Review – Part 1

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1 First-Order Differential Equations

2 Higher Order Differential Equations

Separable Equations: $\frac{dy}{dx} = f(x, y) = g(x)h(y)$

General Procedure of Solving a Separable DE

1. 分別移項: $\dfrac{dy}{h(y)} = g(x)\,dx$. 若分母$h(y) = 0$, singular solutions!
2. 兩邊積分: $\displaystyle\int \frac{dy}{h(y)} = \int g(x)\,dx \implies H(y) = G(x) + c$.
3. 代入條件: $c = H(y_0) - G(x_0)$.
4. 取反函數: $y = H^{-1}(G(x) + H(y_0) - G(x_0))$.
   Don't forget to check singular solutions!

Solving the Linear First-Order ODE

General Procedure of Solving a Linear First-Order ODE

1. 寫成標準式: Rewrite the give ODE into the form $\frac{dy}{dx} + P(x)y = f(x)$.
   若分母$= 0$, exclude the singular points from the interval of solutions.
2. 導出輔助式: Introduce an integrating factor $\mu(x)$ and derive the auxiliary equation of μ to find μ such that $\frac{d(\mu y)}{dx} = \mu(x)f(x)$.
3. 解輔助式: Find one μ satisfying the auxiliary DE $\frac{d\mu}{dx} = P(x)\mu$.
4. 解原式: Plug in the integrating factor $\mu(x)$ we found and solve $\mu(x)y$ by directly integrating $\mu(x)f(x)$.
   Check if the singular points can be included into the interval of solutions.

Discontinuous Coefficients

What if coefficients are discontinuous?

If they are piecewise continuous and only discontinuous at finitely many points, we can solve the equations on each interval and “stitch” them together.

See Example 6 in Section 2-3 on Page 59.

Solving an Exact DE $M(x,y)\,dx + N(x,y)\,dy = 0$

Goal: Find $z = F(x,y)$ such that $dz = M(x,y)\,dx + N(x,y)\,dy = 0$.

General Procedure of Solving an DE

1. Transform DE into the differential form: $M(x,y)\,dx + N(x,y)\,dy = 0$.
2. Verify if it is exact: $\dfrac{\partial M}{\partial y} \stackrel{?}{=} \dfrac{\partial N}{\partial x}$
3. Integrate M with respect to x (or N with respect to y):
   $F(x,y) = \int M dx + g(y)$ (or $F(x,y) = \int N dy + h(x)$)
4. Take partial derivative with respect to y (or x):
   $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M dx\right) + g'(y) = N(x,y) \quad \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\int N dy\right) + h'(x) = M(x,y)$$
   $$\implies g(y) = \int \left(N - \frac{\partial}{\partial y}\int M dx\right) dy \quad \implies h(x) = \int \left(M - \frac{\partial}{\partial x}\int N dy\right) dx$$

Nonexact DE $M(x, y)\,dx + N(x, y)\,dy = 0$ Made Exact

Nonexact DE: $M_y - N_x := \Delta(x, y) \neq 0$

Key Idea 1: Introduce a function $\mu(x, y)$ (*integrating factor*) to ensure

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

Key Idea 2: Restrict $\mu(x, y)$ to be $\mu(x)$ or $\mu(y)$.

$$\text{Plan A: } \mu(x, y) = \mu(x) \implies \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = \frac{\Delta}{N}\mu$$
$$\text{Plan B: } \mu(x, y) = \mu(y) \implies \frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu = -\frac{\Delta}{M}\mu$$

Nonexact DE $M(x, y)\,dx + N(x, y)\,dy = 0$ Made Exact

Nonexact DE: $M_y - N_x := \Delta(x, y) \neq 0$

$$\text{Plan A: } \mu(x, y) = \mu(x) \implies \frac{d\mu}{dx} = \frac{\Delta}{N}\mu$$

$$\text{Plan B: } \mu(x, y) = \mu(y) \implies \frac{d\mu}{dy} = -\frac{\Delta}{M}\mu$$

Key Idea 3: Which plan should we choose? Choose it based on $\Delta(x, y)$:

- If $\dfrac{\Delta}{N}$ only depends on x, then $\dfrac{d\mu}{dx} = \dfrac{\Delta}{N}\mu$ is separable. Plan A!
- If $\dfrac{\Delta}{M}$ only depends on y, then $\dfrac{d\mu}{dy} = -\dfrac{\Delta}{M}\mu$ is separable. Plan B!

Solving a Homogeneous Equation

Definition (Homogeneous Function)

A function $f(x, y)$ is **homogeneous** of degree α if for all x, y,

$$f(tx, ty) = t^{\alpha} f(x, y) \text{ for some } \alpha.$$

To solve a homogeneous equation $M(x, y)\,dx + N(x, y)\,dy = 0$, (both M and N are homogeneous of the same order), set $u := y/x$ and get

$$\boxed{\frac{du}{dx} = -\frac{1}{x}\left\{u + \frac{M(1, u)}{N(1, u)}\right\}}$$

Note: we can also begin with setting $v := x/y$, depending on which will lead to a simpler from.

