Chapter 12: Boundary-Value Problems in Rectangular Coordinates

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In this lecture, we focus on solving some *classical* partial differential equations in **boundary-value problems**.

Instead of solving the general solutions, we are only interested in finding *useful* particular solutions.

We focus on linear second order PDE: ($A, \cdots, G$: functions of x, y)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

notation: for example, $u_{xy} := \frac{\partial^2 u}{\partial x \partial y}$.

Method: Separation of variables – convert a PDE into two ODE's

Types of Equations:

- Heat Equation
- Wave Equation
- Laplace Equation

Classification of Linear Second Order PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

notation: for example, $u_{xy} := \frac{\partial^2 u}{\partial x \partial y}$.

1. Homogeneous vs. Nonhomogeneous

Homogeneous	$\Longleftrightarrow$	$G = 0$
Nonhomogeneous	$\Longleftrightarrow$	$G \neq 0.$

2. Hyperbolic, Parabolic, and Elliptic: $A, B, C, \cdots, G$: constants,

Hyperbolic	$\Longleftrightarrow$	$B^2 - 4AC > 0$
Parabolic	$\Longleftrightarrow$	$B^2 - 4AC = 0$
Elliptic	$\Longleftrightarrow$	$B^2 - 4AC < 0$

Superposition Principle

Theorem

If $u_1(x,y), u_2(x,y), \ldots, u_k(x,y)$ are solutions of a homogeneous linear PDE, then a linear combination

$$u(x,y) := \sum_{n=1}^{k} c_n u_n(x,y)$$

is also a solution.

Note: We shall assume without rigorous argument that the linear combination can be an infinite series

$$u(x,y) := \sum_{n=1}^{\infty} c_n u_n(x,y)$$

Separation of Variables

To find a particular solution of an PDE, one method is **separation of variables**, that is, assume that the solution $u(x, y)$ takes the form of a product of a x-function and a y-function:

$$u(x, y) = X(x)\,Y(y).$$

Then, with the following, *sometimes* the PDE can be converted into an ODE of X and an ODE of Y:

$$u_x = \frac{dX}{dx}Y = X'Y, \qquad u_y = X\frac{dY}{dy} = XY'$$

$$u_{xx} = \frac{d^2X}{dx^2}Y = X''Y, \qquad u_{yy} = X\frac{d^2Y}{dy^2} = XY'', \qquad u_{xy} = X'Y'$$

Note: Derivatives are with respect to different independent variables. For example, $X' := \dfrac{dX}{dx}$.

Convert a PDE into Two ODE's

Example

Use separation of variables to convert the PDE below into two ODE's.

$$x^2 u_{xx} + (x+1)u_y + (x+xy)u = 0$$

With $u(x,y) = X(x)Y(y)$, the original PDE becomes

$$x^2 X'' Y + (x+1)XY' + (x+1)yXY = 0$$
$$\implies x^2 X'' Y = -(x+1)X(Y' + yY)$$
$$\implies \frac{x^2 X''}{(x+1)X} = -\frac{Y'}{Y} - y \quad = \lambda \quad \text{separation constant}$$

Left-hand side is a function of x, independent of y; Right-hand side is a function of y, independent of x. Hence, the above is equal to something independent of x and y

Convert a PDE into Two ODE's

Example

Use separation of variables to convert the PDE below into two ODE's.

$$x^2 u_{xx} + (x+1)u_y + (x+xy)u = 0$$

With $u(x,y) = X(x)\,Y(y)$, the original PDE becomes

$$\begin{aligned}
& x^2 X'' Y + (x+1)XY' + (x+1)yXY = 0 \\
\Longrightarrow\ & x^2 X'' Y = -(x+1)X(Y' + yY) \\
\Longrightarrow\ & \frac{x^2 X''}{(x+1)X} = -\frac{Y'}{Y} - y \quad = \lambda \quad \text{separation constant} \\
\Longrightarrow\ & \begin{cases} x^2 X''(x) - \lambda(x+1)X(x) = 0 \\ Y'(y) + (y+\lambda)\,Y(y) = 0 \end{cases}
\end{aligned}$$

