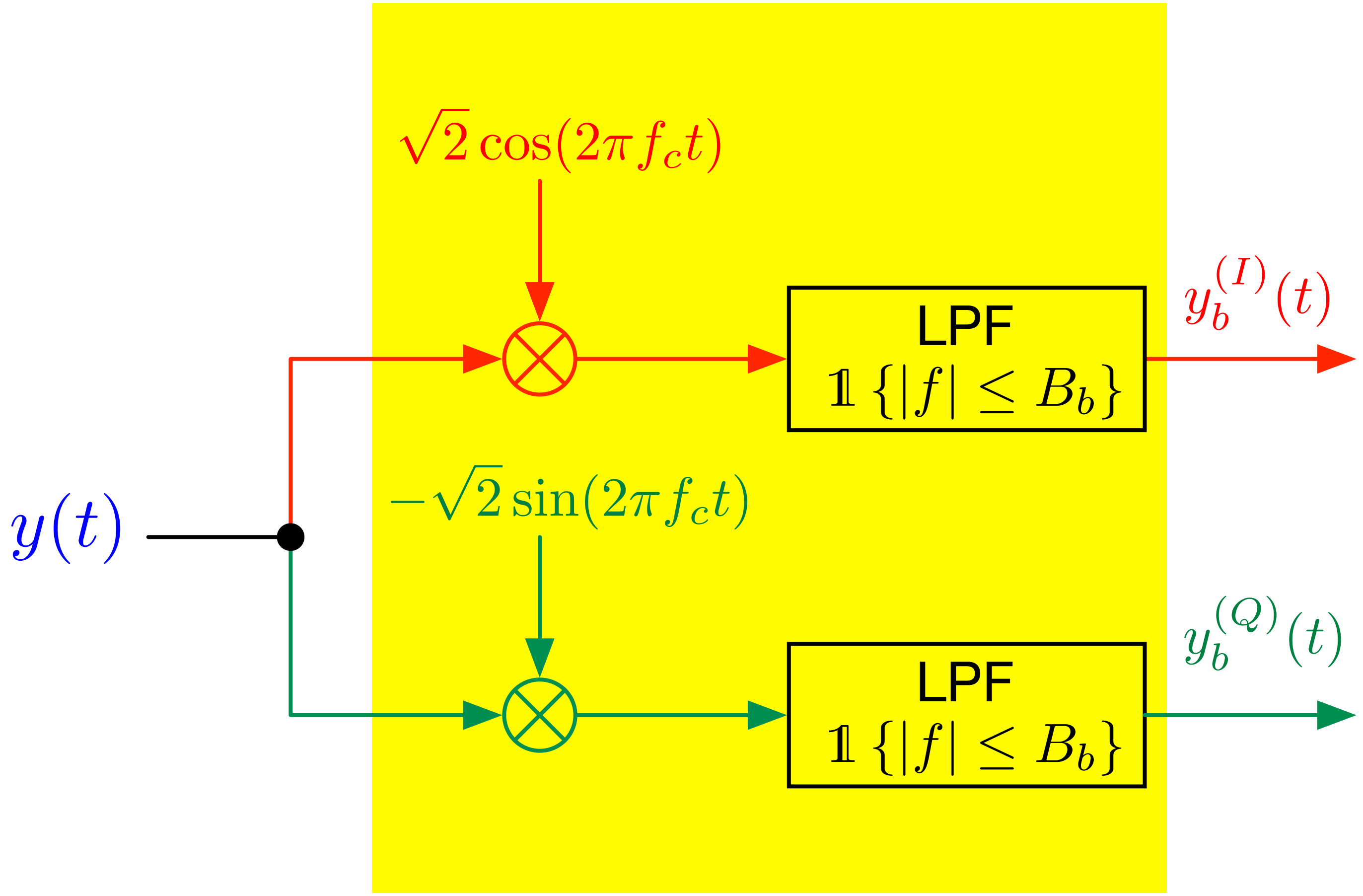


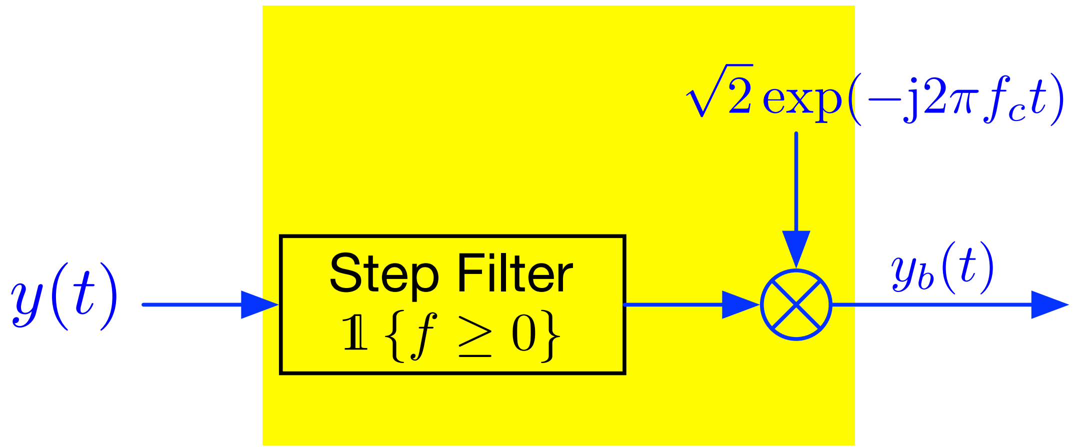


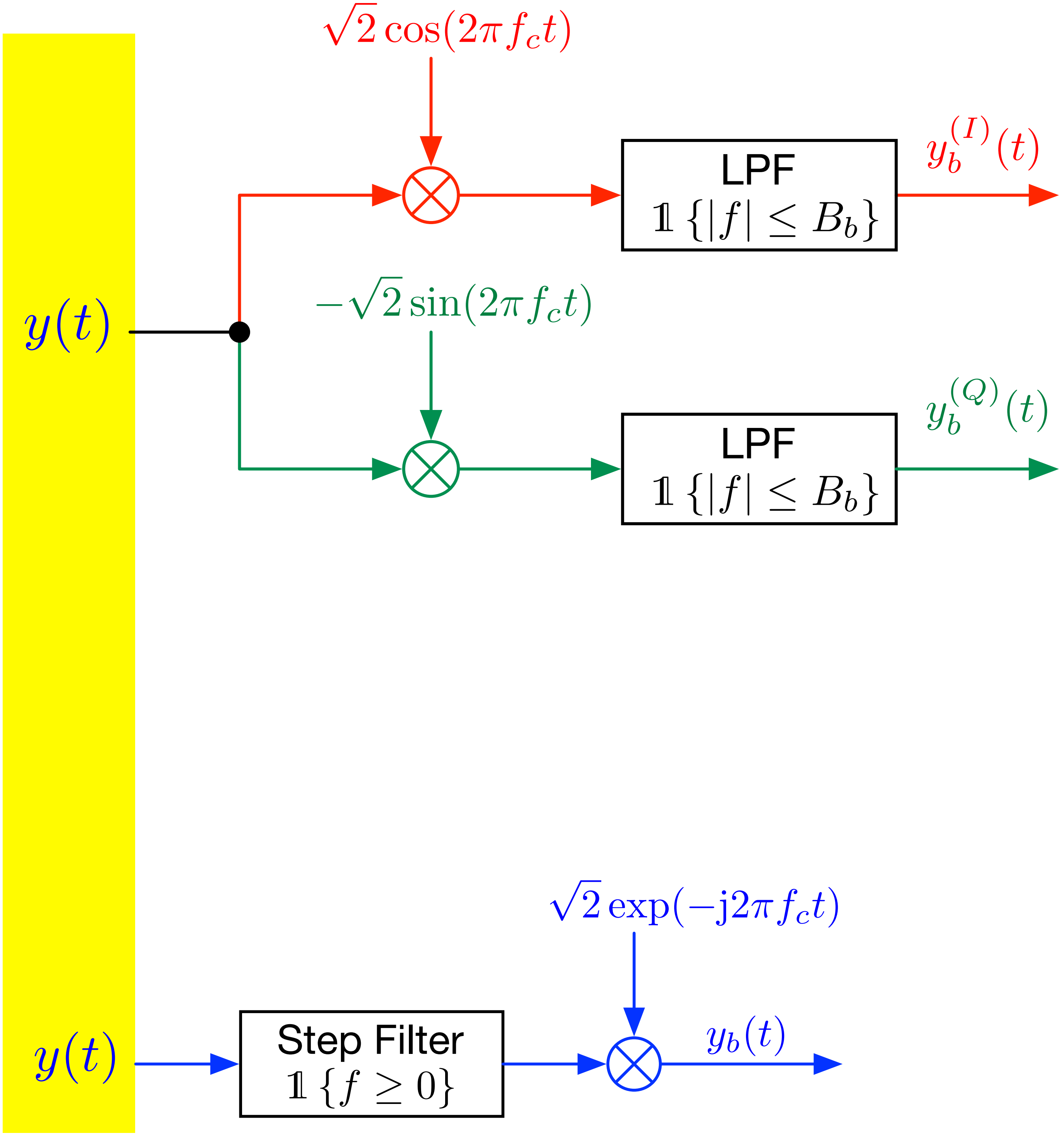
$$\mathrm{Re}\{s(t)\} \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{2}(\breve{s}(f) + \breve{s}^*(-f))$$



