Chapter 14: Fourier Transforms and Boundary Value Problems in an Unbounded Region

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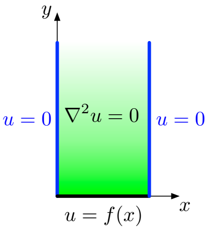
So far we have seen how to solve boundary-value problems within a bounded region, where the boundary conditions are given at finite boundaries, e.g.,

- (one-dimensional) heat equation, wave equation: $x \in [0, L]$
- (two-dimensional) Laplace's equation: $(x, y) \in [0, a] \times [0, b]$

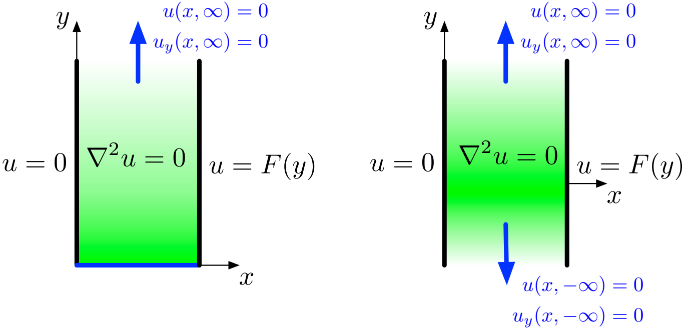
Exception: a BVP on semi-finite plate



Note: we are able to solve this via Fourier series because the homogeneous boundary conditions are still at finite boundaries $x = 0, a$.

From the homogeneous boundary conditions, we can conclude that the solution $u(x, t)$ is a **Fourier cosine/sine series** in x.

What if the homogeneous boundary conditions are given at *infinite boundaries*, i.e., $\pm\infty$?



Separation of variables: unable to use the homogeneous boundary conditions to find the possible values of the separation constant λ.

We introduce **Fourier Transforms** to deal with this issue.

From Fourier Series to Fourier Integral

Recall: For a function $f(x)$ defined on $(-p, p)$, its Fourier series is

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{2p} \int_{-p}^{p} f(x) e^{-i\frac{n\pi}{p}x}\, dx \right) e^{i\frac{n\pi}{p}x}$$

Let $\alpha_n := \dfrac{n\pi}{p}$ and $\Delta\alpha := \alpha_{n+1} - \alpha_n = \dfrac{\pi}{p}$.

Taking $p \to \infty$, $\Delta\alpha \to 0$, and the Fourier series becomes

$$\frac{1}{2\pi} \lim_{p\to\infty} \sum_{n=-\infty}^{\infty} \overbrace{\left(\int_{-p}^{p} f(x) e^{-i\alpha_n x}\, dx \right)}^{F_p(\alpha_n)} e^{i\alpha_n x} \Delta\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \lim_{p\to\infty} F_p(\alpha) \right\} e^{i\alpha x}\, d\alpha \quad \Delta\alpha = \frac{\pi}{p} \to 0 \text{ when } p \to \infty$$

$$= \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx \right\} e^{i\alpha x}\, d\alpha}$$

Fourier Integral and Fourier Transform

Definition (Fourier Integral and Fourier Transform)

The **Fourier integral** of a function $f(x)$ defined on $(-\infty, \infty)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha, \quad \text{where } F(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx.$$

The **Fourier transform** of $f(x)$ is

$$\mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx := F(\alpha).$$

The **inverse Fourier transform** of a function $F(\alpha)$ is

$$\mathscr{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha := f(x).$$

Fourier and Inverse Fourier Transforms

$$f(x) \quad \xrightarrow{\mathscr{F}} \quad F(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

$$F(\alpha) \quad \xrightarrow{\mathscr{F}^{-1}} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha$$

Fourier Coefficients and Fourier Series

$$f(x) \quad \xrightarrow{\mathcal{FS}} \quad c_n = \frac{1}{2p} \int_{-p}^{p} f(x)\, e^{-i\frac{n\pi}{p}x}\, dx$$

$$c_n \quad \xrightarrow{\mathcal{FS}^{-1}} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{p}x}$$

Examples

Example

Find the Fourier integral representation and the Fourier transform of the function $f(x) = \begin{cases} 1, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$.

