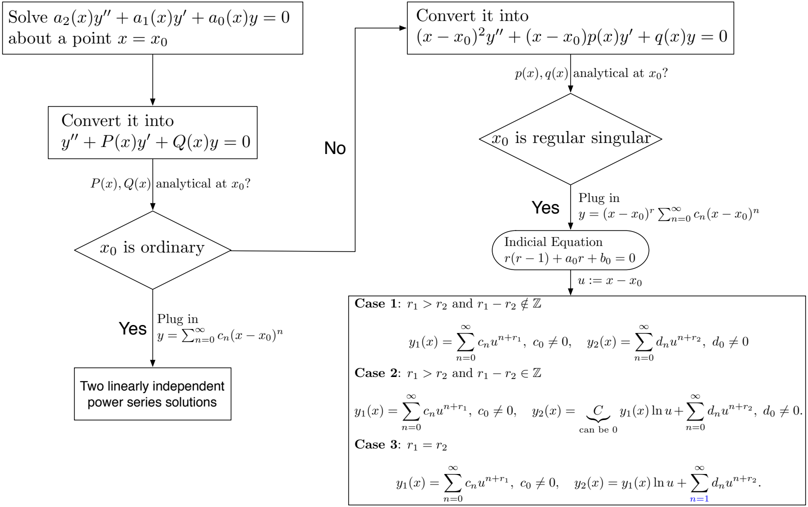
Review – Part 2

王奕翔

Department of Electrical Engineering
National Taiwan University

ihwang@ntu.edu.tw

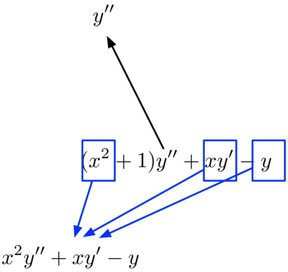
January 4, 2014


Solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ about a point $x = x_0$
Convert it into $y'' + P(x)y' + Q(x)y = 0$
$P(x), Q(x)$ analytical at x_0?
x_0 is ordinary
No
Yes
Plug in $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$
Two linearly independent power series solutions
Convert it into $(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$
$p(x), q(x)$ analytical at x_0?
x_0 is regular singular
Yes
Plug in $y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n$
Indicial Equation $r(r-1) + a_0 r + b_0 = 0$
$u := x - x_0$
Case 1: $r_1 > r_2$ and $r_1 - r_2 \notin \mathbb{Z}$
$$y_1(x) = \sum_{n=0}^{\infty} c_n u^{n+r_1}, \; c_0 \neq 0, \quad y_2(x) = \sum_{n=0}^{\infty} d_n u^{n+r_2}, \; d_0 \neq 0$$
Case 2: $r_1 > r_2$ and $r_1 - r_2 \in \mathbb{Z}$
$$y_1(x) = \sum_{n=0}^{\infty} c_n u^{n+r_1}, \; c_0 \neq 0, \quad y_2(x) = \underbrace{C}_{\text{can be } 0} y_1(x) \ln u + \sum_{n=0}^{\infty} d_n u^{n+r_2}, \; d_0 \neq 0.$$
Case 3: $r_1 = r_2$
$$y_1(x) = \sum_{n=0}^{\infty} c_n u^{n+r_1}, \; c_0 \neq 0, \quad y_2(x) = y_1(x) \ln u + \sum_{n=1}^{\infty} d_n u^{n+r_2}.$$

Deriving the Recursive Formula

$$(x^2+1)y''+xy'-y \qquad \sum_{n=0}^{\infty}\left\{(n^2-1)c_n+(n+2)(n+1)c_{n+2}\right\}x^n$$