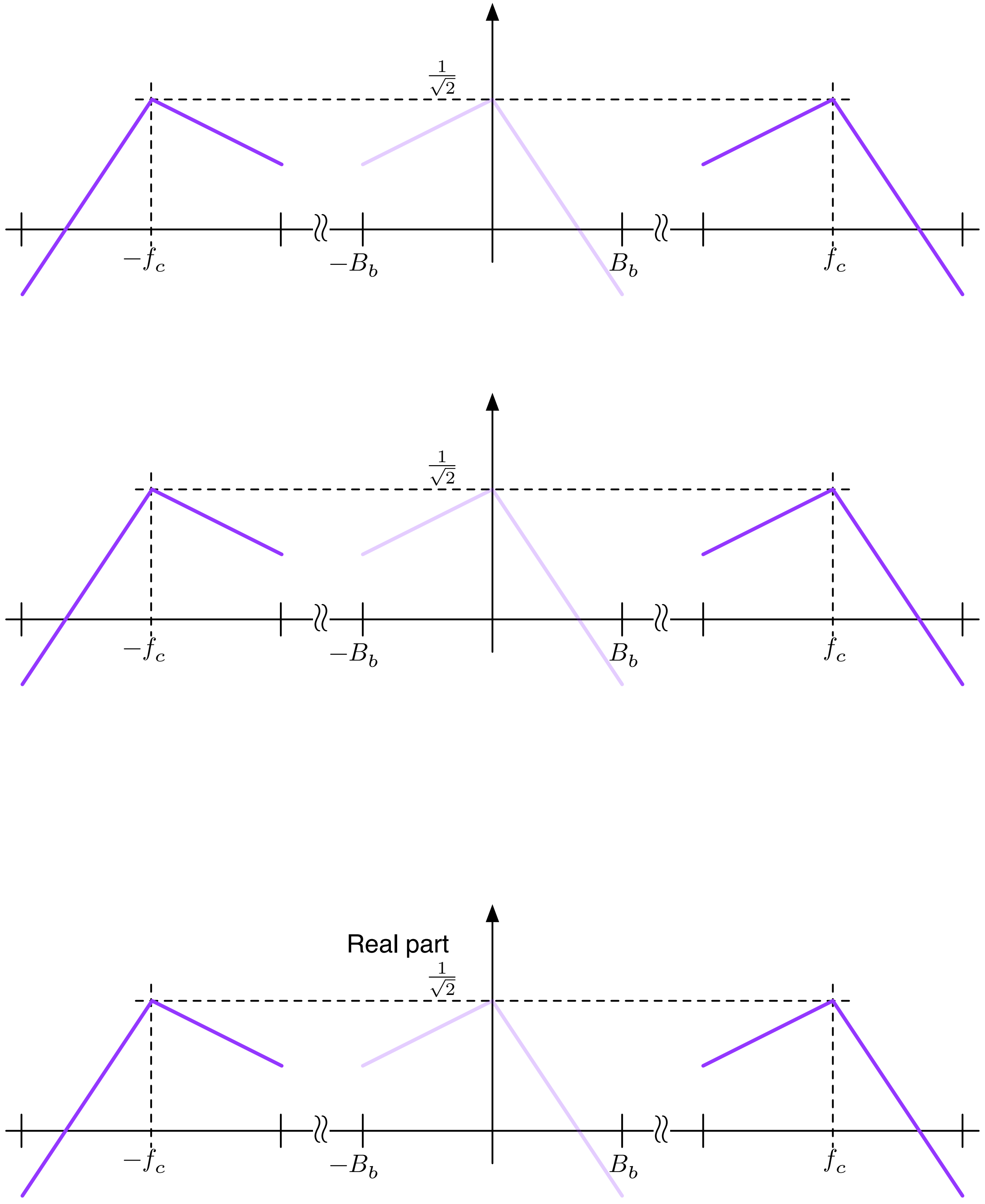
Down conversion



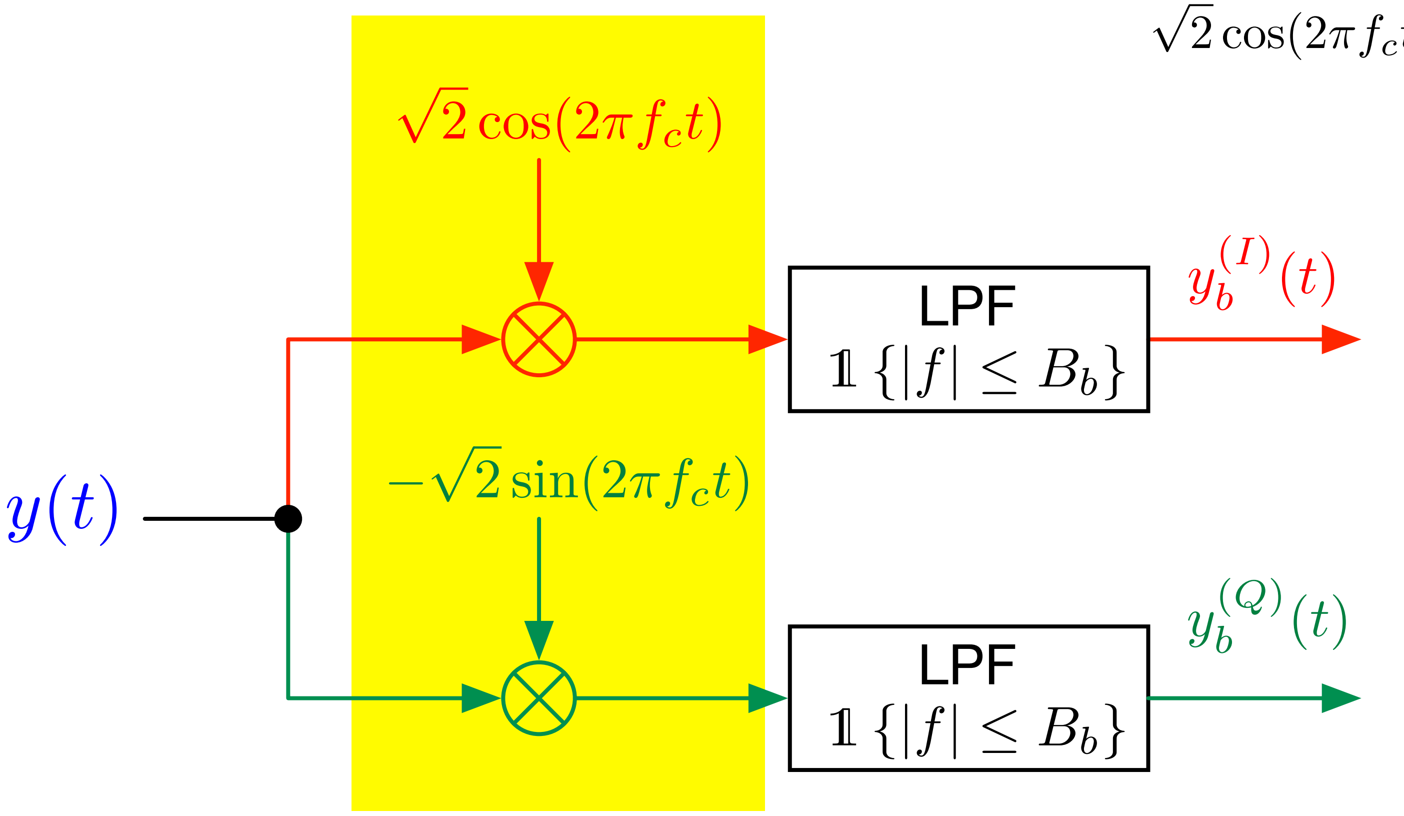




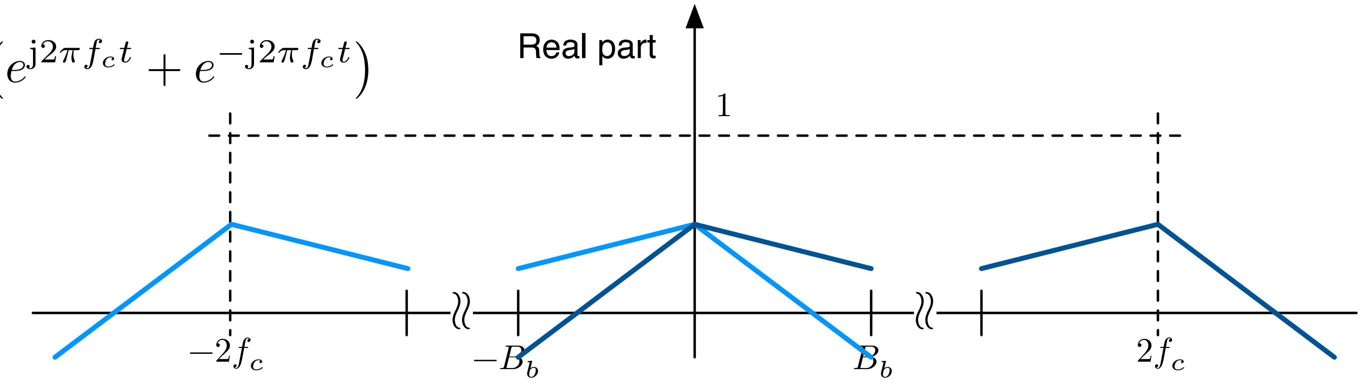
$y(t)$
$\sqrt{2}\cos(2\pi f_c t)$
LPF
$\mathbb{1}\{|f| \le B_b\}$
$y_b^{(I)}(t)$
$-\sqrt{2}\sin(2\pi f_c t)$
LPF
$\mathbb{1}\{|f| \le B_b\}$
$y_b^{(Q)}(t)$
$y(t)$
Step Filter
$\mathbb{1}\{f \ge 0\}$
$\sqrt{2}\exp(-\mathrm{j}2\pi f_c t)$
$y_b(t)$

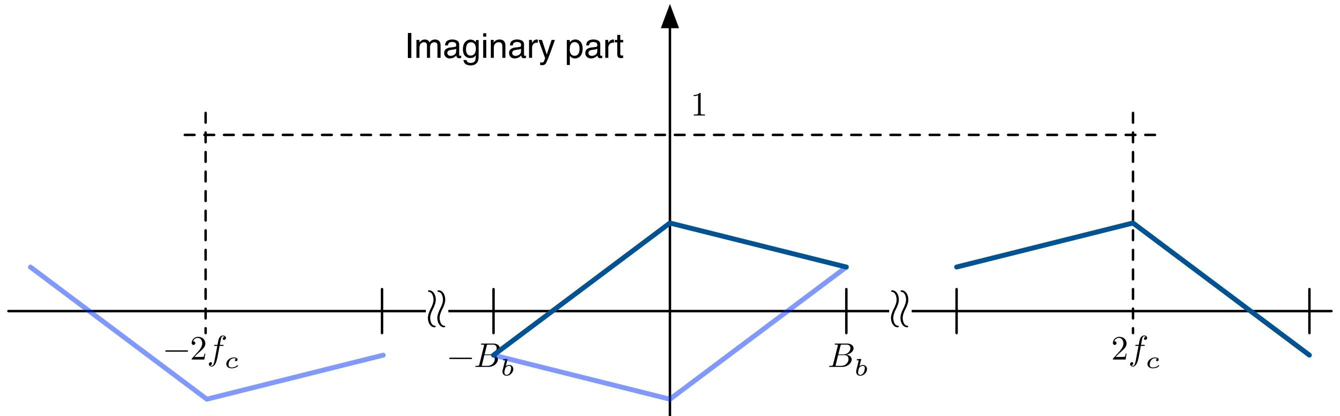


$\frac{1}{\sqrt{2}}$
$-f_c$
$-B_b$
B_b
f_c
$\frac{1}{\sqrt{2}}$
$-f_c$
$-B_b$
B_b
f_c
Real part
$\frac{1}{\sqrt{2}}$
$-f_c$
$-B_b$
B_b
f_c

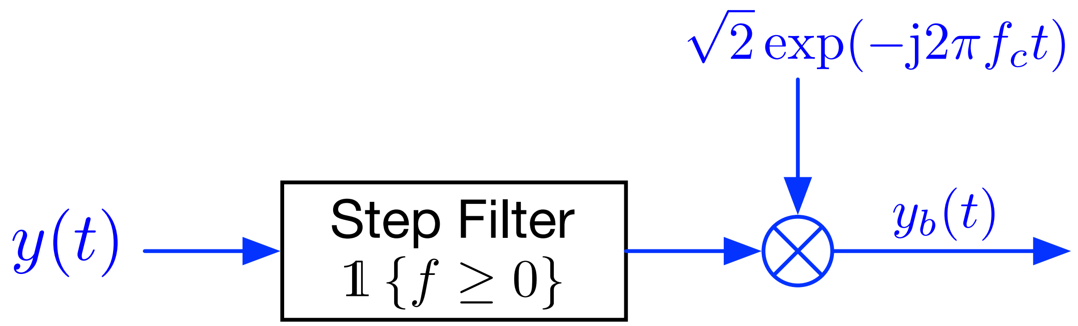


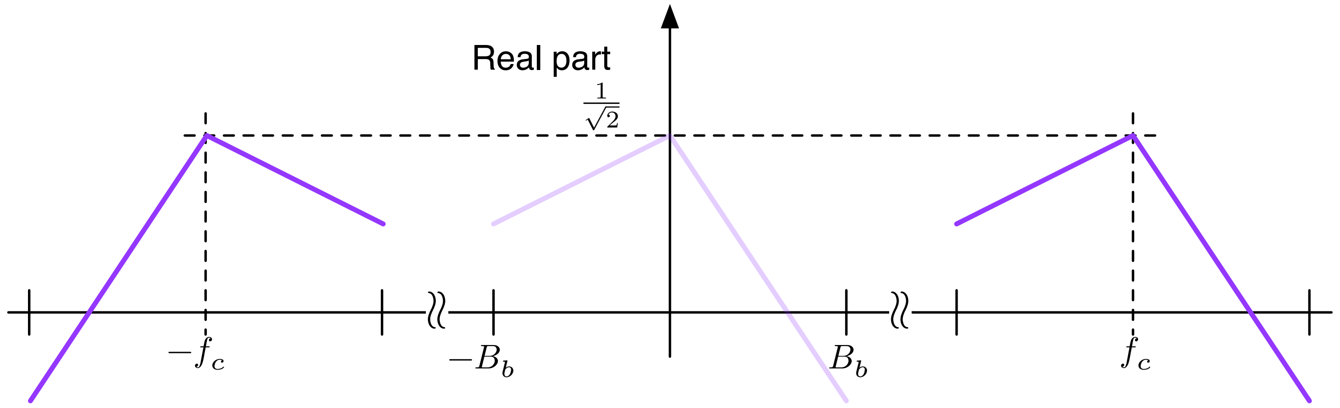
$$\sqrt{2}\cos(2\pi f_c t) = \frac{1}{\sqrt{2}}\left(e^{\mathrm{j}2\pi f_c t} + e^{-\mathrm{j}2\pi f_c t}\right)$$