Bernoulli's Equation

Definition (Bernoulli's Equation)

The DE $\frac{dy}{dx} + P(x)y = f(x)y^r$ where $r \in \mathbb{R}$ is any real number.

For $r = 0, 1$, the equation is linear.

For $r \neq 0, 1$, we shall use the substitution $u := y^{1-r}$ to make it linear:

1 First-Order Differential Equations

2 Higher Order Differential Equations

Solving Linear Higher-Order Differential Equations

The steps to find the general solution of a linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

1. Find the general solution of its homogeneous counterpart ($g(x) = 0$):

   $$y_c = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \ c_i \in \mathbb{R}, \ \forall \ i = 1, 2, \ldots, n.$$

   Here $\{f_1, f_2, \ldots, f_n\}$ is a fundamental set of solutions.

2. Find a particular solution y_p such that it satisfies (1).

3. The general solution of (1) is

   $$y = y_c + y_p = y_p + c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \ c_i \in \mathbb{R}, \ \forall \ i.$$

Types of Equations and Methods Covered

Regarding how to find the general solutions of homogeneous linear DE, we discussed two types of equations:

- Linear equations with constant coefficients
- Cauchy-Euler equation

Regarding how to find a particular solution of a nonhomogeneous linear DE, we presented two methods:

- Undetermined Coefficients
- Variation of Parameters

Reduction of Order

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Suppose we already have a solution $y = f_1(x)$. How do we find another solution $y = f_2(x)$, such that f_1 and f_2 are linearly independent?

Idea: Let $\boxed{f_2(x) = u(x)f_1(x)}$, and make use of the fact that f_1 is a solution of the original DE to reduce the second order DE into a first order DE of u !

Finding the General Solution of $p(D)y = 0$

High-level Idea: let $p(D)$ have n_1 distinct real roots $\{m_i \mid i \in [1:n_1]\}$, and n_2 distinct pairs of conjugate complex roots $\{\alpha_j \pm i\beta_j \mid j \in [1:n_2]\}$.

1. Factorize $p(D) = \sum_{i=0}^{n} a_i D^i$ as

$$p(D) = a_n \left(\prod_{i=1}^{n_1} \overbrace{(D - m_i)^{k_i}}^{p_i(D)} \right) \left(\prod_{j=1}^{n_2} \overbrace{\left(D^2 - 2\alpha_j D + \alpha_j^2 + \beta_j^2\right)^{l_j}}^{q_j(D)} \right)$$

$$= a_n \prod_{i=1}^{n_1} p_i(D) \prod_{j=1}^{n_2} q_j(D), \text{ where } n = \sum_{i=1}^{n_1} k_i + 2\sum_{j=1}^{n_2} l_j.$$

2. For each $i \in [1:n_1]$, find k_i linearly independent solutions of $p_i(D)y = 0$.
3. For each $j \in [1:n_2]$, find $2l_j$ linearly independent solutions of $q_j(D)y = 0$.
4. Combine them all to get n linearly independent solutions of $p(D)y = 0$.

Solve $(D-m)^k\, y = 0$

Here are k linearly independent solutions:

$$\boxed{f_1(x) = e^{mx},\ f_2(x) = xe^{mx},\ f_3(x) = x^2\, e^{mx}, \ldots,\ f_k(x) = x^{k-1}\, e^{mx}}.$$

Solve $\left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^l \; y = 0$

Here are $2l$ linearly independent real-valued solutions:

$$\boxed{\left\{ x^{j-1} e^{\alpha x} \cos \beta x, \;\; x^{j-1} e^{\alpha x} \sin \beta x \;\middle|\; j = 1, 2, \ldots, l \right\}}$$

However, sometimes it is a lot easier to work in the complex domain with $2l$ linearly independent complex-valued solutions:

$$\boxed{\left\{ x^{j-1} e^{(\alpha + i\beta)x}, \;\; x^{j-1} e^{(\alpha - i\beta)x} \;\middle|\; j = 1, 2, \ldots, l \right\}}$$

Cauchy-Euler Eq. $\longrightarrow$ Eq. with Constant Coefficients

With the substitution $\boxed{x = e^t}$, we convert a Cauchy-Euler Equation

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0$$

into a linear differential equation (with respect to t) with constant coefficients

$$L_t\{y\} = 0, \quad \boxed{L_t := \sum_{i=0}^{n} a_i D_t(D_t - 1) \cdots (D_t - i + 1)}$$

Mapping of Solutions: with $x = e^t$,

t **Domain**	$t^k e^{mt}$	$t^k e^{\alpha t} \cos \beta t$	$t^k e^{\alpha t} \sin \beta t$
x **Domain**	$(\ln x)^k x^m$	$(\ln x)^k x^\alpha \cos(\beta \ln x)$	$(\ln x)^k x^\alpha \sin(\beta \ln x)$

Cauchy-Euler Eq. – Solutions for $x < 0$

Idea: Change of variable **–** substitute $x = -u$, and solve the new Cauchy-Euler Equation for $u > 0$.

