# Lecture 4: Demodulation with Noise: Performance Analysis

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## Outline

Detection Performance Analysis

1. M-ary PAM
2. QAM
3. PSK
4. Symbol Error Rate (SER) and Bits Error Rate (BER) v.s. $E_b/N_0$.

Reliable Communication

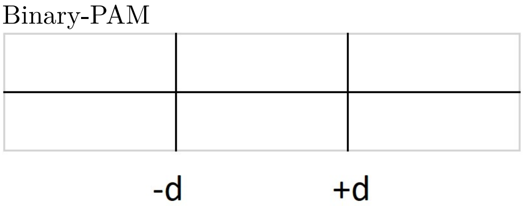
1. Repetition coding
2. Energy-Efficient reliable communication
3. Rate-Efficient

## Recap

under iid Gaussian noise to all dimensions, ML detection in multi-dimensional space $\equiv$ MD detection.

## Performance Analysis

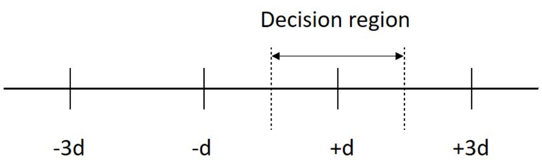
1. Binary-PAM




$y = x + z,\ x \in A = \{-d,\ d\},\ z \approx N(0, \sigma^2)$

$P_e = P_r\{x \neq \hat{x}\} = Q(d/\sigma) = Q(SNR^{1/2})$

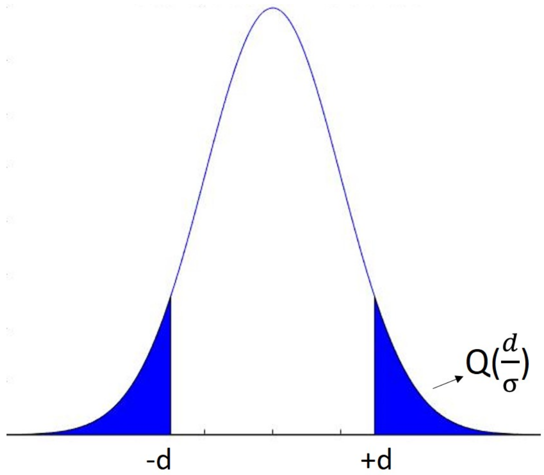
2. M-ary PAM
$M = 2^l\ (l = 2)$




$y = x + z,\ x \in A$ Symbol probability of error, symbol error rate.

$$\begin{aligned} P_e = P_r\{x \neq \hat{x}\} = & P_r\{\hat{x} \neq d | x = d\} \times P_r\{x = d\} \\ & + P_r\{\hat{x} \neq 3d | x = 3d\} \times P_r\{x = 3d\} \\ & + P_r\{\hat{x} \neq -d | x = -d\} \times P_r\{x = -d\} \\ & + P_r\{\hat{x} \neq -3d | x = -3d\} \times P_r\{x = -3d\} \end{aligned}$$

假設事前機率 $P_r\{x=d\}=P_r\{x=3d\}=P_r\{x=-d\}=P_r\{x=-3d\}=1/4$
Given $x=d$, $\{\hat{x}\neq d\}\Leftrightarrow y\notin(0,2d)$, assuming $y\approx N(d,\sigma^2)$



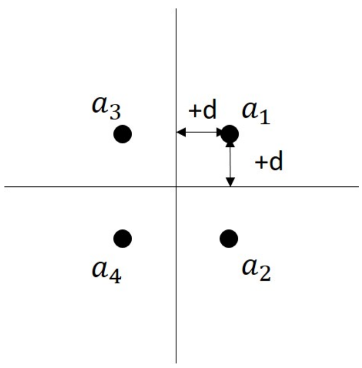

Given $x=3d$
$\{\hat{x}\neq 3d\}\Leftrightarrow\{y<2d\}\Leftrightarrow\{y-3d<-d\}\Leftrightarrow\{y-3d/\sigma<-d/\sigma\}$
$P_r\{\hat{x}\neq 3d|x=3d\}=P_r\{N(0,1)<-d/\sigma\}=Q(d/\sigma)$

Hence, $P_e=2Q(d/\sigma)\times 1/4+Q(d/\sigma)\times 1/4+Q(d/\sigma)\times 1/4+2Q(d/\sigma)\times 1/4=\frac{3}{2}Q(d/\sigma)$