Some Remarks

1. The method of separation of variables can only solve for *some* linear second order PDE's, not all of them.

2. For the PDE's considered in this lecture, the method works.

3. The method may work for both homogeneous ($G = 0$) and nonhomogeneous ($G \neq 0$) PDE's

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

Three Classical PDE's

In this lecture we focus on solving boundary-value problems of the following three classical PDE's that arise frequently in physics, mechanics, and engineering:

1. (One-Dimensional) Heat Equation/Diffusion Equation

$$\boxed{ku_{xx} = u_t, \ \ k > 0}$$

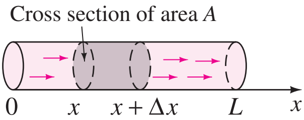
2. (One-Dimensional) Wave Equation/Telegraph Equation

$$\boxed{a^2 u_{xx} = u_{tt}}$$

3. (Two-Dimensional) Laplace Equation

$$\boxed{u_{xx} + u_{yy} = 0}$$

Heat Transfer within a Thin Rod: Heat Equation



Assumptions:

- Heat only flows in x-direction.
- No heat escapes from the surface.
- No heat is generated in the rod.
- Rod is homogeneous with density ρ.

Let $u(x, t)$ denote the temperature of the rod at location x at time t.

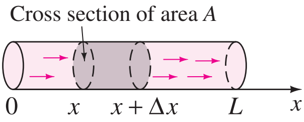
Consider the quantity of heat with $[x, x + dx]$: (γ : 比熱)

$$dQ = \gamma\,(\rho A\,dx)\,u \implies Q_x = \gamma\rho A u \implies Q_{xt} = \gamma\rho A u_t$$

Heat transfer rate through the cross section $= -KAu_x$, and hence the net heat rate inside $[x, x + dx]$ is $dQ_t = -KAu_x(x, t) - (-KAu_x(x + dx, t))$

$$dQ_t = KAu\,(u_x(x + dx, t) - u_x(x, t)) = KAu_{xx}dx$$
$$\implies Q_{tx} = KAu_{xx}$$

Heat Transfer within a Thin Rod: Heat Equation



Assumptions:

- Heat only flows in x-direction.
- No heat escapes from the surface.
- No heat is generated in the rod.
- Rod is homogeneous with density ρ.

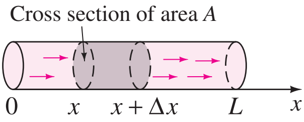
Let $u(x, t)$ denote the temperature of the rod at location x at time t.

Hence,

$$\begin{cases} Q_{xt} = \gamma\rho A u_t \\ Q_{tx} = K A u_{xx} \end{cases} \implies \gamma\rho \cancel{A} u_t = K \cancel{A} u_{xx} \implies \left(\frac{K}{\gamma\rho}\right) u_{xx} = u_t$$

$$\implies \boxed{k u_{xx} = u_t, \ \ k > 0}$$

Heat Equation: Initial and Boundary Conditions



Initial Condition:
Provides the spatial distribution of the temperature at time $t = 0$.

$$u(x, 0) = f(x), \; 0 < x < L$$

Boundary Conditions:
At the end points $x = 0$ and $x = L$, give the constraints on

- u: (Dirchlet condition), for example, (u_0: constant)

$$u(L, t) = u_0 \quad \text{Temperature at the right end is held at constant.}$$

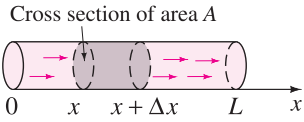
- u_x: (Neumann condition), for example,

$$u_x(L, t) = 0 \quad \text{The right end is insulated.}$$

- $u_x + hu$: (Robin condition), for example, ($h > 0$, u_m: constants)

$$u_x(L, t) = -h\{u(L, t) - u_m\} \quad \text{Heat is lost from the right end.}$$

Heat Equation: Boundary-Value Problems



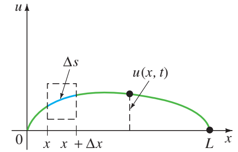
- A problem involving both initial and boundary conditions is called a **boundary-value** problem
- At the two boundaries $x - 0$ and $x = L$, one can use different kinds of conditions.