Deriving the Recursive Formula



$$\sum_{n=0}^{\infty}\left\{(n^2-1)c_n+(n+2)(n+1)c_{n+2}\right\}x^n$$

Deriving the Recursive Formula

$$y''$$

$$(x^2 + 1)y'' + xy' - y$$

$$\sum_{n=0}^{\infty} \left\{ (n^2 - 1)c_n + (n+2)(n+1)c_{n+2} \right\} x^n$$

$$x^2y'' + xy' - y$$

$$x^2y'' = x^2 \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} c_n n(n-1)x^n$$

$$xy' = x \sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_n n x^n$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Deriving the Recursive Formula

$$y'' = \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k$$

y''

$$(x^2+1)y'' + xy' - y$$

$$\sum_{n=0}^{\infty} \left\{ (n^2-1)c_n + (n+2)(n+1)c_{n+2} \right\} x^n$$

$$x^2y'' + xy' - y$$

$$x^2y'' = x^2 \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} c_n n(n-1)x^n$$

$$xy' = x\sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_n n x^n$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Deriving the Recursive Formula

$$(x^2+1)y'' + xy' - y$$

$$y'' = \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k$$

$$x^2y'' + xy' - y$$

$$x^2y'' = x^2 \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} c_n n(n-1)x^n$$

$$xy' = x \sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_n n x^n$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=0}^{\infty} \left\{(n^2-1)c_n + (n+2)(n+1)c_{n+2}\right\} x^n$$

Method of Frobenius: Indicial Equation

Consider a series solution of the following DE about the regular singular point $x = 0$:

$$x^2 y'' + xp(x)y' + q(x)y = 0$$

where $p(x) = \sum_{n=0}^{\infty} a_n x^n$ and $q(x) = \sum_{n=0}^{\infty} b_n x^n$.

For a series solution $\sum_{n=0}^{\infty} c_n x^{n+r}$, $c_0 \neq 0$, $\{c_n\}$ and r have to satisfy

$$\boxed{c_n I(n+r) + \sum_{k=0}^{n-1} c_k \{a_{n-k}(k+r) + b_{n-k}\} = 0, \quad n = 0, 1, \ldots,}$$

where $I(r) = r(r-1) + a_0 r + b_0$.

For $n = 0$, the condition reduces to

$$\boxed{I(r) = r(r-1) + a_0 r + b_0 = 0}.$$

Notes

1. **Radius of Convergence**:

   When writing the solution in the form of a series, do not forget to specify the radius of convergence.

2. **From Series Solution to Analytic Expression**:

   - Once you found the power series or generalized power series solution, try your best to convert it back to an analytic expression using known Taylor series.
   - Do not forget to plug it back to see if the analytic solution you found is indeed a solution.

3. **Method of Frobenius and Reduction of Order**:

   If the roots of the indicial equation, $r = r_1, r_2$, differ by an integer ($r_1 > r_2$, $r_1 - r_2 \in \mathbb{Z}$), solve for r_1 first. For r_2, try to find a series solution first; if the found solution is the same as that for r_2, try to use the formula on Page 2 or use the method of reduction of order.

Laplace Transforms of Some Basic Functions

$f(t)$	$F(s)$	Domain of $F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$	$s > 0$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$	$s > 0$
$\sinh(kt)$	$\dfrac{k}{s^2-k^2}$	$s > \|k\|$
$\cosh(kt)$	$\dfrac{s}{s^2-k^2}$	$s > \|k\|$

Limiting Behavior

Theorem ($F(s)$ at $s = \infty$)

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order, and $\mathscr{L}\{f(t)\} = F(s)$, then $\lim_{s\to\infty} F(s) = 0$.

Theorem (Initial and Final Values)

If $f(t)$ is a function for which $f'(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order, and $\mathscr{L}\{f(t)\} = F(s)$, then $f(0) = \lim_{s\to\infty} sF(s)$.

Furthermore, if $f(\infty) := \lim_{t\to\infty} f(t)$ exists, $f(\infty) = \lim_{s\to 0} sF(s)$.