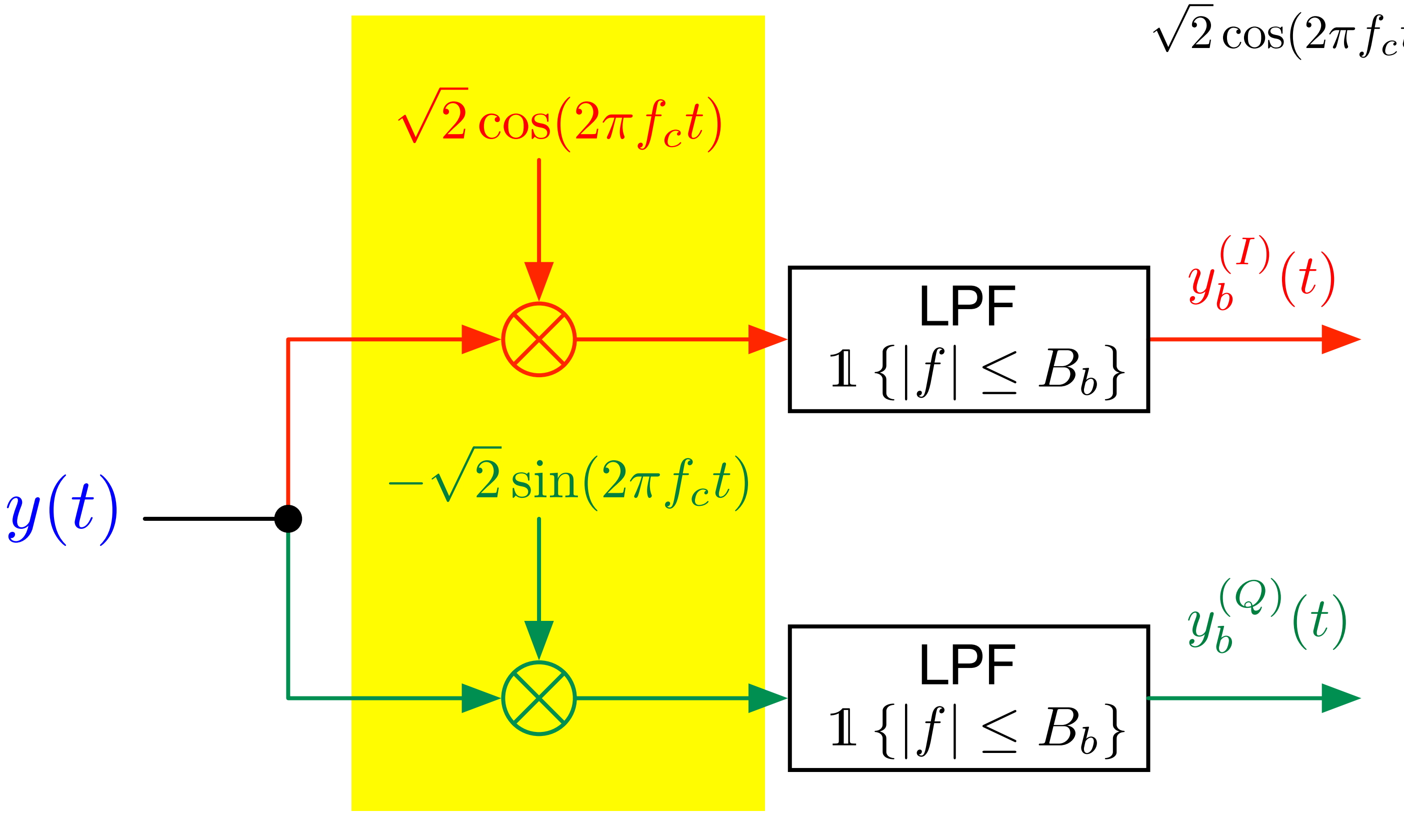




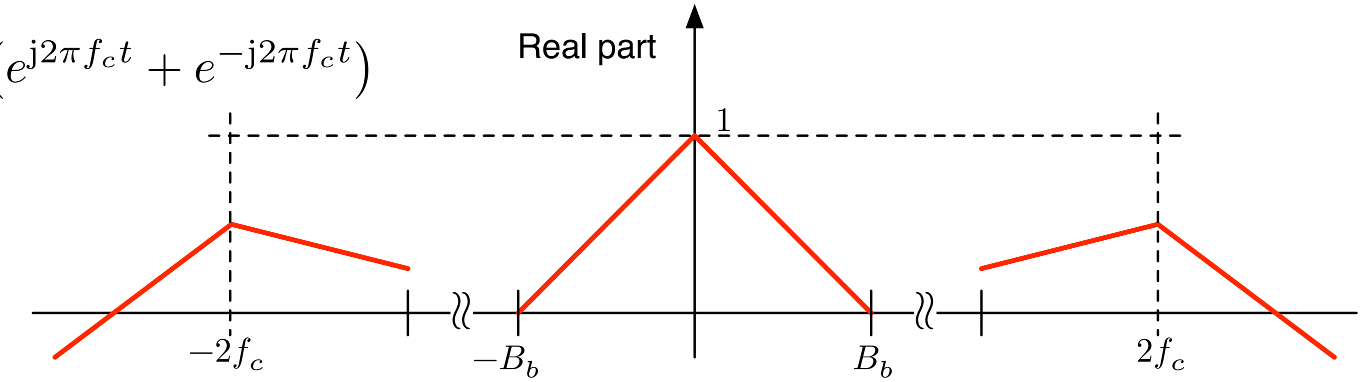
$$-\sqrt{2}\sin(2\pi f_c t) = \frac{\mathrm{j}}{\sqrt{2}}\left(e^{\mathrm{j}2\pi f_c t} - e^{-\mathrm{j}2\pi f_c t}\right)$$

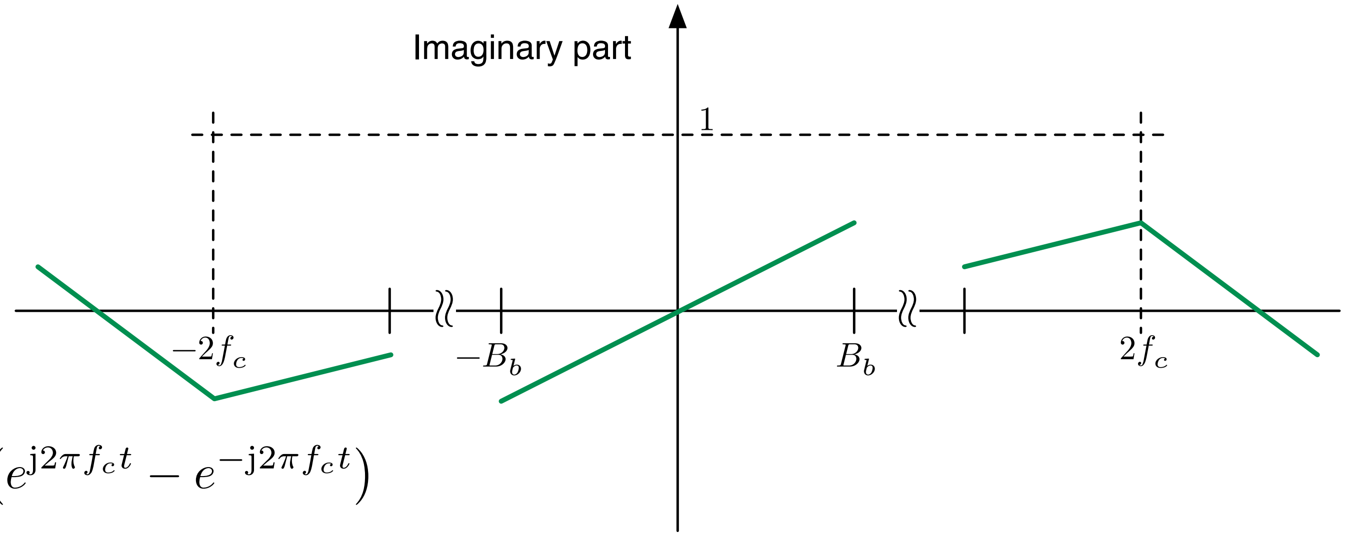




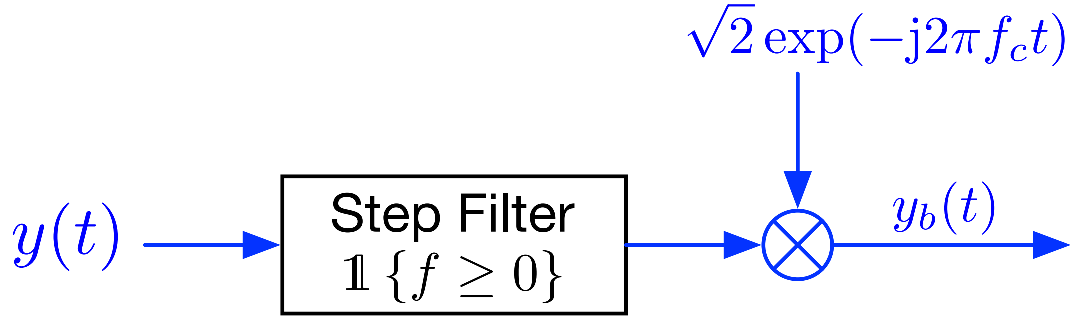


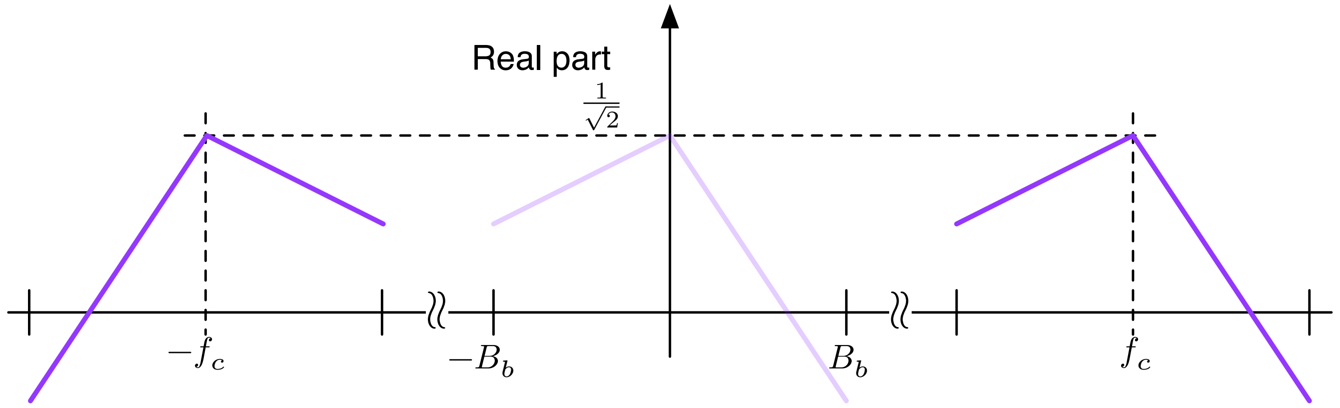
$$\sqrt{2}\cos(2\pi f_c t) = \frac{1}{\sqrt{2}}\left(e^{\mathrm{j}2\pi f_c t} + e^{-\mathrm{j}2\pi f_c t}\right)$$