Examples:

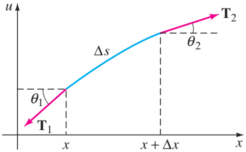
$ku_{xx} = u_t, \quad 0 < x < L, \quad t > 0$	Heat equation
$u(0,t) = u_0, \quad u_x(L,t) = -h\left\{u(L,t) - u_m\right\}, \quad t > 0$	Boundary condition
$u(x,0) = f(x), \quad 0 < x < L$	Initial condition

$ku_{xx} = u_t, \quad 0 < x < L, \quad t > 0$	Heat equation
$u_x(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$	Boundary condition
$u(x,0) = f(x), \quad 0 < x < L$	Initial condition

Dynamics of a String Fixed at Two Ends: Wave Equation



(a) Segment of string



Assumptions:

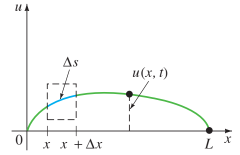
- No external force.
- Tension force is large compared to gravity and is the same at all points.
- Slope of the curve is very small at all points.
- Vertical displacement $\ll$ string length.
- String has mass per unit length ρ.

Let $u(x, t)$ denote the vertical position (displacement) of the string at location x at time t.

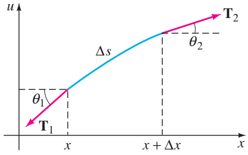
Consider the string in $[x, x + dx]$. Net vertical force is

$$\begin{aligned} T(\sin\theta_2 - \sin\theta_1) &\approx T(\tan\theta_2 - \tan\theta_1) \\ &= T\{u_x(x + dx, t) - u_x(x, t)\} \\ &= Tu_{xx}dx \end{aligned}$$

Dynamics of a String Fixed at Two Ends: Wave Equation



(a) Segment of string



Assumptions:

- No external force.
- Tension force is large compared to gravity and is the same at all points.
- Slope of the curve is very small at all points.
- Vertical displacement $\ll$ string length.
- String has mass per unit length ρ.

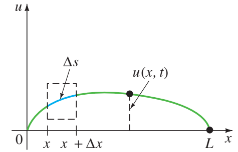
Let $u(x, t)$ denote the vertical position (displacement) of the string at location x at time t.

Since the slope is small, the mass $\approx \rho dx$. Hence

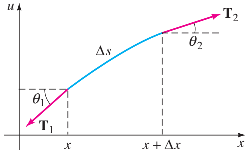
$$T u_{xx} \cancel{dx} = (\rho \cancel{dx})\, u_{tt} \implies \frac{T}{\rho} u_{xx} = u_{tt}$$

$$\implies \boxed{a^2 u_{xx} = u_{tt}}$$

Wave Equation: Initial and Boundary Conditions



(a) Segment of string



Initial Conditions:
Provide the initial displacement u and velocity u_t at time $t = 0$.

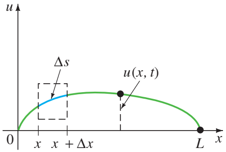
$$u(x, 0) = f(x), \; u_t(x, 0) = g(x), \; 0 < x < L$$

Boundary Conditions:
At the end points $x = 0$ and $x = L$, give the constraints on u, u_x, or $u_x + hu$. Usually in the scenario of strings, the boundary conditions are

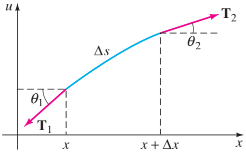
$$u(0, t) = 0, \; u(0, L) = 0, \; t > 0 \quad \text{Both ends are fixed.}$$

$$u_x(0, t) = 0, \; u_x(0, L) = 0, \; t > 0 \quad \text{Free-ends condition}$$

Wave Equation: Boundary-Value Problems



(a) Segment of string



Examples:

Both ends are fixed:

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0$$ Wave equation

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$$ Boundary condition

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < L$$ Initial condition

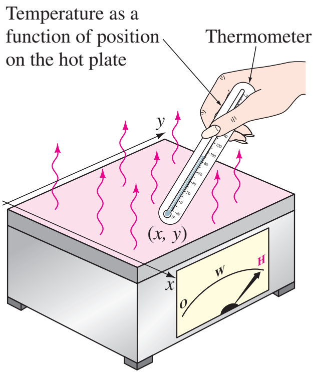
Free Ends:

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0$$ Wave equation

$$u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad t > 0$$ Boundary condition