Translation and Scaling

Translation:

$$e^{at}f(t) \xrightarrow{\mathscr{L}} F(s-a)$$

$$e^{-as}F(s) \xrightarrow{\mathscr{L}^{-1}} f(t-a)\mathcal{U}(t-a)$$

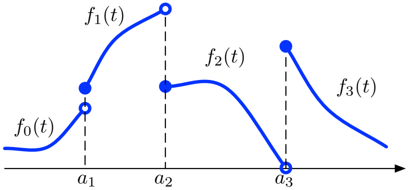
Scaling:

$$f(at) \xrightarrow{\mathscr{L}} \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$F(as) \xrightarrow{\mathscr{L}^{-1}} \frac{1}{a}f\left(\frac{t}{a}\right)$$

Piecewise-Defined Function

$$f(t) = \begin{cases} f_0(t), & t < a_1 \\ f_1(t), & a_1 \le t < a_2 \\ \vdots & \vdots \\ f_n(t), & t \ge a_n \end{cases}$$



$$\begin{aligned} &= f_0(t)\{1 - \mathcal{U}(t-a_1)\} + f_1(t)\{\mathcal{U}(t-a_1) - \mathcal{U}(t-a_2)\} \\ &\quad + f_2(t)\{\mathcal{U}(t-a_2) - \mathcal{U}(t-a_3)\} + \cdots + f_n(t)\mathcal{U}(t-a_n). \end{aligned}$$

Derivatives, Integrals, and Convolution

Derivatives:

$$f^{(n)}(t) \quad \xrightarrow{\mathscr{L}} \quad s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$$

$$F^{(n)}(s) \quad \xrightarrow{\mathscr{L}^{-1}} \quad (-t)^n f(t)$$

Integrals:

$$\int_0^t f(\tau) g(t-\tau)\, d\tau \quad \xrightarrow{\mathscr{L}} \quad F(s)G(s)$$

$$\int_0^t f(\tau)\, d\tau \quad \xrightarrow{\mathscr{L}} \quad \frac{F(s)}{s}$$

$$\int_s^\infty F(u)\, du \quad \xrightarrow{\mathscr{L}^{-1}} \quad \frac{f(t)}{t}$$

Periodic Functions, Dirac Delta

Periodic Function:

$f(t)$, period T $\quad \xrightarrow{\mathscr{L}} \quad \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T f(t)e^{-st}dt$

Dirac Delta Function:

$\delta(t-t_0)$, $t_0 \geq 0$ $\quad \xrightarrow{\mathscr{L}} \quad e^{-st_0}$

General Procedure of Solving IVP with Laplace Transform

Initial Value Problem

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = g(t)$$

$$y(0) = y_0,\ y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$$

↓ Laplace Transform

$$\mathscr{L}\left\{y^{(k)}(t)\right\} = s^k Y(s) - \left(s^{k-1} y_0 + s^{k-2} y_1 + \cdots + y_{k-1}\right)$$

An Algebraic Equation

$$P(s)Y(s) = Q(s) + G(s)$$

$$P(s) := \sum_{k=0}^{n} a_k s^k,$$

$$Q(s) = \sum_{k=0}^{n} a_k \sum_{j=0}^{k-1} y_{k-1-j} s^j$$

→ Solve the Algebraic Equation

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

↑ Partial Fraction Decomposition

↑ Inverse Laplace Transform

$$y(t)$$

Functions as Vectors: Inner Product

Definition (Inner Product of Functions)

The inner product of $f_1(x)$ and $f_2(x)$ on an interval $[a, b]$ is defined as

$$\langle f_1, f_2 \rangle := \int_a^b f_1(x) f_2(x)\, dx$$

Definition (Norm of a Function)

The norm of a function $f(x)$ on an interval $[a, b]$ is

$$||f(x)|| := \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b (f(x))^2 \ dx}$$

.

Orthogonality of Functions

Definition (Orthogonal Functions)

$f_1(x)$ and $f_2(x)$ are **orthogonal** on an interval $[a, b]$ if $\langle f_1, f_2 \rangle = 0$.

Definition (Orthogonal Set)

$\{\phi_0(x), \phi_1(x), \cdots\}$ are **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n.$$

Definition (Orthonormal Set)

$\{\phi_0(x), \phi_1(x), \cdots\}$ are **orthonomal** on an interval $[a, b]$ if they are orthogonal and $||\phi_n(x)|| = 1$ for all n.