Exercise: compute the symbol error rate $P_e=P_r\{\hat{x}\neq x\}$ for $x\in A$, $A: s^l$-ary PAM, where the distance between neighboriong constellation point is 2d

Performance Analysis:
computation of symbol energy, define SNR $=E_s/\sigma^2$
$l=1$: $E_s=d^2,\ SNR=d^2/\sigma^2,\ P_e=Q(d/\sigma)=Q(SNR^{1/2})\approx e^{-SNR/2}$
$l=2$: $E_s=1/2\times(d^2+(3d)^2)=5d^2,\ SNR=5d^2/\sigma^2,\ P_e=\frac{3}{2}Q(d/\sigma)=Q(SNR^{1/2})\approx e^{-SNR/10}$
若要得到相同 $P_e$，需讓距離減少。

3. M-ary QAM



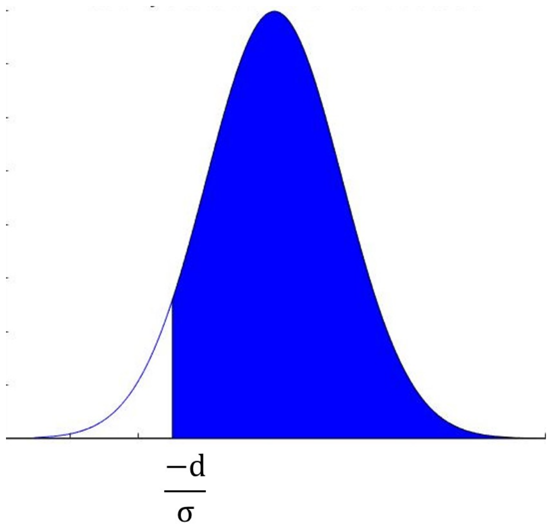

- 4 QAM

  $M=2^{2l}$
  $\mathbf{x}=(x_1,x_2),\ \mathbf{y}=(y_1,y_2)$

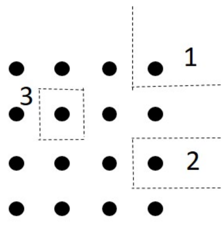
$$\begin{aligned}
P_r\{\hat{x} \neq a_1 | x = a_1\} &= P_r\{x \notin First\ Quadrant | x = a_1\} \\
&= P_r\{y_1 < 0\ or\ y_2 < 0 | x_1 = d, x_2 = d\} \\
&= 1 - P_r\{y_1 > 0\ and\ y_2 > 0 | x_1 = d, x_2 = d\} \\
&= 1 - P_r\{N(d, \sigma^2) > 0\}^2 \\
&= 1 - P_r\{N(0,1) > -d/\sigma\}^2 \\
&= 1 - (1 - Q(d/\sigma))^2 \\
&= 2Q(d/\sigma) - (Q(d/\sigma))^2
\end{aligned}$$

$SNR = E_s/(2\sigma^2) = 2d^2/(2\sigma^2) = d^2/\sigma^2$
union bound: $P_r\{A \cup B\} \leq P_r\{A\} + P_r\{B\}$

$$\begin{aligned}
P_{e,4QAM} &= 2Q(d/\sigma) - Q(d/\sigma)^2 \\
&= 2Q(SNR^{1/2}) - Q(SNR^{1/2})^2 \\
&\approx 2Q(SNR^{1/2})
\end{aligned}$$




- 16 QAM (M=16, l=2)




分成三種 case, 利用 union bound:

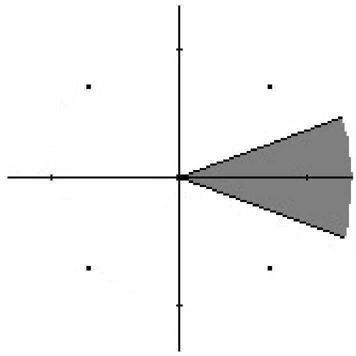
1 $\leq 2Q(d/\sigma)$
2 $\leq Q(d/\sigma) + 2Q(d/\sigma) = 3Q(d/\sigma)$
3 $2Q(d/\sigma) + 2Q(d/\sigma) = 4Q(d/\sigma)$

Exercise: for M-ary QAM, $M = 2^{2l}$
Compute

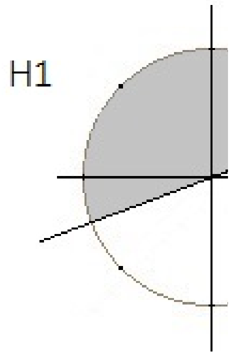
(1) the exact SER
(2) the approximate SER using union bound