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < L$$ Initial condition

Laplace's Equation



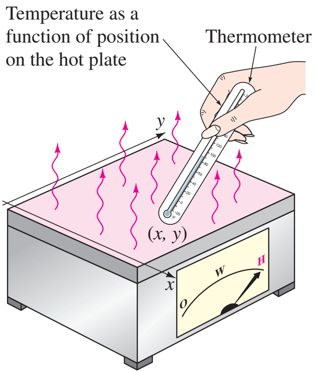
- Laplace's equation usually occurs in *time-independent* problems involving **potentials**.
- Its solution can also be interpreted as a steady-state temperature distribution.
- Two-dimensional Laplace Equation

$$\nabla^2 u := u_{xx} + u_{yy} = 0$$

- Three-dimensional Laplace Equation

$$\nabla^2 u := u_{xx} + u_{yy} + u_{zz} = 0$$

Laplace's Equation: Boundary Conditions



Boundary Conditions:
In the x-direction, at the end points $x = 0$ and $x = a$, give the constraints on u, u_x, or $u_x + hu$.

In the y-direction, at the end points $y = 0$ and $y = b$, give the constraints on u, u_y, or $u_y + hu$.

Examples:

- Both ends in x are insulated

$$u_x(0, y) = 0, \quad u_x(a, y) = 0$$

- Temperatures of two ends in y are held at different distributions

$$u(x, 0) = f(x), \quad u(x, b) = g(x)$$

Laplace's Equation: Boundary-Value Problems

Example:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b$$

Laplace's equation

$$u_x(0, y) = 0, \quad u_x(a, y) = 0, \quad 0 < y < b$$

Boundary condition

$$u(x, 0) = f(x), \quad u(x, b) = g(x), \quad 0 < x < a$$

Boundary condition

Modifications of Heat and Wave Equations

In the derivation of the heat equation and the wave equation, we assume that there is no internal or external influences. For example, *no heat escapes from the surface, no heat is generated in the rod, no external force act on the string, etc.*

Taking external and internal influences into account, more general forms of the heat equation and the wave equation are the following:

$$ku_{xx} + G(x, t, u, u_x) = u_t \qquad \text{Heat Equation}$$

$$a^2 u_{xx} + F(x, t, u, u_t) = u_{tt} \qquad \text{Wave Equation}$$

Example:

$$ku_{xx} - h(u - u_m) = u_t$$

heat transfers from the surface to an environment with constant temperature u_m

$$a^2 u_{xx} + f(x, t) = u_{tt}$$

External force f acts on the string

Homogeneous vs. Nonhomogeneous Boundary Conditions

Homogeneous Boundary Condition:

$$u_x(0, y) = 0, \quad u_x(a, y) = 0, \quad u(x, 0) = 0, \quad u(0, L) = 0$$

Nonhomogeneous Boundary Condition:

$$u_x(0, y) = f(y), \quad u_x(a, y) = g(y), \quad u(x, L) = u_m$$

Typically, when using separation of variables, start with the independent variable associated with homogeneous boundary conditions, to determine the value of the **separation constant**.

Wave Equation: a Boundary-Value Problem

$$\begin{array}{ll} \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0 \\ \text{subject to}: & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L \end{array}$$

We focus on solving the above BVP (both ends are fixed).

Step 1: Separation of variables:
Assume that the solution $u(x,t) = X(x)T(t)$, $X, T \neq 0$. Then,

$$a^2 u_{xx} = u_{tt} \implies a^2 X'' T = XT'' \implies \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

$$\implies \begin{cases} X'' + \lambda X & = 0 \\ T'' + a^2 \lambda T & = 0 \end{cases}$$

The 2 homogeneous boundary conditions become $X(0) = X(L) = 0$.

Solve in the x-Dimension and Find λ

$$\begin{aligned}
&\text{Solve :} \quad X'' + \lambda X = 0, \quad T'' + a^2\lambda T = 0 \\
&\text{subject to :} \quad X(0) = 0, \quad X(L) = 0 \\
&\qquad X(x)\,T(0) = f(x), \quad X(x)\,T'(0) = g(x), \quad 0 < x < L
\end{aligned}$$

Step 2: λ remains to be determined. What values should λ take?