Conversion:

$$x^k D_x^k = (-u)^k (-1)^k D_u^k = u^k D_u^k$$

Method of Undetermined Coefficients

Let $L(D) := \sum_{i=0}^{n} a_i D^i$.

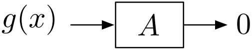
Goal: Find a particular solution y_p such that $L\{y_p\} = g(x)$

Assumption: $\exists$ a polynomial of D, $A(D)$, such that $A\{g(x)\} = 0$.

Procedure:

1. Find a fundamental set of solutions of $P\{y\} = 0$, $\mathcal{P}$, where $P(D) := L(D)A(D)$.
2. Find a fundamental set of solutions of $L\{y\} = 0$, $\mathcal{L}$, and $\mathcal{L} \subseteq \mathcal{P}$.
3. Plug in $L\{y_p\} = g(x)$ with $y_p =$ a linear combination of the functions in $\mathcal{P} \setminus \mathcal{L}$.
4. Solve the undetermined coefficients in the linear combination.

$g(x)$ and its Annihilator A



A : polynomial of D with constant coefficients

For $k = 0, 1, \ldots$, and $m, \alpha, \beta \in \mathbb{R}$,

$g(x)$	A
x^k	D^{k+1}
$x^k e^{mx}$	$(D-m)^{k+1}$
$x^k \sin \beta x$	$\left(D^2+\beta^2\right)^{k+1}$
$x^k \cos \beta x$	$\left(D^2+\beta^2\right)^{k+1}$
$x^k e^{\alpha x} \sin \beta x$	$\left(D^2-2\alpha D+\alpha^2+\beta^2\right)^{k+1}$
$x^k e^{\alpha x} \cos \beta x$	$\left(D^2-2\alpha D+\alpha^2+\beta^2\right)^{k+1}$

Variation of Parameters

Variation of parameters is a powerful method to find a particular solution y_p of **any** linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

given that the general solution of the corresponding homogeneous DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (2)$$

can be found.

No restrictions on $g(x)$! $g(x)$ can be $1/x$, $\csc x$, $\ln x$, etc.

n-th Order DE $L\{y\} = g(x),\ L := \sum_{i=0}^{n} a_i(x) D^i$

Suppose $\{f_1, f_2, \ldots, f_n\}$ form a fundamental set of solutions of the homogeneous linear DE $L\{y\} = 0$.

Then a particular solution $\boxed{y_p = \sum_{i=0}^{n} u_i(x) f_i(x)}$ can be found by the following formula regarding $\{u_1{}', u_2{}', \ldots, u_n{}'\}$:

$$\boxed{u_i{}' = \frac{W_i}{W}, \qquad W = \begin{vmatrix} f_1 & \cdots & f_n \\ f_1{}' & \cdots & f_n{}' \\ \vdots & & \vdots \\ f_1{}^{(n-1)} & \cdots & f_n{}^{(n-1)} \end{vmatrix}}$$

W_i is W with the i-th column replaced by $\begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{g(x)}{a_n(x)} \end{bmatrix}^T$.

Solving a System of Linear DE with Constant Coefficients

1. Convert it into the following form:

$$\begin{cases} L_{11}\{y_1\} + L_{12}\{y_2\} + \cdots + L_{1k}\{y_k\} = g_1(t) \\ L_{21}\{y_1\} + L_{22}\{y_2\} + \cdots + L_{2k}\{y_k\} = g_2(t) \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ L_{k1}\{y_1\} + L_{k2}\{y_2\} + \cdots + L_{kk}\{y_k\} = g_k(t) \end{cases}$$

2. Use Cramer's rule to get $L\{y_j\} = \widetilde{g}_j(t)$, $j = 1, \ldots, k$, where

$$L = \begin{vmatrix} L_{11} & L_{12} & \cdots & L_{1k} \\ L_{21} & L_{22} & \cdots & L_{2k} \\ \vdots & \vdots & & \vdots \\ L_{k1} & L_{k2} & \cdots & L_{kk} \end{vmatrix}, \quad \widetilde{g}_j(t) = L\Big|_{j\text{-th column replaced by } \begin{bmatrix} g_1 & \cdots & g_k \end{bmatrix}^T}$$

3. Solve each $y_j(t)$, $j = 1, \ldots, k$.
4. Plug into the initial system, find additional constraints on the coefficients in the complimentary solutions $\{y_{1c}, y_{2c}, \ldots, y_{kc}\}$, and finalize.

Notes and Tips

- It is very important to plug the general solutions found for $\{y_1, y_2, \dots, y_k\}$ back to the original system of equations to find additional constraints on the coefficients in the complimentary solutions $\{y_{1c}, y_{2c}, \dots, y_{kc}\}$ (see example).
- When solving $L\{y_j\} = \widetilde{g}_j(t)$, sometimes we can eliminate redundant operators (see example).
- When $k = 2$, that is, only two dependent variables to be solved, after solving one dependent variable, it may save some time if we plug the general solution we found back to the original system and find the solution of the other dependent variable (see example).