The Fourier transform

$$F(\alpha) = \mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx = \int_{0}^{2} e^{-i\alpha x}\, dx = \frac{1}{-i\alpha}\left(e^{-2i\alpha} - 1\right)$$

The Fourier integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x}\, d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i}{\alpha}\left(e^{-2i\alpha} - 1\right) e^{i\alpha x}\, d\alpha$$

Sufficient Condition of Convergence

Theorem (Convergence of Fourier Integral)

Let f and f' be piecewise continuous on every finite interval and let f be absolutely integrable on $(-\infty, \infty)$ (i.e., $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges). Then, its Fourier integral converges to

- $f(x)$ *at a point where* $f(x)$ *is continuous*
- $\frac{1}{2}\left(f(x+) + f(x-)\right)$ *at a point where* $f(x)$ *is discontinuous.*

Alternative Form of the Fourier Integral

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,(\cos\alpha x + i\sin\alpha x)\; d\alpha \\
&= \frac{1}{2\pi}\left\{\int_{0}^{\infty} F(\alpha)\,(\cos\alpha x + i\sin\alpha x)\; d\alpha + \int_{-\infty}^{0} F(\alpha)\,(\cos\alpha x + i\sin\alpha x)\; d\alpha\right\} \\
&= \frac{1}{2\pi}\left\{\int_{0}^{\infty} F(\alpha)\,(\cos\alpha x + i\sin\alpha x)\; d\alpha + \int_{0}^{\infty} F(-\alpha)\,(\cos\alpha x - i\sin\alpha x)\; d\alpha\right\} \\
&= \frac{1}{2\pi}\int_{0}^{\infty} \left\{ [F(\alpha) + F(-\alpha)]\cos\alpha x + i\,[F(\alpha) - F(-\alpha)]\sin\alpha x \right\} d\alpha \\
&= \boxed{\frac{1}{\pi}\int_{0}^{\infty}\left\{\left[\int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx\right]\cos\alpha x + \left[\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx\right]\sin\alpha x\right\}\, d\alpha}
\end{aligned}
$$

$$
F(\alpha) + F(-\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx + \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx = 2\int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx
$$

$$
F(\alpha) - F(-\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx - \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx = -2i\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx
$$

Alternative Form of the Fourier Integral

$$f(x) = \frac{1}{\pi}\int_0^\infty \{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\}\ d\alpha,$$

where $A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx, \quad B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx.$

Fourier Integral of an Even Function

$$f(x) = \frac{1}{\pi}\int_0^\infty \{A(\alpha)\cos\alpha x + \cancel{B(\alpha)\sin\alpha x}\}\ d\alpha,$$

where $A(\alpha) = 2\int_0^\infty f(x)\cos\alpha x\,dx, \quad B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx = 0.$

Fourier Integral of an Odd Function

$$f(x) = \frac{1}{\pi}\int_0^\infty \{\cancel{A(\alpha)\cos\alpha x} + B(\alpha)\sin\alpha x\}\ d\alpha,$$

where $A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx = 0, \quad B(\alpha) = 2\int_0^\infty f(x)\sin\alpha x\,dx.$

Fourier Cosine Integral and Fourier Cosine Transform

Definition (Fourier Cosine Integral and Fourier Cosine Transform)

The **Fourier cosine integral** of a function $f(x)$ defined on $(0, \infty)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\alpha) \cos \alpha x \, d\alpha, \quad \text{where } F(\alpha) = \int_0^\infty f(x) \cos \alpha x \, dx.$$

The **Fourier cosine transform** of $f(x)$ is

$$\mathscr{F}_c \{f(x)\} = \int_0^\infty f(x) \cos \alpha x \, dx = F(\alpha).$$

The **inverse Fourier cosine transform** of a function $F(\alpha)$ is

$$\mathscr{F}_c^{-1} \{F(\alpha)\} = \frac{2}{\pi} \int_0^\infty F(\alpha) \cos \alpha x \, d\alpha = f(x).$$

Fourier Sine Integral and Fourier Sine Transform

Definition (Fourier Sine Integral and Fourier Sine Transform)

The **Fourier sine integral** of a function $f(x)$ defined on $(0, \infty)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\alpha)\sin\alpha x\,d\alpha, \quad \text{where } F(\alpha) = \int_0^\infty f(x)\sin\alpha x\,dx.$$

The **Fourier sine transform** of $f(x)$ is

$$\mathscr{F}_s\{f(x)\} = \int_0^\infty f(x)\sin\alpha x\,dx = F(\alpha).$$

The **inverse Fourier sine transform** of a function $F(\alpha)$ is

$$\mathscr{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\sin\alpha x\,d\alpha = f(x).$$

Examples

Example

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise} \end{cases}.$$

$f(x)$ is an even function. Hence, its Fourier integral is a cosine integral

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \left\{\int_0^\infty f(x)\cos\alpha x\,dx\right\}\cos\alpha x\,d\alpha \\ &= \frac{2}{\pi}\int_0^\infty \left\{\int_0^a \cos\alpha x\,dx\right\}\cos\alpha x\,d\alpha \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha a\cos\alpha x}{\alpha}\,d\alpha \end{aligned}$$

Examples

Example

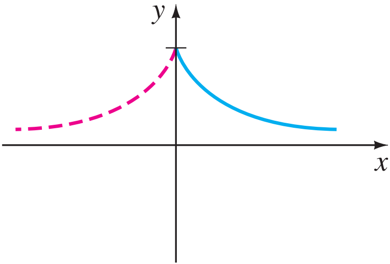
On $(0, \infty)$, represent $f(x) = e^{-x}$ (a) by a Fourier cosine integral, and (b) by a Fourier sine integral.