1. $\lambda = 0$: $X(x) = c_1 + c_2 x$. $X(0) = X(L) = 0 \implies c_1 = c_2 = 0$.
2. $\lambda = -\alpha^2 < 0$: $X(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$.
   Plug in $X(0) = X(L) = 0$, we get $c_1 = c_2 = 0$.
3. $\lambda = \alpha^2 > 0$: $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.
   Plug in $X(0) = X(L) = 0$, we get $c_1 = 0$, and $c_2 \sin(\alpha L) = 0$. Hence, $c_2 \neq 0$ only if $\alpha L = n\pi$.

Since $X \neq 0$, pick $\boxed{\lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots} \implies X(x) = c_2 \sin \frac{n\pi}{L} x.$

Solve in t-Dimension and Superposition

$$\begin{aligned} &\text{Solve :} \quad X'' + \lambda X = 0, \quad T'' + a^2 \lambda T = 0 \\ &\text{subject to :} \quad X(0) = 0, \quad X(L) = 0 \\ &\qquad X(x)\,T(0) = f(x), \quad X(x)\,T'(0) = g(x), \quad 0 < x < L \end{aligned}$$

Step 3: Once we fix $\lambda = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$, we obtain

$$X(x) = c_2 \sin\left(\frac{n\pi}{L}x\right), \quad T(t) = c_3 \cos\left(\frac{n\pi a}{L}t\right) + c_4 \sin\left(\frac{n\pi a}{L}t\right)$$

$$\implies u_n(x,t) = \left\{A_n \cos\left(\frac{n\pi a}{L}t\right) + B_n \sin\left(\frac{n\pi a}{L}t\right)\right\} \sin\left(\frac{n\pi}{L}x\right),$$

$$(A_n := c_2 c_3, \ B_n := c_2 c_4)$$

$$\implies u(x,t) := \sum_{n=1}^{\infty} u_n(x,t) \text{ is a solution, by the superposition principle.}$$

Plug in Initial Condition, Revoke Fourier Series, and Done

$$\begin{array}{lll} \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0 \\ \text{subject to}: & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L \end{array}$$

Step 4: Plug in the initial conditions and find $\{A_n, B_n \mid n = 1, 2, \ldots\}$.

$$u(x,0) = f(x), \quad u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right\} \sin\left(\frac{n\pi}{L} x\right)$$

$$\implies f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right), \quad 0 < x < L$$

From the Fourier sine series expansion on $(0, L)$, we get

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx.$$

Plug in Initial Condition, Revoke Fourier Series, and Done

$$\begin{array}{ll} \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0 \\ \text{subject to}: & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L \end{array}$$

Step 4: Plug in the initial conditions and find $\{A_n, B_n \mid n = 1, 2, \ldots\}$.

$$u_t(x,0) = g(x), \quad u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right\} \sin\left(\frac{n\pi}{L} x\right)$$

$$\implies g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin\left(\frac{n\pi}{L} x\right), \quad 0 < x < L$$

From the Fourier sine series expansion on $(0, L)$, we get

$$B_n \frac{n\pi a}{L} = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx. \implies B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx.$$

Final Solution

$$\begin{array}{lll} \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0 \\ \text{subject to}: & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L \end{array}$$

Step 5: The final solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right\} \sin\left(\frac{n\pi}{L} x\right)$$

$$= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi a}{L} t + \phi_n\right) \sin\left(\frac{n\pi}{L} x\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)\, dx, \quad B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right)\, dx$$

$$C_n = \sqrt{A_n^2 + B_n^2}, \quad \sin\phi_n = \frac{A_n}{C_n}, \quad \cos\phi_n = \frac{B_n}{C_n}$$

Standing Waves

The final solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi a}{L} t + \phi_n\right) \sin\left(\frac{n\pi}{L} x\right)$$

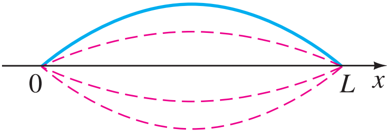
is a linear combination of **standing waves** or **normal modes**

$$u_n(x,t) = C_n \sin\left(\frac{n\pi a}{L} t + \phi_n\right) \sin\left(\frac{n\pi}{L} x\right), \ \ n = 1, 2, \ldots$$

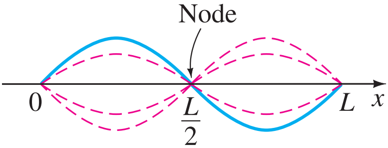
For a normal mode n, at a fixed location x, the string moves with