Orthogonal Series Expansion

Projecting $f(x)$ onto the space spanned by an infinite orthogonal set $\{\phi_n(x) \mid n = 0, 1, \ldots\}$ on some interval $[a, b]$ results in

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \qquad c_m = \frac{\langle f, \phi_m \rangle}{||\phi_m||^2}}.$$

Definition of Fourier Series

Definition

The **Fourier series** of a function $f(x)$ defined on the interval $(a, a+2p)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right) \right\},$$

$$a_n = \frac{1}{p}\int_a^{a+2p} f(x)\cos\left(\frac{n\pi}{p}x\right)\, dx, \quad b_n = \frac{1}{p}\int_a^{a+2p} f(x)\sin\left(\frac{n\pi}{p}x\right)\, dx.$$

Complex Form:

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{p}x}, \quad \text{where } c_n = \frac{1}{2p}\int_a^{a+2p} f(x)\, e^{-\frac{in\pi}{p}x}\, dx.$$

Fourier Cosine and Sine Series

Definition

The **Fourier cosine series** of a function $f(x)$ defined on $(0, p)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right), \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)\, dx.$$

Definition

The **Fourier sine series** of a function $f(x)$ defined on $(0, p)$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right), \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)\, dx.$$

Half-Range Expansions

3 options to expand a function $f(x)$ defined on the interval $(0, L)$:

1. **Fourier Cosine Series**: Take $p := L$, and expand it as

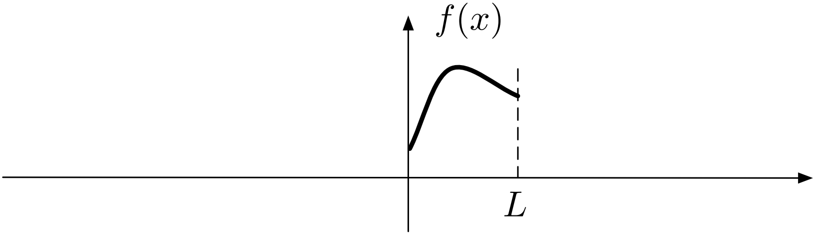
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx.$$

2. **Fourier Sine Series**: Take $p := L$, and expand it as

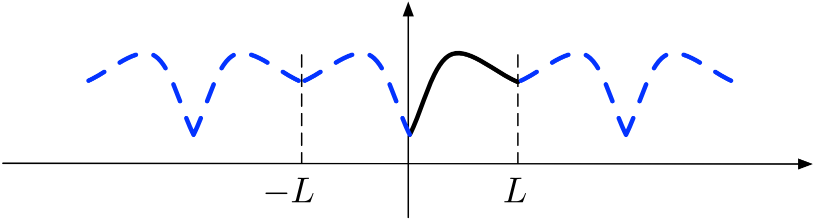
$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

3. **Fourier Series**: Take $a := 0,\ 2p := L$, and expand it as

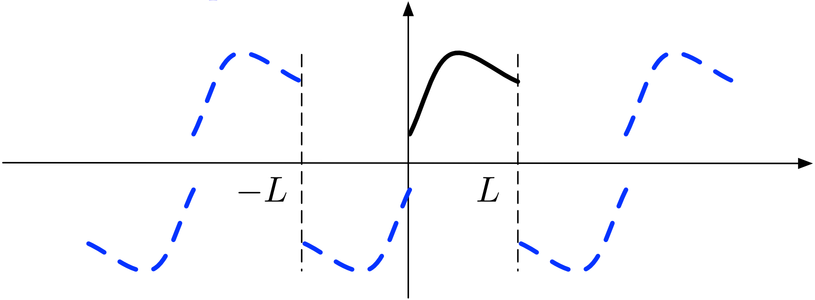
$$\sum_{n=-\infty}^{\infty} c_n e^{i\frac{2n\pi}{L}x}, \quad \text{where } c_n = \frac{1}{L}\int_0^L f(x)\,e^{-i\frac{2n\pi}{L}x}\,dx.$$