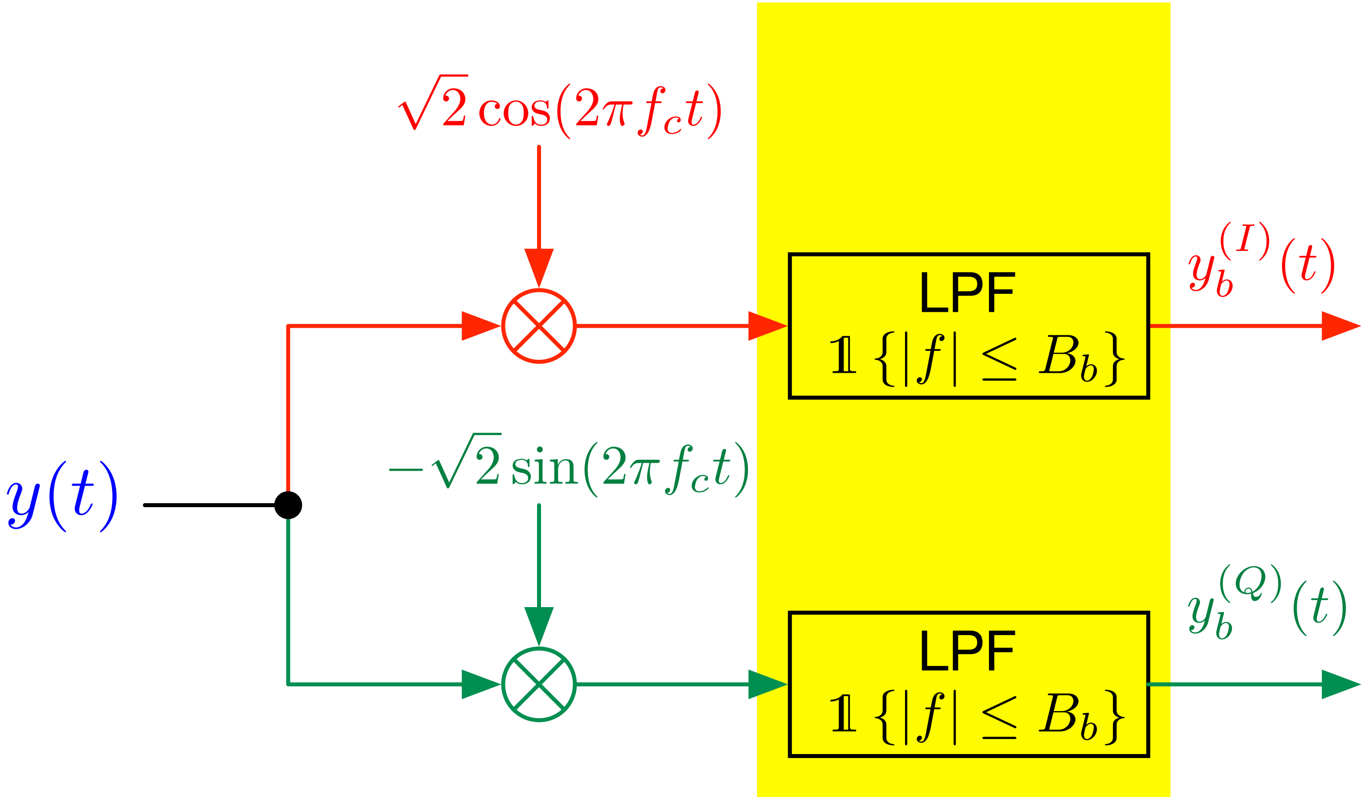


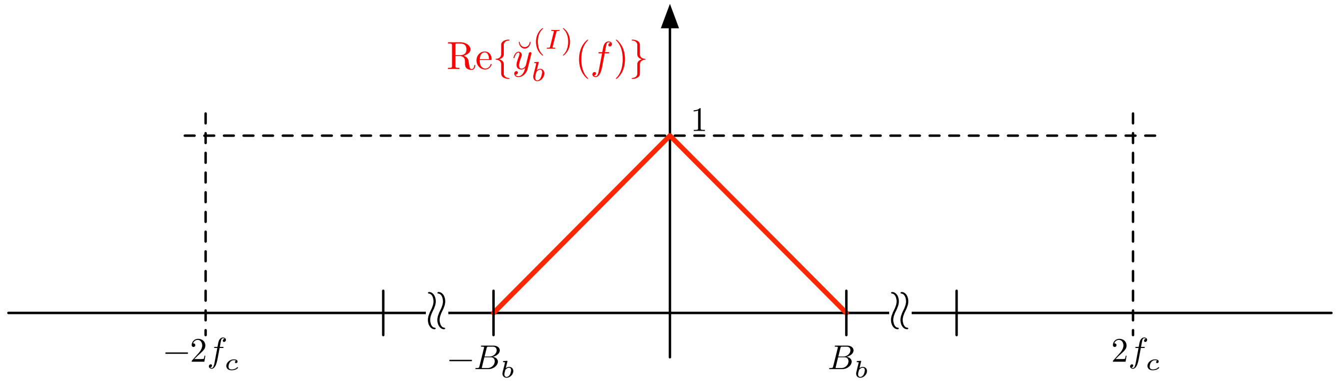


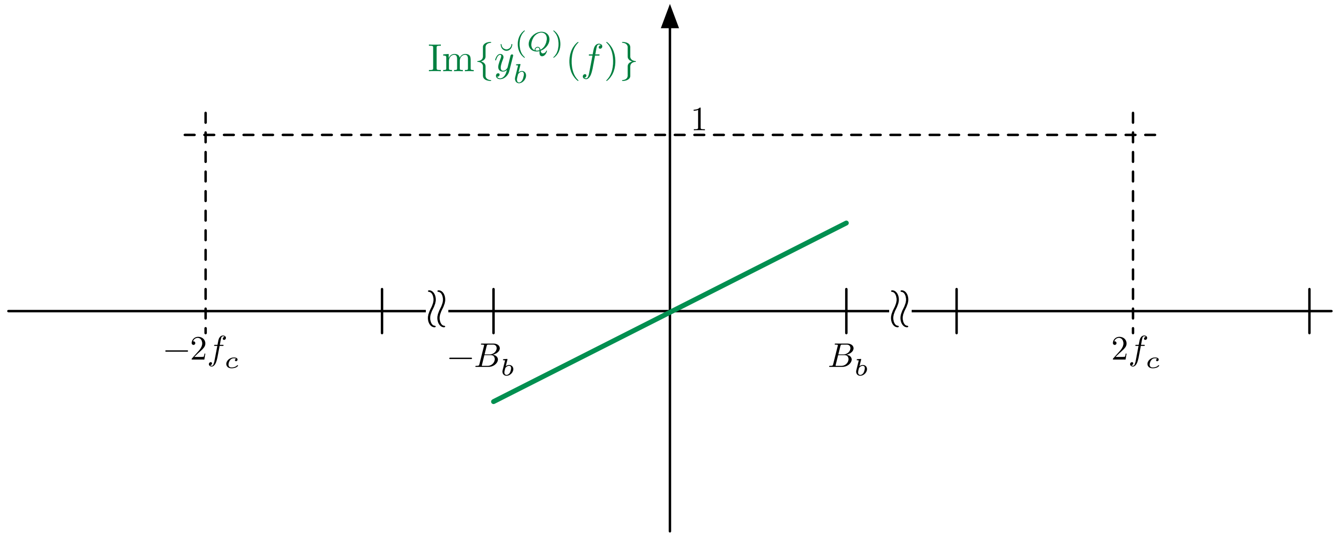
$$-\sqrt{2}\sin(2\pi f_c t) = \frac{\mathrm{j}}{\sqrt{2}}\left(e^{\mathrm{j}2\pi f_c t} - e^{-\mathrm{j}2\pi f_c t}\right)$$

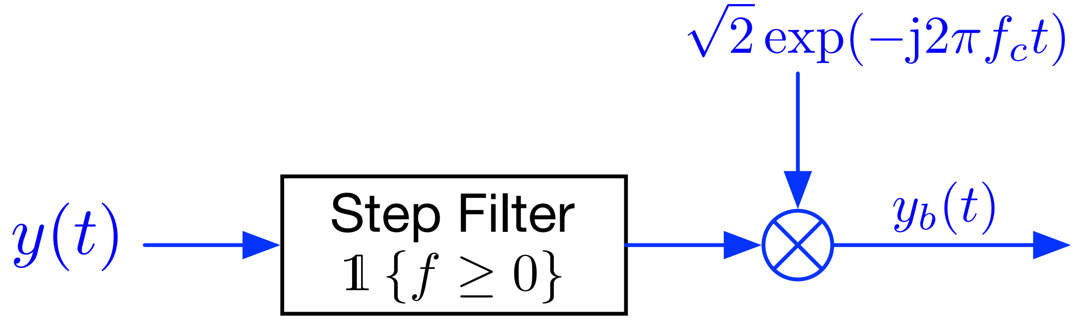


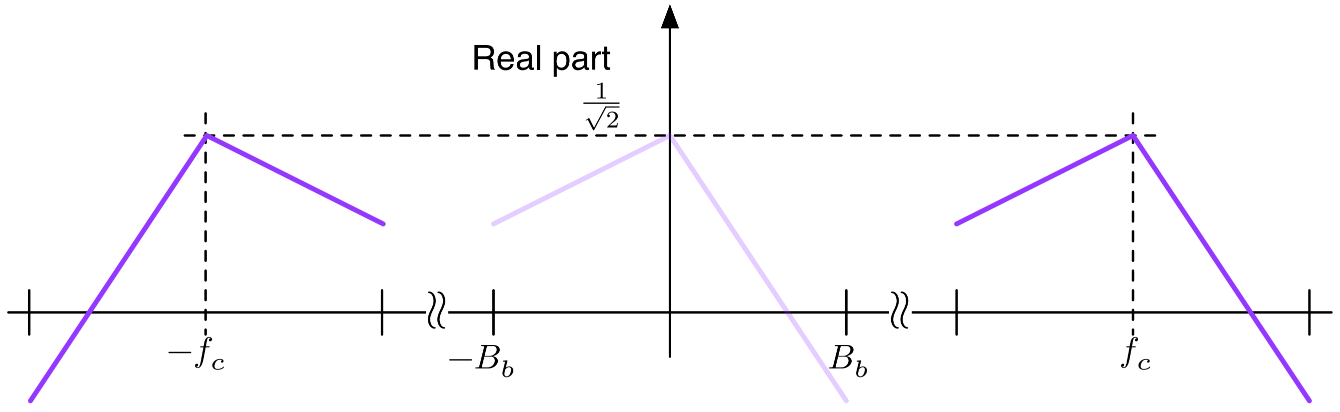


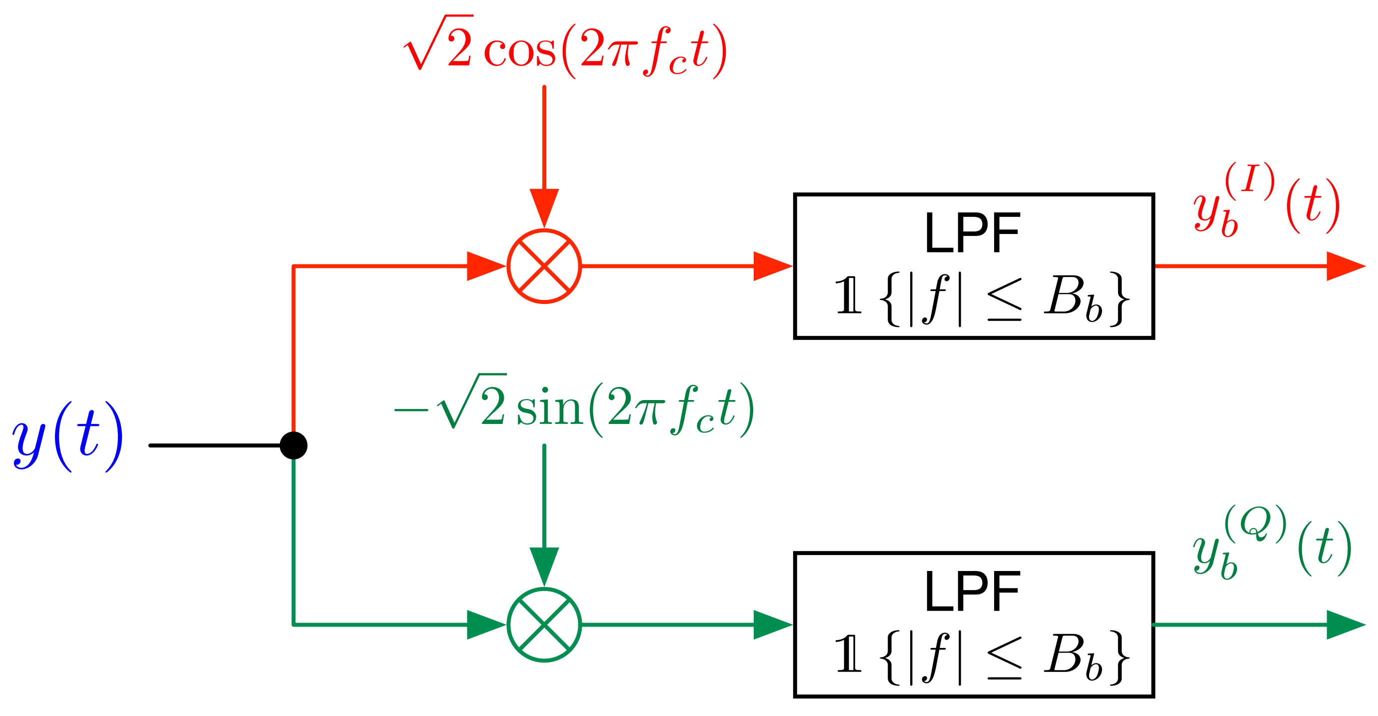

$\sqrt{2}\cos(2\pi f_c t)$
$y(t)$
LPF
$\mathbb{1}\{|f| \le B_b\}$
$y_b^{(I)}(t)$
$-\sqrt{2}\sin(2\pi f_c t)$
LPF
$\mathbb{1}\{|f| \le B_b\}$
$y_b^{(Q)}(t)$