- time-varying amplitude $C_n \sin\left(\frac{n\pi}{L} x\right)$
- frequency $f_n := \frac{n\pi a/L}{2\pi} = \frac{na}{2L}$

Fundamental Frequency: $f_1 := \frac{\pi a/L}{2\pi} = \frac{a}{2L}$



(a) First standing wave



(b) Second standing wave

Laplace's Equation: a Boundary-Value Problem

$$\begin{array}{lll} \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, & 0<x<a, \quad 0<y<b \\ \text{subject to}: & u_x(0,y)=0, \quad u_x(a,y)=0, & 0<y<b \\ & u(x,0)=0, \quad u(x,b)=f(x), & 0<x<a \end{array}$$

We focus on solving the above BVP (both ends $x=0$ and $x=a$ are insulated).

Step 1: Separation of variables:
Assume that the solution $u(x,y)=X(x)\,Y(y)$, $X, Y \neq 0$. Then,

$$u_{xx}+u_{yy}=0 \implies X''Y+XY''=0 \implies \frac{X''}{X}=-\frac{Y''}{Y}=-\lambda$$

$$\implies \begin{cases} X''+\lambda X &= 0 \\ Y''-\lambda Y &= 0 \end{cases}$$

The 3 homogeneous boundary conditions become $X'(0)=X'(a)=Y(0)=0$.

Solve in the x-Dimension and Find λ

$$\begin{array}{lll} \text{Solve :} & X'' + \lambda X = 0, & Y'' - \lambda Y = 0 \\ \text{subject to :} & X'(0) = 0, & X'(a) = 0 \\ & Y(0) = 0, & X(x)\,Y(b) = f(x), \quad 0 < x < a \end{array}$$

Step 2: λ remains to be determined. What values should λ take?

1. $\lambda = 0$: $X(x) = c_1 + c_2 x$. $X'(0) = X'(a) = 0 \implies c_2 = 0$.
2. $\lambda = -\alpha^2 < 0$: $X(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$.
   Plug in $X'(0) = X'(a) = 0$, we get $c_1 = c_2 = 0$.
3. $\lambda = \alpha^2 > 0$: $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.
   Plug in $X'(0) = X'(a) = 0$, we get $c_2 = 0$, and $c_1 \alpha \sin(\alpha a) = 0$.
   Hence, $c_1 \neq 0$ only if $\alpha a = n\pi$.

Since $X \neq 0$, pick $\boxed{\lambda = \dfrac{n^2\pi^2}{a^2},\ n = 0, 1, 2, \ldots} \implies X(x) = c_1 \cos\left(\frac{n\pi}{a} x\right)$.

Solve in y-Dimension and Superposition

$$\begin{array}{lll} \text{Solve :} & X'' + \lambda X = 0, & Y'' - \lambda Y = 0 \\ \text{subject to :} & X'(0) = 0, & X'(a) = 0 \\ & Y(0) = 0, & X(x)\,Y(b) = f(x), \quad 0 < x < a \end{array}$$

Step 3: Once we fix $\lambda = \frac{n^2\pi^2}{a^2}$, $n = 0, 1, 2, \ldots$, we obtain $X(x) = c_1 \cos\left(\frac{n\pi}{a}x\right)$

$$Y(y) = \begin{cases} \cancel{c_3} + c_4 y, & n = 0 \\ \cancel{c_3 \cosh\left(\frac{n\pi}{a}y\right)} + c_4 \sinh\left(\frac{n\pi}{a}y\right), & n \geq 1 \end{cases} \quad (Y(0) = 0 \implies c_3 = 0)$$

$$\implies u_n(x, y) = \begin{cases} A_0 y, & n = 0 \\ A_n \cos\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right), & n \geq 1 \end{cases}, \quad (A_n := c_1 c_4)$$

$\implies u(x, y) := \sum_{n=0}^{\infty} u_n(x, y)$ is a solution, by the superposition principle.