(a)

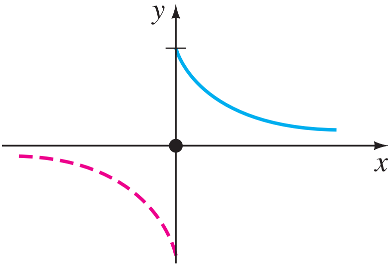
$$f(x) = \frac{2}{\pi} \int_0^\infty \left\{ \int_0^\infty e^{-x} \cos \alpha x \, dx \right\} \cos \alpha x \, d\alpha = \frac{2}{\pi} \int_0^\infty \frac{1}{1+\alpha^2} \cos \alpha x \, d\alpha$$

(b)

$$f(x) = \frac{2}{\pi} \int_0^\infty \left\{ \int_0^\infty e^{-x} \sin \alpha x \, dx \right\} \sin \alpha x \, d\alpha = \frac{2}{\pi} \int_0^\infty \frac{\alpha}{1+\alpha^2} \sin \alpha x \, d\alpha$$



(a) cosine integral



(b) sine integral

Fourier Transforms of Derivatives

The Fourier transform has many operational properties, and many of them resemble those of the Laplace transform.

In this lecture we only focus on the Fourier transform of derivatives, as it is useful in solving BVP's of PDE's.

Fact

If $f(x), f'(x) \to 0$ *as* $x \to \pm\infty$*, then*

$$\begin{aligned} \mathscr{F}\left\{f'(x)\right\} &= i\alpha\mathscr{F}\left\{f(x)\right\}, & \mathscr{F}\left\{f''(x)\right\} &= -\alpha^2\mathscr{F}\left\{f(x)\right\} \\ \mathscr{F}_s\left\{f'(x)\right\} &= -\alpha\mathscr{F}_c\left\{f(x)\right\}, & \mathscr{F}_s\left\{f''(x)\right\} &= -\alpha^2\mathscr{F}_s\left\{f(x)\right\} + \alpha f(0) \\ \mathscr{F}_c\left\{f'(x)\right\} &= \alpha\mathscr{F}_s\left\{f(x)\right\} - f(0), & \mathscr{F}_c\left\{f''(x)\right\} &= -\alpha^2\mathscr{F}_c\left\{f(x)\right\} - f'(0) \end{aligned}$$

$$\begin{aligned}
\mathscr{F}\left\{f'(x)\right\} &= \int_{-\infty}^{\infty} f'(x)\, e^{-i\alpha x}\, dx = \int_{-\infty}^{\infty} e^{-i\alpha x}\, d\left(f(x)\right) \\
&= \left[f(x)\, e^{-i\alpha x}\right]_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \\
&= i\alpha \mathscr{F}\left\{f(x)\right\} \qquad f(x) \to 0 \text{ as } x \to \pm\infty \\
\mathscr{F}_s\left\{f'(x)\right\} &= \int_0^{\infty} f'(x) \sin \alpha x\, dx = \int_0^{\infty} \sin \alpha x\, d\left(f(x)\right) \\
&= \left[f(x) \sin \alpha x\right]_0^{\infty} - \alpha \int_0^{\infty} f(x) \cos \alpha x\, dx \\
&= -\alpha \mathscr{F}_c\left\{f(x)\right\} \qquad f(x) \to 0 \text{ as } x \to \infty \\
\mathscr{F}_s\left\{f'(x)\right\} &= \int_0^{\infty} f'(x) \cos \alpha x\, dx = \int_0^{\infty} \cos \alpha x\, d\left(f(x)\right) \\
&= \left[f(x) \cos \alpha x\right]_0^{\infty} + \alpha \int_0^{\infty} f(x) \sin \alpha x\, dx \\
&= \alpha \mathscr{F}_s\left\{f(x)\right\} - f(0) \qquad f(x) \to 0 \text{ as } x \to \infty
\end{aligned}$$