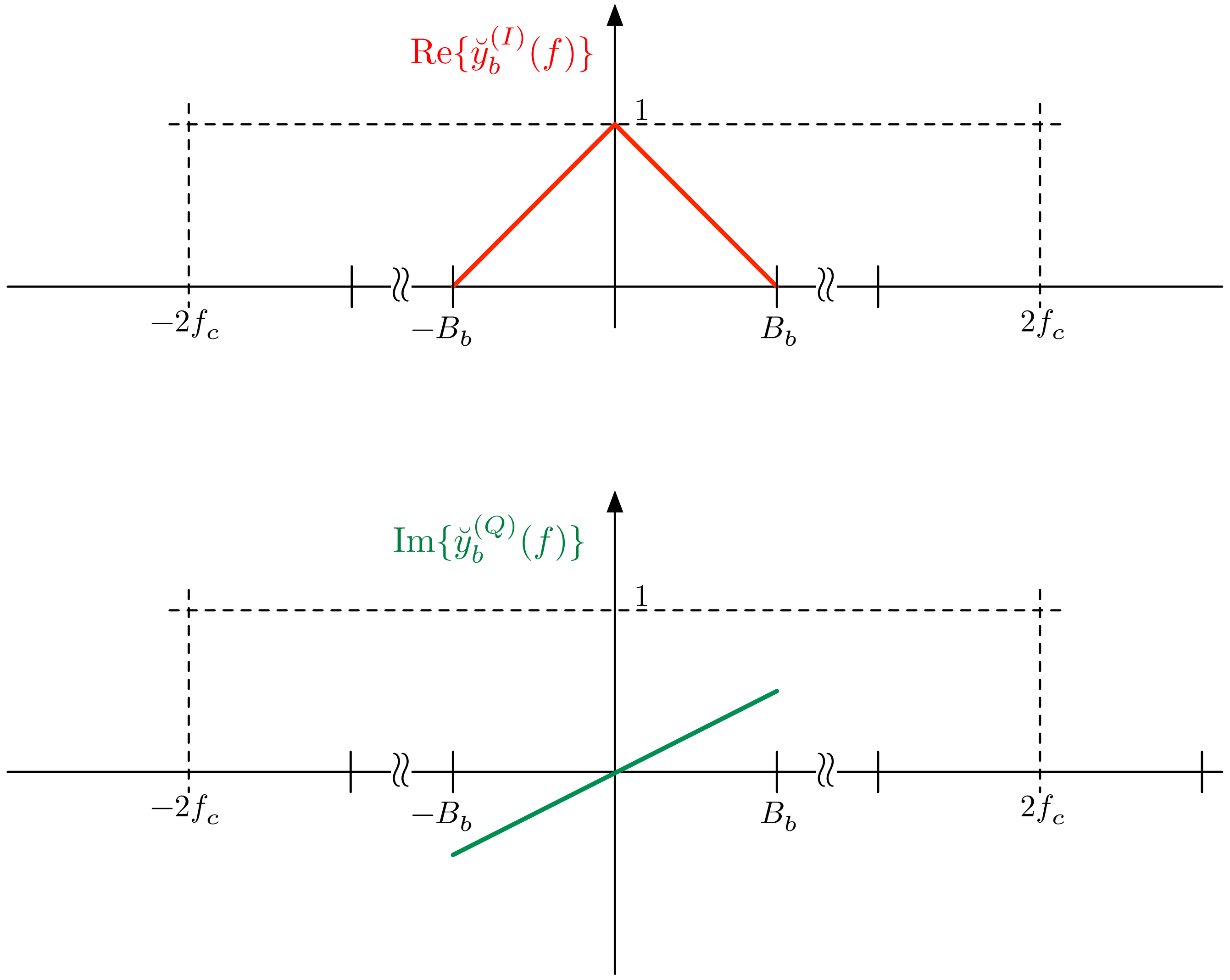

$\mathrm{Re}\{\breve{y}_b^{(I)}(f)\}$
1
$-2f_c$
$-B_b$
B_b
$2f_c$

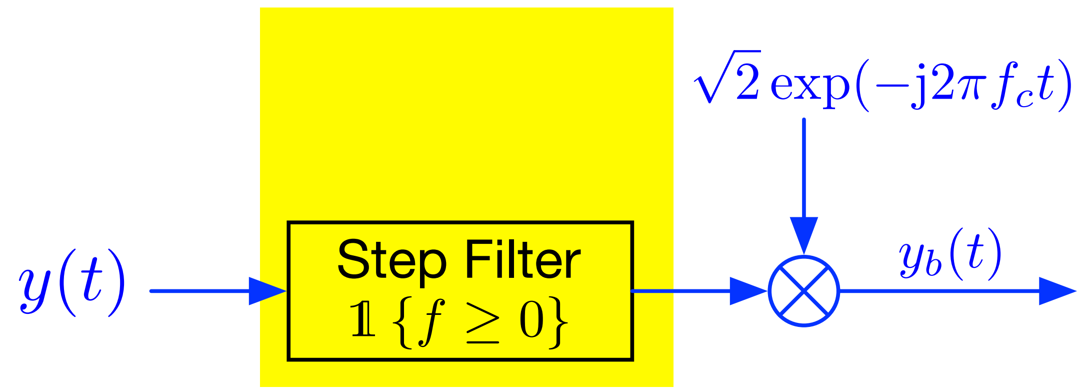

$\mathrm{Im}\{\breve{y}_b^{(Q)}(f)\}$
1
$-2f_c$
$-B_b$
B_b
$2f_c$

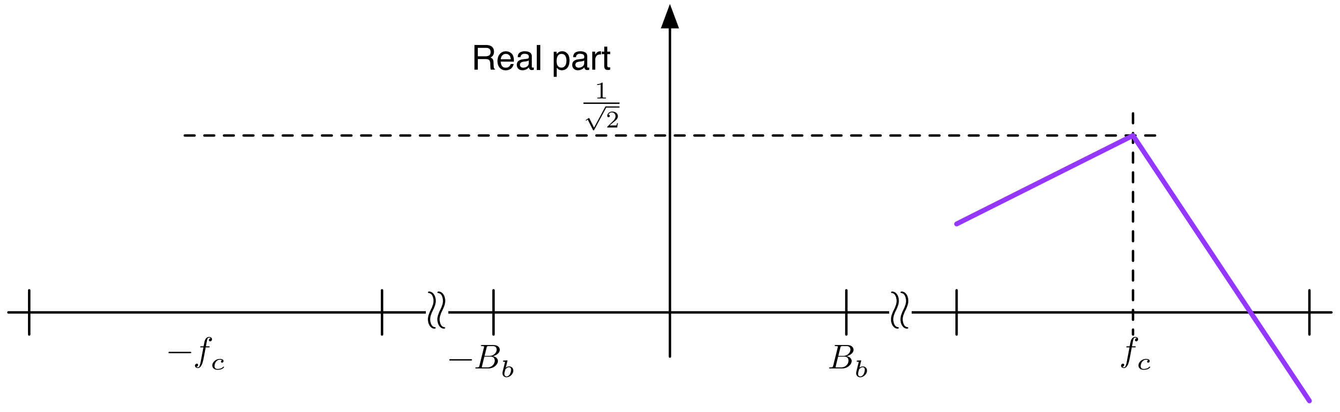

$\sqrt{2}\exp(-\mathrm{j}2\pi f_c t)$
$y(t)$
Step Filter
$\mathbb{1}\{f \ge 0\}$
$y_b(t)$

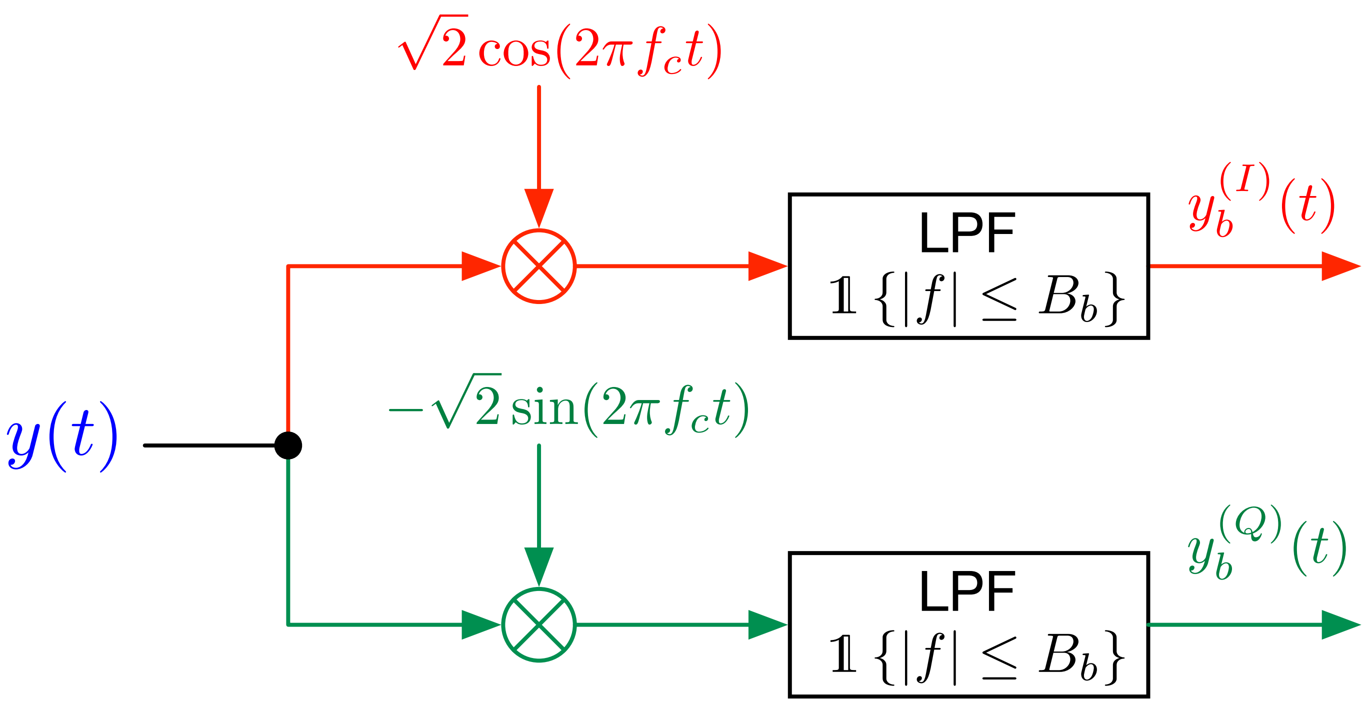

Real part
$\frac{1}{\sqrt{2}}$
$-f_c$
$-B_b$
B_b
f_c


$\sqrt{2}\cos(2\pi f_c t)$
LPF
$\mathbb{1}\{|f| \le B_b\}$
$y_b^{(I)}(t)$
$y(t)$
$-\sqrt{2}\sin(2\pi f_c t)$
LPF
$\mathbb{1}\{|f| \le B_b\}$
$y_b^{(Q)}(t)$