Plug in Initial Condition, Revoke Fourier Series, and Done

$$\begin{array}{|rlll|}\hline \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, & 0<x<a, & 0<y<b \\ \text{subject to}: & u_x(0,y)=0, & u_x(a,y)=0, & 0<y<b \\ & u(x,0)=0, & u(x,b)=f(x), & 0<x<a \\ \hline \end{array}$$

Step 4: Plug in the initial conditions and find $\{A_n \mid n=1,2,\ldots\}$.

$$u(x,b)=f(x), \quad u(x,y)=A_0 y+\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$$\implies f(x)=A_0 b+\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right), \quad 0<x<a$$

From the Fourier cosine series expansion on $(0,a)$, we get

$$2A_0 b=\frac{2}{a}\int_0^a f(x)\,dx, \quad A_n \sinh\left(\frac{n\pi}{a}b\right)=\frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)\,dx$$

Final Solution

$$\begin{array}{lll} \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \text{subject to}: & u_x(0,y)=0, \quad u_x(a,y)=0, \quad 0<y<b \\ & u(x,0)=0, \quad u(x,b)=f(x), \quad 0<x<a \end{array}$$

Step 5: The final solution is

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$

$$A_0 = \frac{1}{ab}\int_0^a f(x)\,dx$$

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)\,dx, \quad n \geq 1$$

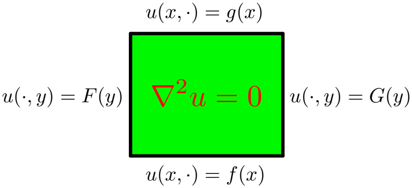
Superposition Principle

(So far) Key steps in solving a boundary-value problem of a PDE using separation of variables:

- Identify for which "dimension" (independent variable) (in our previous example, x), the given conditions are all homogeneous.
- Translate these homogeneous conditions into conditions on the single-argument function $X(x)$).
- Solve the associated ODE $(X'' + \lambda X = 0)$ under these conditions, and find the value of the separation constant λ that leads to non-trivial solutions.

Question: What if all dimensions contain some nonhomogeneous condition?

Solve $u(x,y)$:	$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b$
subject to :	$u(0,y) = F(y), \quad u(a,y) = G(y), \quad 0 < y < b$
	$u(x,0) = f(x), \quad u(x,b) = g(x), \quad 0 < x < a$



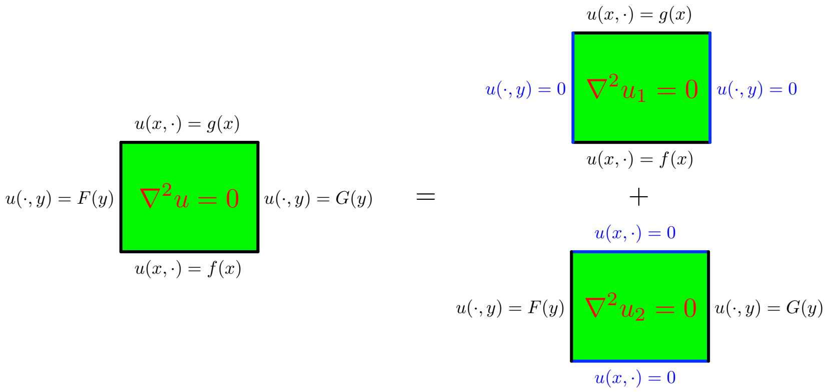
$$\begin{array}{ll} \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \text{subject to}: & u(0,y)=F(y), \quad u(a,y)=G(y), \quad 0<y<b \\ & u(x,0)=f(x), \quad u(x,b)=g(x), \quad 0<x<a \end{array}$$

The solution $u(x,y)=u_1(x,y)+u_2(x,y)$, where u_1, u_2 are the solutions of the following 2 BVP's respectively.

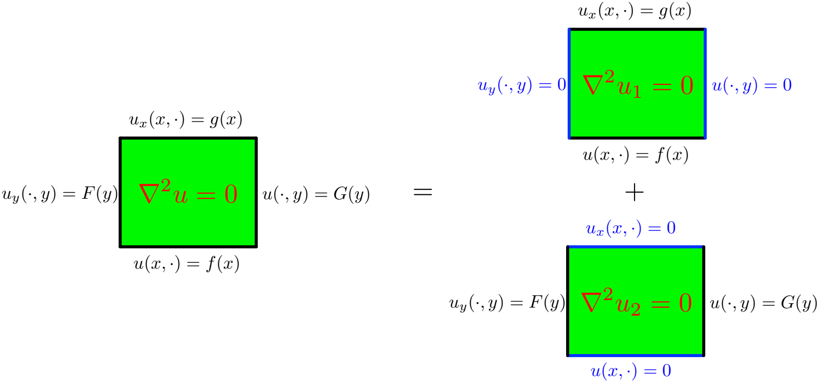
$$\begin{array}{ll} \text{Solve } u_1(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \text{subject to}: & u(0,y)=0, \quad u(a,y)=0, \quad 0<y<b \\ & u(x,0)=f(x), \quad u(x,b)=g(x), \quad 0<x<a \end{array}$$