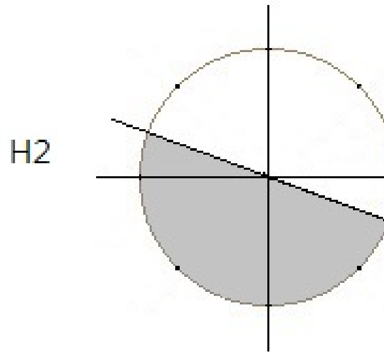
Compare the two nume

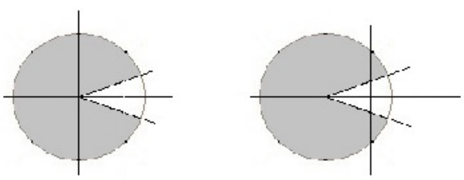
4. M-ary PSK



- 8 PSK (M=8, l=3)
  $M = 2^l$
  $E_s = d^2,\ y = x + z, z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$
  $SNR = \frac{E_s}{2\sigma^2} = \frac{d^2}{2\sigma^2}$

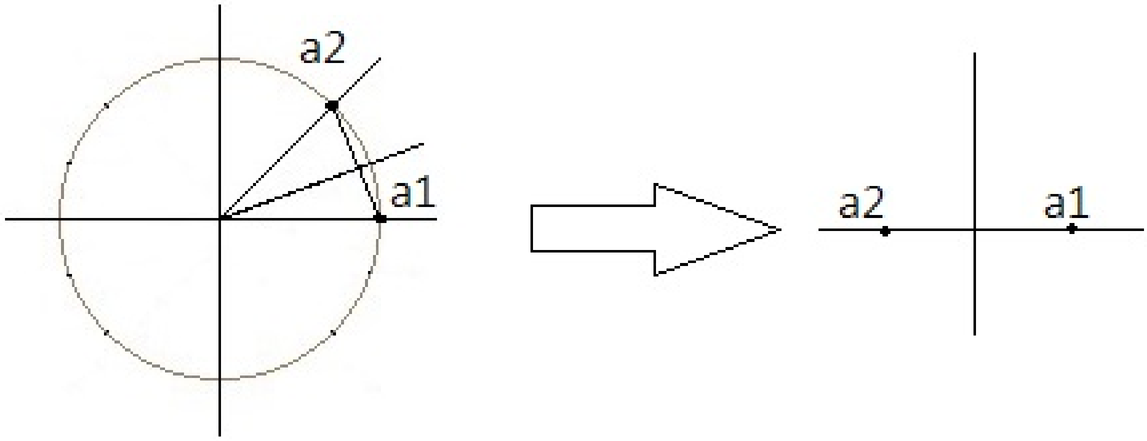


$= P_r\{y \in H_1 \cup H_2 | x = a_1\}$
$\leq P_r\{y \in H_1 | x = a_1\} + P_r\{y \in H_2 | x = a_1\}$
$y \in H_1 \iff \|y - a_1\| > \|y - a_2\|$

Consider a new binary detection problem in 2-dimension space in Fig6




$P_e = Q(\frac{\|a_1 - a_2\|}{2\sigma})$
$z\ N[0, \sigma^2 I_n]\ z' = Rz\ R = rotationmatrix$
$\Rightarrow P_e \leq 2Q(\frac{d \sin \frac{\pi}{8}}{\sigma})$
$P_r\{y \in H_1 | x = a_1\}$
$= Q(\frac{\|a_1 - a_2\|}{2\sigma})$
$= Q(\frac{d \sin \frac{\pi}{8}}{\sigma}) \leq P_r\{y \in (H_1 \cup H_2) | x = a_1\}$
$\Rightarrow Q(\frac{d \sin \frac{\pi}{8}}{\sigma}) \leq P_e \leq 2Q(\frac{d \sin \frac{\pi}{8}}{\sigma})$

5. Symbol Error Rate (SER) and Bits Error Rate (BER) v.s. $E_b/N_0$.

   (a) Energy $E_b \equiv$ energy per bit $= \frac{E_s}{l}$
   $l$ =Number of bits per symbol