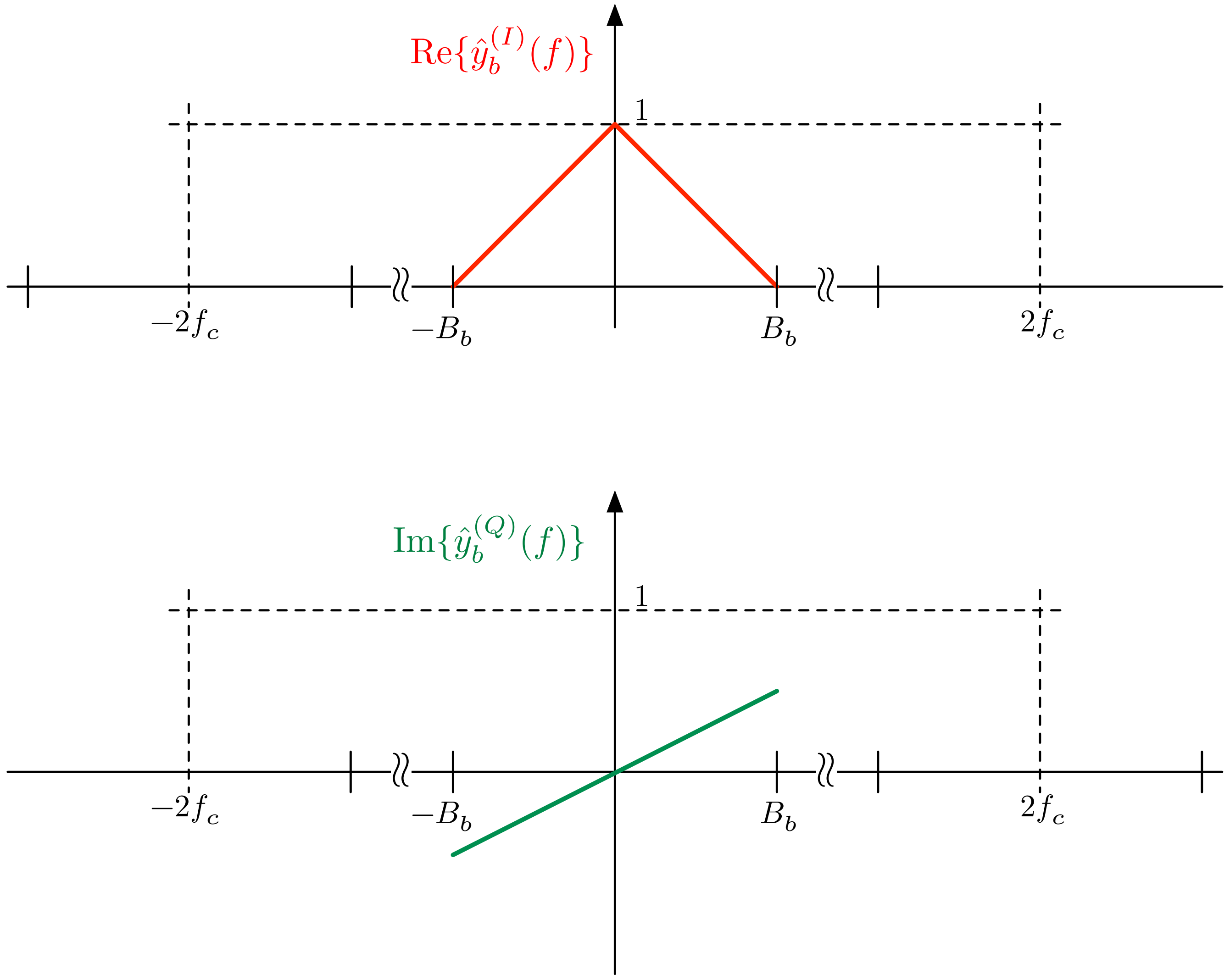

$\mathrm{Re}\{\breve{y}_b^{(I)}(f)\}$
1
$-2f_c$
$-B_b$
B_b
$2f_c$
$\mathrm{Im}\{\breve{y}_b^{(Q)}(f)\}$
1
$-2f_c$
$-B_b$
B_b
$2f_c$


$y(t)$
Step Filter
$\mathbb{1}\{f \ge 0\}$
$\sqrt{2}\exp(-j2\pi f_c t)$
$y_b(t)$

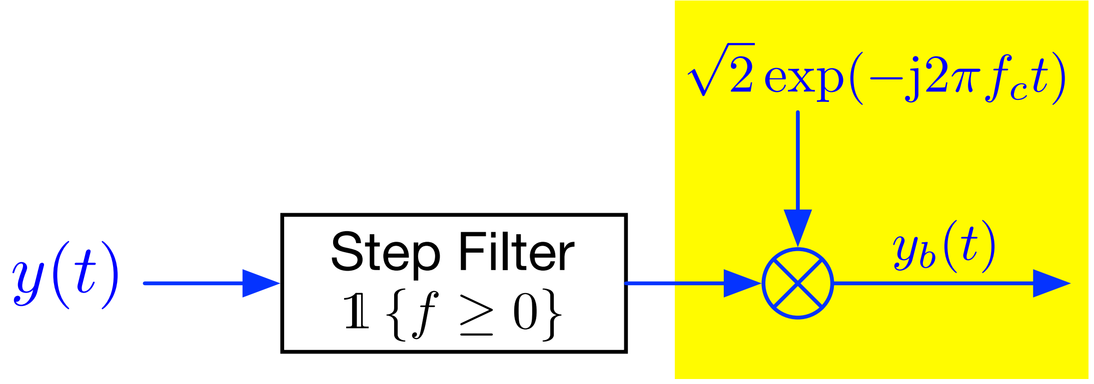

Real part
$\frac{1}{\sqrt{2}}$
$-f_c$
$-B_b$
B_b
f_c



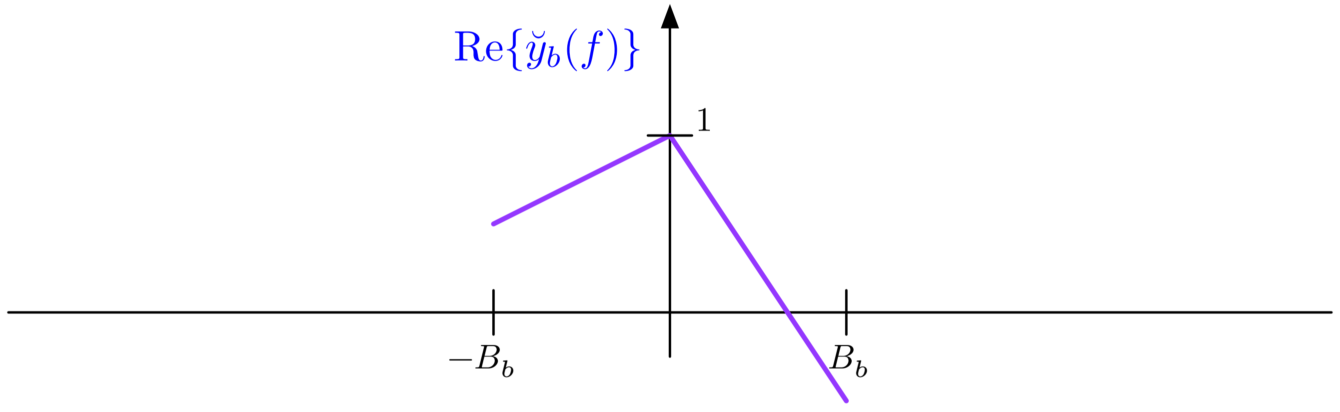
$\sqrt{2}\cos(2\pi f_c t)$
LPF
$\mathbb{1}\{|f| \leq B_b\}$
$y_b^{(I)}(t)$
$y(t)$
$-\sqrt{2}\sin(2\pi f_c t)$
LPF
$\mathbb{1}\{|f| \leq B_b\}$
$y_b^{(Q)}(t)$



$\mathrm{Re}\{\hat{y}_b^{(I)}(f)\}$
1
$-2f_c$
$-B_b$
B_b
$2f_c$
$\mathrm{Im}\{\hat{y}_b^{(Q)}(f)\}$
1
$-2f_c$
$-B_b$
B_b
$2f_c$

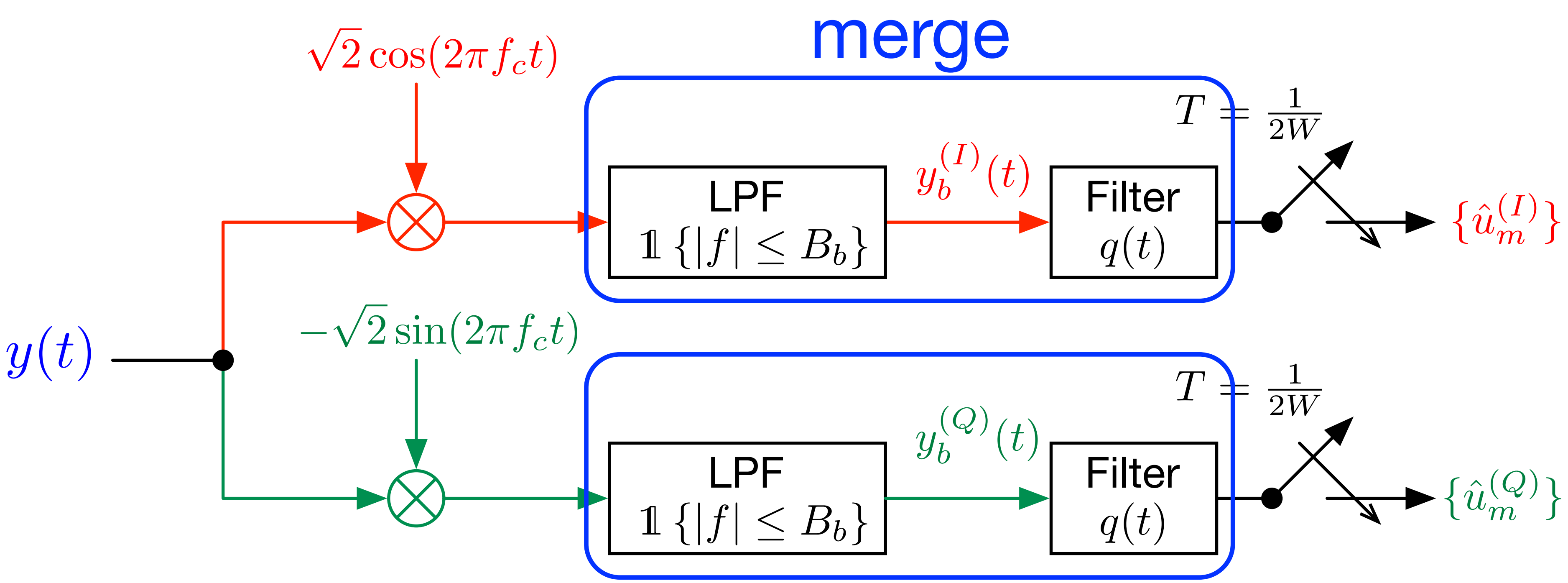


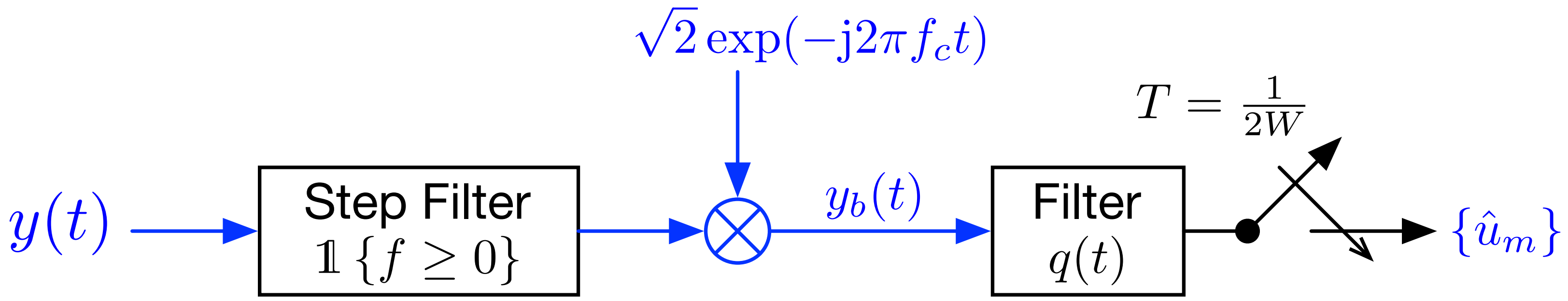
$y(t)$
Step Filter
$\mathbb{1}\{f \geq 0\}$
$\sqrt{2}\exp(-\mathrm{j}2\pi f_c t)$
$y_b(t)$