Heat Equation in an Infinite Rod

$$\begin{array}{|rl|}\hline \text{Solve } u(x,t): & ku_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0 \\ \text{subject to}: & u(\pm\infty, t) = 0, \quad u_x(\pm\infty, t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad -\infty < x < \infty \\ \hline \end{array}$$

Step 1: Take the Fourier transform w.r.t. x:

Let $u(x,t) \xrightarrow{\mathscr{F}} U(\alpha, t)$. The original problem becomes

$$-k\alpha^2 U(\alpha, t) = \frac{dU}{dt} \quad \text{subject to: } U(\alpha, 0) = F(\alpha)$$

Note: The condition $u(\pm\infty, t) = 0, u_x(\pm\infty, t) = 0$ is used to conclude that $u_{xx} \xrightarrow{\mathscr{F}} -\alpha^2 U(\alpha, t)$

Heat Equation in an Infinite Rod

$$\begin{array}{|lll|}\hline \text{Solve } u(x,t): & ku_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0 \\ \text{subject to}: & u(\pm\infty, t) = 0, \quad u_x(\pm\infty, t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad -\infty < x < \infty \\ \hline \end{array}$$

Step 2: Solve $U(\alpha, t)$:

$$-k\alpha^2 U(\alpha, t) = \frac{dU}{dt} \implies U(\alpha, t) = C(\alpha) e^{-k\alpha^2 t}.$$

Plug in $U(\alpha, 0) = F(\alpha)$, we get $C(\alpha) = F(\alpha)$. Hence,

$$U(\alpha, t) = F(\alpha) e^{-k\alpha^2 t}.$$

Heat Equation in an Infinite Rod

$$\begin{array}{rlll} \text{Solve } u(x,t): & k u_{xx} = u_t, & -\infty < x < \infty, & t > 0 \\ \text{subject to}: & u(\pm\infty, t) = 0, & u_x(\pm\infty, t) = 0, & t > 0 \\ & u(x,0) = f(x), & -\infty < x < \infty & \end{array}$$

Step 3: Take inverse Fourier transform to find $u(x,t)$:

$$u(x,t) = \mathscr{F}^{-1}\{U(\alpha,t)\} = \mathscr{F}^{-1}\left\{F(\alpha)\, e^{-k\alpha^2 t}\right\}$$

$$= \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx\right) e^{i\alpha x} e^{-k\alpha^2 t}\, d\alpha}$$

Laplace's Equation in a Semi-Infinite Plate

$$\begin{array}{lll} \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, & 0<y<\infty \\ \text{subject to}: & u_x(0,y)=f(y), \quad u_x(a,y)=g(y), & 0<y<\infty \\ & u_y(x,0)=0, \quad u(x,\infty)=u_y(x,\infty)=0, & 0<x<a \end{array}$$

Step 1: Take the Fourier **cosine** transform w.r.t. y:

Let $u(x,y) \xrightarrow{\mathscr{F}_c} U(x,\alpha)$. The original problem becomes

$$\frac{d^2 U}{dx^2} - \alpha^2 U(x,\alpha) - u_y(x,0) = 0 \quad \text{s.t. } U(0,\alpha)=F(\alpha), \; U(a,\alpha)=G(\alpha)$$

Note: The condition $u(x,\infty)=u_y(x,\infty)=0$ is used to conclude that $u_{yy} \xrightarrow{\mathscr{F}_c} -\alpha^2 U(x,\alpha) - u_y(x,0)$

Laplace's Equation in a Semi-Infinite Plate

Solve $u(x, y)$:	$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < \infty$	
subject to :	$u_x(0, y) = f(y), \quad u_x(a, y) = g(y), \quad 0 < y < \infty$	
	$u_y(x, 0) = 0, \quad u(x, \infty) = u_y(x, \infty) = 0, \quad 0 < x < a$	

Step 2: Solve $U(x, \alpha)$:

$$\frac{d^2 U}{dx^2} - \alpha^2 U(x, \alpha) = 0 \implies U(x, \alpha) = C_1(\alpha) \cosh \alpha x + C_2(\alpha) \sinh \alpha x.$$

Plug in $U(0, \alpha) = F(\alpha)$, $U(a, \alpha) = G(\alpha)$, we get

$$C_1(\alpha) = F(\alpha), \quad C_2(\alpha) = \frac{G(\alpha) - F(\alpha) \cosh \alpha a}{\sinh \alpha a}.$$

$$\implies U(x, \alpha) = F(\alpha) \cosh \alpha x + \left(\frac{G(\alpha)}{\sinh \alpha a} - \frac{F(\alpha)}{\tanh \alpha a} \right) \sinh \alpha x.$$