$f(x)$
L

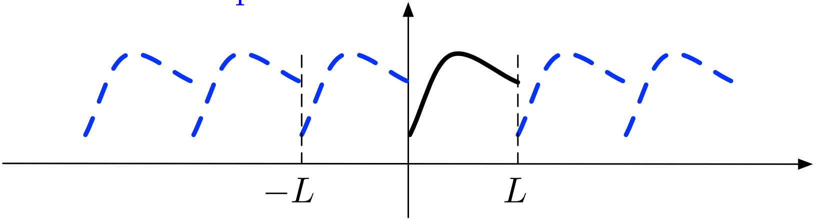
Expansion in Fourier cosine series



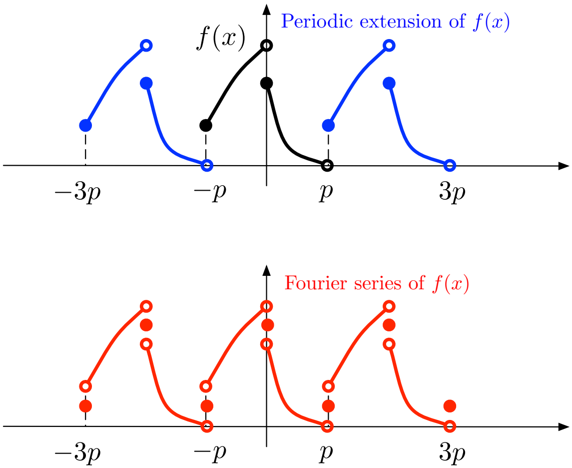
Expansion in Fourier sine series



Expansion in Fourier series



At Discontinuities



Three Classical PDE's

1. (One-Dimensional) Heat Equation/Diffusion Equation

$$\boxed{ku_{xx} = u_t, \;\; k > 0}$$

2. (One-Dimensional) Wave Equation/Telegraph Equation

$$\boxed{a^2 u_{xx} = u_{tt}}$$

3. (Two-Dimensional) Laplace's Equation

$$\boxed{u_{xx} + u_{yy} = 0}$$

Dirchlet Problem: Solution is a Fourier Sine Series

1. Heat Equation $ku_{xx} = u_t,\ \ k > 0, \quad u(0, t) = u(L, t) = 0$:

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)}.$$

2. Wave Equation $a^2 u_{xx} = u_{tt}, \quad u(0, t) = u(L, t) = 0$:

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} \left\{A_n \cosh\left(\frac{n\pi a}{L}t\right) + B_n \sinh\left(\frac{n\pi a}{L}t\right)\right\} \sin\left(\frac{n\pi}{L}x\right)}.$$

3. Laplace's Equation $u_{xx} + u_{yy} = 0 \quad u(0, y) = u(a, y) = 0$:

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} \left\{A_n \cosh\left(\frac{n\pi}{a}y\right) + B_n \sinh\left(\frac{n\pi}{a}y\right)\right\} \sin\left(\frac{n\pi}{a}x\right)}.$$

Neumann Problem: Solution is a Fourier Cosine Series

1. Heat Equation $ku_{xx} = u_t, \ k > 0, \quad u_x(0, t) = u_x(L, t) = 0$:

$$\boxed{u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k\frac{n^2\pi^2}{L^2}t} \cos\left(\frac{n\pi}{L}x\right)}.$$

2. Wave Equation $a^2 u_{xx} = u_{tt}, \quad u_x(0, t) = u_x(L, t) = 0$:

$$\boxed{u(x, t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \left\{A_n \cosh\left(\frac{n\pi a}{L}t\right) + B_n \sinh\left(\frac{n\pi a}{L}t\right)\right\} \cos\left(\frac{n\pi}{L}x\right)}.$$