   (b) P.S.D. of WGN process: $\frac{N_0}{2}$each real dimension at symbol level $z^{(I)}[m]\ N(0, \sigma^2)$
   $z^{(Q)}[m]\ N(0, \sigma^2)\sigma^2 = \frac{N_0}{2}$

   (c) $SNR_{symbol} = SNR_{bit} * l$
   Example: 4-QAM(4-PSK)

      i. Gray Mapping: $P_{e,b1}$
      $= \frac{1}{4} P_r\{b1' \neq 0 | b1 = 0, b2 = 0\}$
      $= \frac{1}{4} P_r\{b1' \neq 0 | b1 = 0, b2 = 1\}$
      $= \frac{1}{4} P_r\{b1' \neq 1 | b1 = 1, b2 = 1\}$
      $= \frac{1}{4} P_r\{b1' \neq 1 | b1 = 1, b2 = 0\}$
      $= \frac{1}{4} Q(\frac{d}{\sigma}) * 4$

      $P_{e,b2}$
      $= \frac{1}{4} P_r\{b2' \neq 0 | b1 = 0, b2 = 0\}$
      $= \frac{1}{4} P_r\{b2' \neq 0 | b1 = 0, b2 = 1\}$
      $= \frac{1}{4} P_r\{b2' \neq 1 | b1 = 1, b2 = 1\}$
      $= \frac{1}{4} P_r\{b2' \neq 1 | b1 = 1, b2 = 0\}$
      $= P_{e,b1}$

      $BER = P_{e,total} = \frac{1}{2}(P_{e,b1} + P_{e,b2}) = Q(\frac{d}{\sigma})$

      ii. Nature Mapping: $P_{e,b1}$
      $= \frac{1}{4} P_r\{b1' \neq 0 | b1 = 0, b2 = 0\}$
      $= \frac{1}{4} P_r\{b1' \neq 0 | b1 = 0, b2 = 1\}$
      $= \frac{1}{4} P_r\{b1' \neq 1 | b1 = 1, b2 = 1\}$
      $= \frac{1}{4} P_r\{b1' \neq 1 | b1 = 1, b2 = 0\}$
      $= \frac{1}{4} Q(\frac{d}{\sigma}) * 4$

$P_{e,b2} = 4 * P_r\{b2' \neq 0|b1 = 0, b2 = 0\}$
$\leq 4 * \frac{1}{4} 2Q(\frac{d}{\sigma})$
$= 2Q(\frac{d}{\sigma})$
$P_{eb} \approx \frac{3}{2} Q(\frac{d}{\sigma})$

The Principle in designing bit to symbol mapping:

(a) equal-protection at all information bits.

(b) minimize the number of "bit-level" nearest neighbor.
$\Rightarrow$ Gray Mapping

(a) PAM: l bits per symbol $2^l$-PAM (1-dimision space)

(b) QAM: 2l bits per symbol $2^{2l}$-QAM (2-dimision space)

(c) PSK: l bits per symbol $2^l$-PSK(2-dimision space)

Consider sending a "file" using the method discussed above

(a) Divide the file into chunks, each chunk corresponds to a symbol

(b) Each symbol has symbol-error-rate(SER),$P_{e,s}$
$P_{e,F} = 1 - (1 - P_{e,s})^n$ n:number of symbols// Example: $P_{e,s} = 10^{-4}$ by using sufficient amount of power
$n = 250 \quad P_{e,F} = 1 - (1 - 10^{-4})^{250} \approx \frac{25}{100}$

# Reliable Communication

1. Repetition Coding
bit $\rightarrow$ Repetition Coding
$\rightarrow$ { bit bit bit ......bit}(n times)
$\rightarrow$ Modulation(Binary PAM)$\rightarrow$ $y = [x_1, x_2......x_n]$

$$b = 0 \quad x = \begin{bmatrix} -1 \\ -1 \\ . \\ . \\ . \\ -1 \end{bmatrix} * d = u_0$$

$$b = 1 \quad x = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} * d = u_1$$

$P_e = Q(\frac{\|u_0 - u_1\|}{2\sigma}) = Q(\sqrt{\frac{nd^2}{\sigma^2}}) \rightarrow 0$ as n$\rightarrow$ 0
$\approx e^{\frac{-nd^2}{2\sigma^2}} = e^{\frac{-n}{2} * SNR}$
$Rate : \frac{1}{n} bit/symbol \rightarrow 0$ as n$\rightarrow \infty$
$Energyperbit : nd^2 \rightarrow \infty$ as n$\rightarrow \infty$