$$\begin{array}{ll} \text{Solve } u_2(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \text{subject to}: & u(0,y)=F(y), \quad u(a,y)=G(y), \quad 0<y<b \\ & u(x,0)=0, \quad u(x,b)=0, \quad 0<x<a \end{array}$$

Superposition Principle

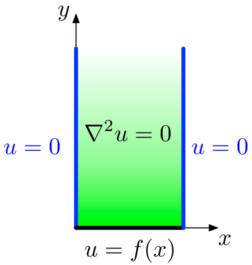


Superposition Principle



Semi-Finte Plate

$$\begin{aligned} &\text{Solve } u(x,y): && u_{xx}+u_{yy}=0, \quad 0<x<a, \quad y>0 \\ &\text{subject to}: && u(0,y)=0, \quad u(a,y)=0, \quad y>0 \\ & && u(x,0)=f(x), \quad |u(x,\infty)|<\infty, \quad 0<x<a \end{aligned}$$



Semi-Finte Plate

$$\begin{array}{rl} \text{Solve } u(x,y): & u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad y > 0 \\ \text{subject to}: & u(0,y) = 0, \quad u(a,y) = 0, \quad y > 0 \\ & u(x,0) = f(x), \quad |u(x,\infty)| < \infty, \quad 0 < x < a \end{array}$$

Following the same steps as before (setting $u(x,y) = X(x)\,Y(y)$), we can convert the original problem into

$$\begin{array}{rl} \text{Solve } u(x,y): & X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0 \\ \text{subject to}: & X(0) = 0, \quad X(a) = 0, \quad y > 0 \\ & X(x)\,Y(0) = f(x), \quad |Y(\infty)| < \infty, \quad 0 < x < a \end{array}$$

Step 1: First we solve $X(x) = c_2 \sin\left(\frac{n\pi}{a}x\right)$ and find the possible $\lambda = \frac{n^2\pi^2}{a^2}$, $n = 1, 2, \ldots$.

Semi-Finte Plate

$$\begin{array}{rl} \text{Solve}: & X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0 \\ \text{subject to}: & X(0) = 0, \quad X(a) = 0, \quad y > 0 \\ & X(x)\,Y(0) = f(x), \quad |\,Y(\infty)| < \infty, \quad 0 < x < a \end{array}$$

Step 1: First we solve $X(x) = c_2 \sin\left(\frac{n\pi}{a}x\right)$ $(\lambda = \frac{n^2\pi^2}{a^2})$, $n = 1, 2, \ldots$.

Step 2: Next we solve $Y(y) = c_3 e^{\frac{n\pi}{a}y} + c_4 e^{-\frac{n\pi}{a}y}$.

By the condition $|\,Y(\infty)| < \infty$, we have $c_3 = 0$.

Hence,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{n\pi}{a}y}.$$

Semi-Finte Plate

Solve $u(x, y)$:	$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad y > 0$
subject to :	$u(0, y) = 0, \quad u(a, y) = 0, \quad y > 0$
	$u(x, 0) = f(x), \quad \lvert u(x, \infty)\rvert < \infty, \quad 0 < x < a$

Final Solution:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{n\pi}{a}y}.$$

By the condition $u(x, 0) = f(x), 0 < x < a$, we have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \implies A_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right)\, dx.$$

Short Recap

- Method of Separation of Variables: Convert PDE into two ODE's
- Solve the ODE with homogeneous boundary conditions first, to determine the separation constant
- Fourier Series to determine the undetermined coefficients
- Heat Equation, Wave Equation, Laplace's Equation
- Superposition Principle

Self-Practice Exercises

12-1: 9, 15, 17, 22

12-2: 1, 3, 7, 11

12-4: 3, 7, 9, 11, 14

12-5: 5, 7, 12, 15, 19