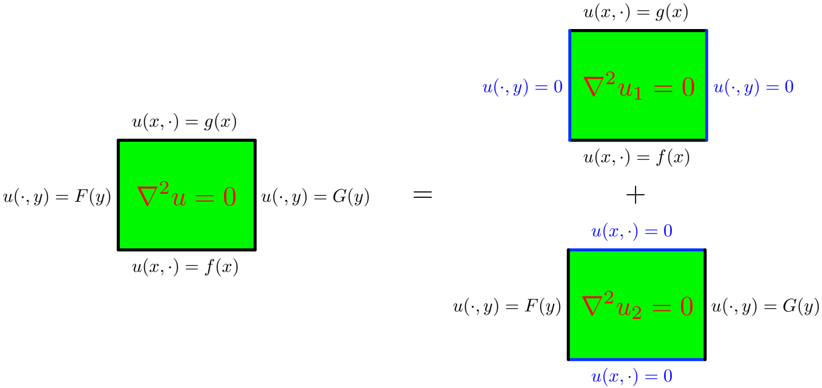
3. Laplace's Equation $u_{xx} + u_{yy} = 0 \quad u_x(0, y) = u_x(a, y) = 0$:

$$\boxed{u(x, y) = A_0 + B_0 y + \sum_{n=1}^{\infty} \left\{A_n \cosh\left(\frac{n\pi}{a}y\right) + B_n \sinh\left(\frac{n\pi}{a}y\right)\right\} \cos\left(\frac{n\pi}{a}x\right)}.$$

Remarks

1. Plug in the rest of the conditions to find out undetermined coefficients $\{A_n,\ B_n\}$, using the formulas of Fourier coefficients.
2. If the homogeneous conditions are not of the same type (i.e., both Dirchlet or both Neumann), use separation of variables step-by-step.

Superposition Principle



1 Series Solution of Linear Equations

2 Laplace Transform

3 Fourier Series

4 Fourier Transform

Fourier Transform

Definition (Fourier Integral and Fourier Transform)

The **Fourier transform** of $f(x)$ is

$$\mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx := F(\alpha).$$

The **inverse Fourier transform** of a function $F(\alpha)$ is

$$\mathscr{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha := f(x).$$

Fourier Cosine Integral and Fourier Cosine Transform

Definition (Fourier Cosine Integral and Fourier Cosine Transform)

The **Fourier cosine transform** of $f(x)$ is

$$\mathscr{F}_c\{f(x)\} = \int_0^\infty f(x)\cos\alpha x\,dx = F(\alpha).$$

The **inverse Fourier cosine transform** of a function $F(\alpha)$ is

$$\mathscr{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos\alpha x\,d\alpha = f(x).$$

Fourier Sine Integral and Fourier Sine Transform

Definition (Fourier Sine Integral and Fourier Sine Transform)

The **Fourier sine transform** of $f(x)$ is

$$\mathscr{F}_s\left\{f(x)\right\} = \int_0^\infty f(x)\sin\alpha x\,dx = F(\alpha).$$

The **inverse Fourier sine transform** of a function $F(\alpha)$ is

$$\mathscr{F}_s^{-1}\left\{F(\alpha)\right\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\sin\alpha x\,d\alpha = f(x).$$

Fourier Transforms of Derivatives

Theorem

If $f(x), f'(x) \to 0$ *as* $x \to \pm\infty$, *then*

$$\begin{aligned}
\mathscr{F}\{f'(x)\} &= i\alpha\mathscr{F}\{f(x)\}, & \mathscr{F}\{f''(x)\} &= -\alpha^2\mathscr{F}\{f(x)\} \\
\mathscr{F}_s\{f'(x)\} &= -\alpha\mathscr{F}_c\{f(x)\}, & \mathscr{F}_s\{f''(x)\} &= -\alpha^2\mathscr{F}_s\{f(x)\} + \alpha f(0) \\
\mathscr{F}_c\{f'(x)\} &= \alpha\mathscr{F}_s\{f(x)\} - f(0), & \mathscr{F}_c\{f''(x)\} &= -\alpha^2\mathscr{F}_c\{f(x)\} - f'(0)
\end{aligned}$$