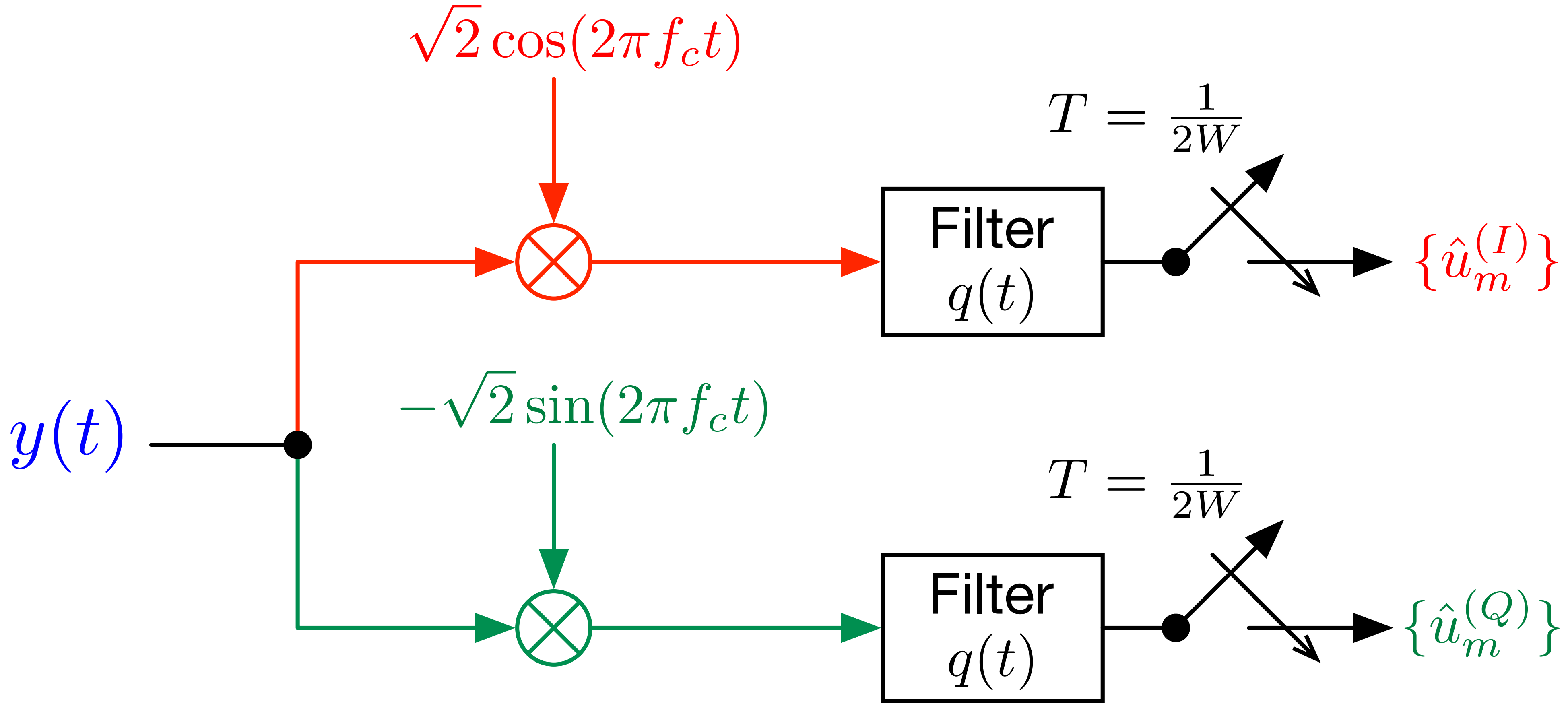
$\mathrm{Re}\{\breve{y}_b(f)\}$
1
$-B_b$
B_b

QAM demodulation





QAM demodulation



Passband expansion

- In summary, under QAM, the transmitted waveform is

$$x(t) = x_b^{(I)}(t)\sqrt{2}\cos(2\pi f_c t) - x_b^{(Q)}(t)\sqrt{2}\sin(2\pi f_c t)$$

$$= \sum_m u_m^{(I)} \underbrace{p(t-mT)\sqrt{2}\cos(2\pi f_c t)}_{\psi_m^{(I)}(t)} - \sum_k u_k^{(Q)} \underbrace{p(t-kT)\sqrt{2}\sin(2\pi f_c t)}_{\psi_m^{(Q)}(t)}.$$

- Identify

$$\begin{aligned} p(t-mT) &\longleftrightarrow \phi_m(t) \\ p(t-mT)\sqrt{2}\cos(2\pi f_c t) &\longleftrightarrow \psi_m^{(I)}(t) \\ -p(t-mT)\sqrt{2}\sin(2\pi f_c t) &\longleftrightarrow \psi_m^{(Q)}(t) \end{aligned}$$

- Rewrite

$$x(t) = \sum_m u_m^{(I)}\psi_m^{(I)}(t) + u_m^{(Q)}\psi_m^{(Q)}(t).$$ ← is this an orthonormal expansion?

Passband expansion

Theorem

Consider an orthonormal set of waveforms $\{\phi_m(t) : m \in \mathbb{Z}\}$. Assume the Fourier transform exists for each $\phi_m(t)$ and is band-limited, that is,

$$\breve{\phi}_m(f) = 0, \quad \forall\, |f| > B_b.$$

Then for a center frequency $f_c > B_b$, $\{\psi_m^{(I)}(t), \psi_m^{(Q)}(t) \mid m \in \mathbb{Z}\}$ also form an orthonormal set, where

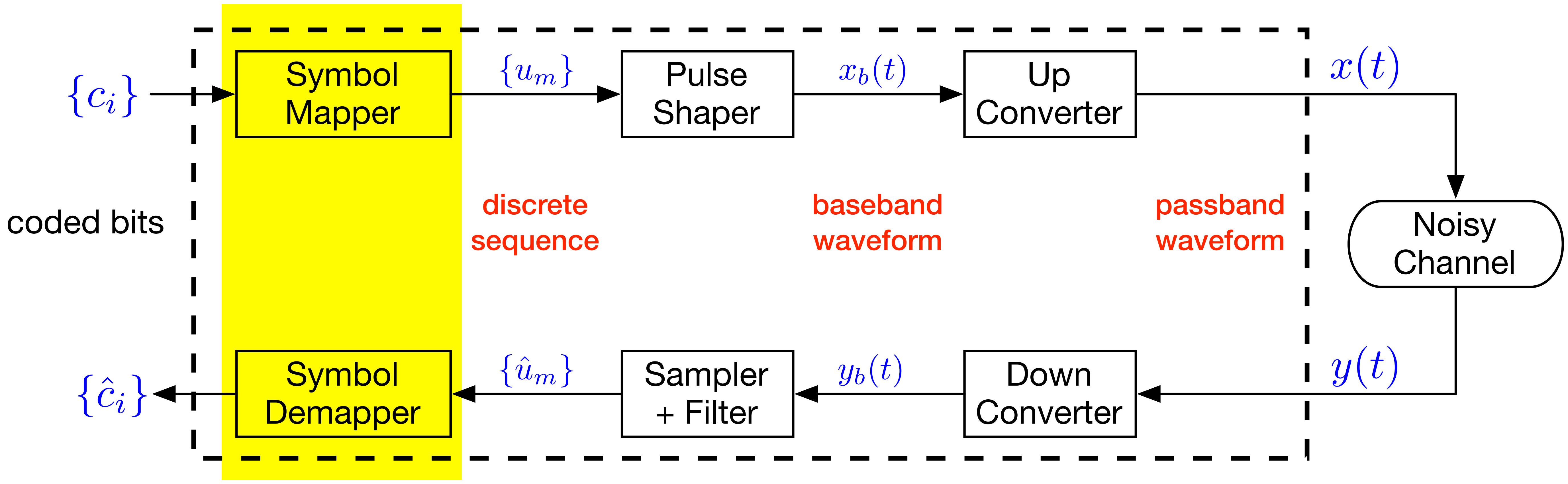
$$\psi_m^{(I)}(t) \triangleq \phi_m(t)\sqrt{2}\,\text{cos}(2\pi f_c t),\ \psi_m^{(Q)}(t) \triangleq -\phi_m(t)\sqrt{2}\,\text{sin}(2\pi f_c t).$$

In words, a baseband orthonormal basis remains orthonormal after up conversion

Part IV. Constellation Set and Symbol Mapping

Standard PAM, QAM, and PSK constellations
Gray mapping

Symbol mapping



- To be designed: the constellation set and how to map bits to symbols
- A standard way:

group ℓ bits and map them to a symbol in a constellation set $\mathcal{A}$

$$(c_1, c_2, \ldots, c_\ell) \quad \mapsto \quad u \in \mathcal{A} \triangleq \{a_1, a_2, \ldots, a_M\}$$

Coded bit sequence 0010001111001010000100001110

↓

Grouping

↓

0010 0011 1100 1010 0001 0000 1110 $\ell = 4$

↓

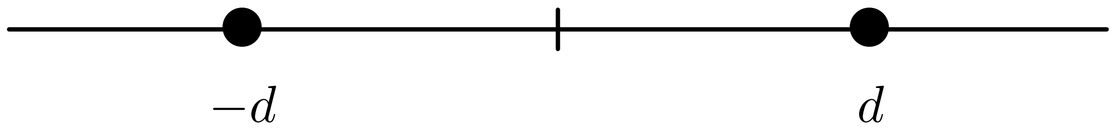
Mapping ← depends on
1) Constellation set
2) Mapping from bits to symbols

↓

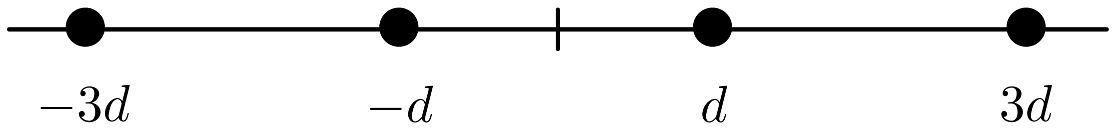
Symbol sequence u_1 u_2 u_3 u_4 u_5 u_6 u_7