Laplace's Equation in a Semi-Infinite Plate

Solve $u(x,y)$: $u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < \infty$

subject to : $u_x(0,y) = f(y), \quad u_x(a,y) = g(y), \quad 0 < y < \infty$

$u_y(x,0) = 0, \quad u(x,\infty) = u_y(x,\infty) = 0, \quad 0 < x < a$

Step 3: Take inverse Fourier cosine transform to find $u(x,y)$:

$$u(x,y) = \mathscr{F}_c^{-1}\left\{U(x,\alpha)\right\}$$

$$= \mathscr{F}_c^{-1}\left\{F(\alpha)\cosh\alpha x + \left(\frac{G(\alpha)}{\sinh\alpha a} - \frac{F(\alpha)}{\tanh\alpha a}\right)\sinh\alpha x\right\}$$

$$= \boxed{\frac{2}{\pi}\int_0^\infty \left\{F(\alpha)\cosh\alpha x + \left(\frac{G(\alpha)}{\sinh\alpha a} - \frac{F(\alpha)}{\tanh\alpha a}\right)\sinh\alpha x\right\}\cos\alpha y\, d\alpha}$$

where $F(\alpha) = \mathscr{F}_c\{f(y)\}$, and $G(\alpha) = \mathscr{F}_c\{g(y)\}$.

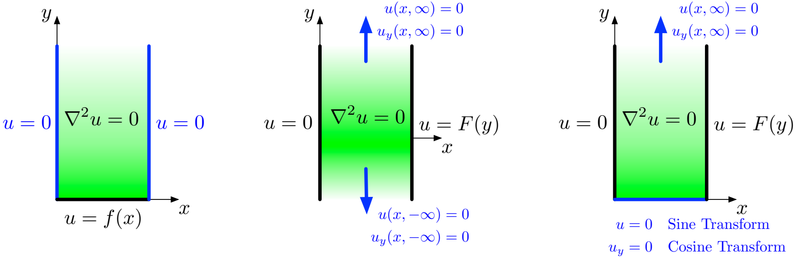
Remarks

1. Boundary conditions at $\pm\infty$, such as $u(\pm\infty, t) = u_x(\pm\infty, t) = 0$ and $u(x, \infty) = u_y(x, \infty) = 0$, are used to guarantee that the Fourier transforms of the second-order partial derivates exist.

2. Which transform to use? Suppose the unbounded range is on x.
   - If the range is $(-\infty, \infty)$, use Fourier transform.
   - If the range is $(0, \infty)$ and at 0 the given condition is on u, use Fourier sine transform.
     (Because $\mathscr{F}_s\{f''(x)\} = -\alpha^2\mathscr{F}_s\{f(x)\} + \alpha f(0)$!)
   - If the range is $(0, \infty)$ and at 0 the given condition is on u_x, use Fourier sine transform.
     (Because $\mathscr{F}_c\{f''(x)\} = -\alpha^2\mathscr{F}_c\{f(x)\} - f'(0)$!)

Fourier Transform or Fourier Series?

1. Use **Fourier Series** if all the homogeneous conditions are given at finite boundaries.
2. Use **Fourier Transform** if all the homogeneous conditions are given at infinite boundaries $\pm\infty$.
3. Use **Fourier Cosine/Sine Transform** if one homogeneous conditions are given at infinite boundaries $+\infty$.



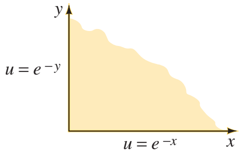
Example

Example

$$\text{Solve } u(x,y):\quad u_{xx}+u_{yy}=0,\quad 0<x<\infty,\quad 0<y<\infty$$

$$\text{subject to}:\quad u(0,y)=e^{-y},\quad u(\infty,y)=u_x(\infty,y)=0,\quad 0<y<\infty$$

$$u(x,0)=e^{-x},\quad u(x,\infty)=u_y(x,\infty)=0,\quad 0<x<\infty$$



Key Ideas:

1. Superposition Principle
2. Fourier Sine Transform

The final answer is

$$u(x,y)=\frac{2}{\pi}\int_0^\infty\left(\frac{\alpha e^{-\alpha y}}{1+\alpha^2}\sin\alpha x+\frac{\alpha e^{-\alpha x}}{1+\alpha^2}\sin\alpha y\right)\,d\alpha.$$

Short Recap

- Fourier Integral and Fourier Transform
- Fourier Cosine/Sine Integral and Fourier Cosine/Sine Transform
- Fourier Transforms to Solve BVP in an Unbounded Region
- Superposition Principle

Self-Practice Exercises

14-3: 3, 5, 9, 11, 17, 19

14-4: 1, 9, 11, 15, 18, 20, 21, 26