Heat Equation in an Infinite Rod

$$\begin{array}{lll} \text{Solve } u(x,t): & ku_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0 \\ \text{subject to}: & u(\pm\infty, t) = 0, \quad u_x(\pm\infty, t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad -\infty < x < \infty \end{array}$$

Step 1: Take the Fourier transform w.r.t. x : $u(x,t) \xrightarrow{\mathscr{F}} U(\alpha, t)$:

$$-k\alpha^2 U(\alpha, t) = \frac{dU}{dt} \quad \text{subject to: } U(\alpha, 0) = F(\alpha).$$

Step 2: Solve $U(\alpha, t)$: $U(\alpha, t) = F(\alpha) e^{-k\alpha^2 t}$.

Step 3: Take inverse Fourier transform to find $u(x,t)$:

$$u(x,t) = \mathscr{F}^{-1}\{U(\alpha, t)\} = \mathscr{F}^{-1}\left\{F(\alpha) e^{-k\alpha^2 t}\right\}$$

Laplace's Equation in a Semi-Infinite Plate

$$\begin{array}{rl} \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<\infty \\ \text{subject to}: & u_x(0,y)=f(y), \quad u_x(a,y)=g(y), \quad 0<y<\infty \\ & u_y(x,0)=0, \quad u(x,\infty)=u_y(x,\infty)=0, \quad 0<x<a \end{array}$$

Step 1: Take the Fourier **cosine** transform w.r.t. y : $u(x,y) \xrightarrow{\mathscr{F}_c} U(x,\alpha)$:

$$\frac{d^2U}{dx^2} - \alpha^2 U(x,\alpha) - u_y(x,0) = 0 \quad \text{s.t. } U'(0,\alpha) = F(\alpha), \; U'(a,\alpha) = G(\alpha).$$

Step 2: Solve $U(x,\alpha)$:

$$U(x,\alpha) = \left(\frac{G(\alpha)}{\alpha\sinh\alpha a} - \frac{F(\alpha)}{\alpha\tanh\alpha a}\right)\sinh\alpha x + \frac{F(\alpha)}{\alpha}\cosh\alpha x.$$

Step 3: Take inverse Fourier cosine transform to find $u(x,y)$:

$$u(x,y) = \mathscr{F}_c^{-1}\left\{\left(\frac{G(\alpha)}{\alpha\sinh\alpha a} - \frac{F(\alpha)}{\alpha\tanh\alpha a}\right)\sinh\alpha x + \frac{F(\alpha)}{\alpha}\cosh\alpha x\right\}.$$

Solving BVP with Fourier Transforms

1. Boundary conditions at $\pm\infty$, such as $u(\pm\infty, t) = u_x(\pm\infty, t) = 0$ and $u(x, \infty) = u_y(x, \infty) = 0$, are used to guarantee that the Fourier transforms of the second-order partial derivates exist.

2. Which transform to use? Suppose the unbounded range is on x.
   - If the range is $(-\infty, \infty)$, use Fourier transform.
   - If the range is $(0, \infty)$ and at 0 the given condition is on u, use Fourier sine transform.
     (Because $\mathscr{F}_s\{f''(x)\} = -\alpha^2 \mathscr{F}_s\{f(x)\} + \alpha f(0)!$)
   - If the range is $(0, \infty)$ and at 0 the given condition is on u_x, use Fourier sine transform.
     (Because $\mathscr{F}_c\{f''(x)\} = -\alpha^2 \mathscr{F}_c\{f(x)\} - f'(0)!$)

Fourier Transform or Fourier Series?

1. Use **Fourier Series** if all the homogeneous conditions are given at finite boundaries.
2. Use **Fourier Transform** if all the homogeneous conditions are given at infinite boundaries $\pm\infty$.
3. Use **Fourier Cosine/Sine Transform** if one homogeneous conditions are given at infinite boundaries $+\infty$.