Standard PAM constellation sets

$M = 2$



$M = 4$

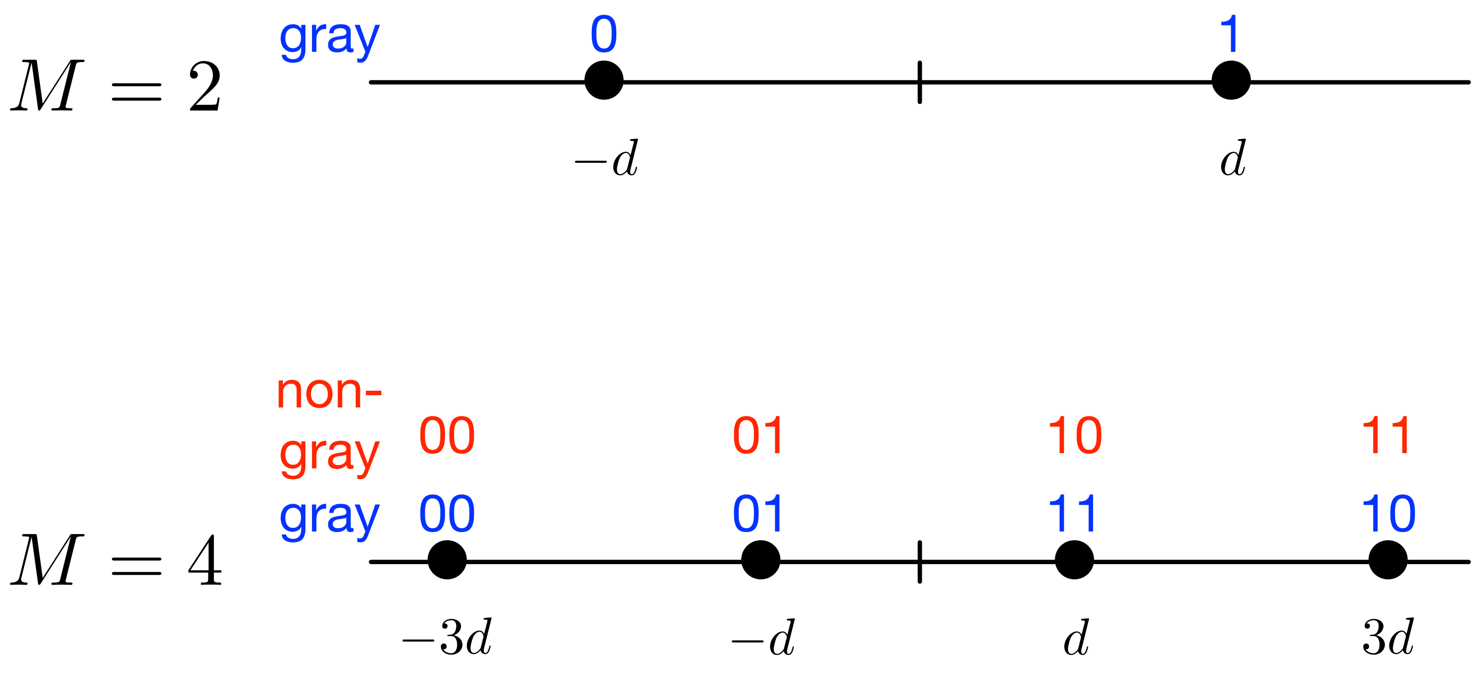


$$\mathcal{A}_{\mathrm{PAM},2^\ell} \triangleq \left\{\pm d, \pm 3d, \ldots, \pm(2^\ell - 1)d\right\}$$

typically $M = 2^\ell$

$$d_{\min} = 2d$$

Gray mapping



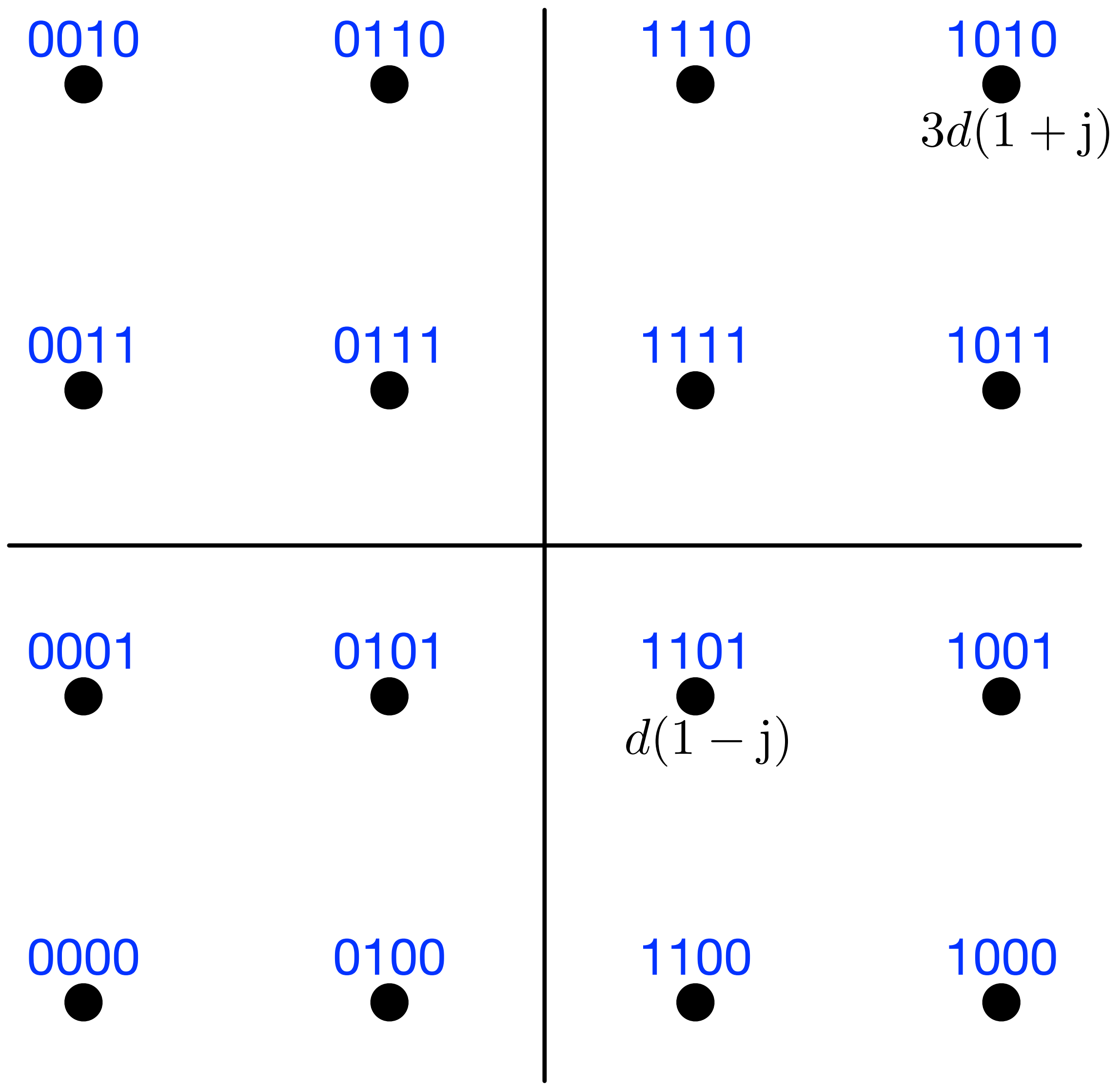
- Gray mapping assigns all possible combinations of ordered ℓ bits to constellation points in a way such that there is only one-bit difference between nearest neighboring points.

Standard QAM constellation sets

$M = 16$

typically $M = 2^{2\ell}$

$d_{\min} = 2d$



direct product of two 2^ℓ-ary standard PAM constellation sets

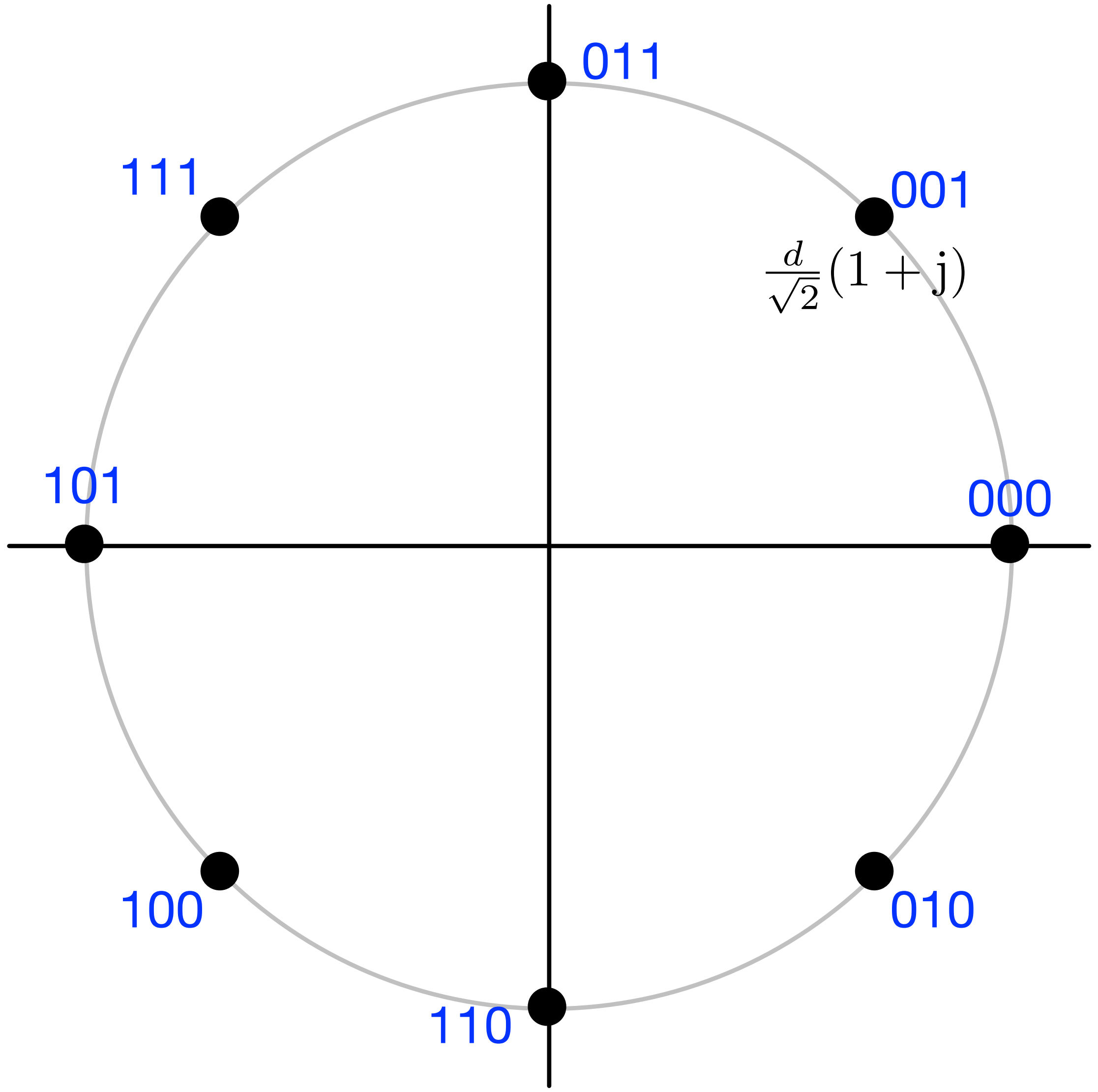
$$\mathcal{A}_{\mathrm{QAM},2^{2\ell}} \triangleq \left\{ a^{(I)} + \mathrm{j}a^{(Q)} \;\middle|\; a^{(I)}, a^{(Q)} \in \mathcal{A}_{\mathrm{PAM},2^\ell} \right\}.$$

Standard PSK constellation set

$$M = 8$$

typically $M = 2^\ell$

$$d_{\min} = 2d \sin\left(\frac{\pi}{M}\right)$$



encode the information on the phase

$$\mathcal{A}_{\mathrm{PSK},M} \triangleq \left\{ d\exp\left(\mathrm{j}\frac{2\pi}{M}k\right) \;\middle|\; k = 0, 1, \ldots, M-1 \right\}.$$

Design principles of constellation sets

- Energy
  - Depends on minimum distance $d_{\min}$ and the total number of points M
  - Increase with M under fixed $d_{\min}$
  - Increase with $d_{\min}$ under fixed M

- Reliability
  - Higher reliability if $d_{\min}$ is larger

- Rate
  - Higher rate if M is larger