# Chapter 11: Fourier Series

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**Fourier Series** is invented by Joseph Fourier, which basically asserts that most periodic functions can be represented by infinite sums of sine and cosine functions.

Jean Baptiste Joseph Fourier, (1768 - 1830).

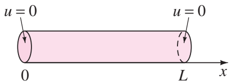
# Fourier's Motivation: Solving the Heat Equation

Solve $u(x,t)$ : $\quad k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0$

subject to : $\quad u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$ — Boundary condition

$\quad u(x,0) = f(x), \quad 0 < x < L$ — Initial condition

The above is called the **Heat Equation**, which can be derived from heat transfer theory.




Prior to Fourier, there is no known solution to the BVP if $f(x)$ (initial temperature distribution over the space) is general.

Below, let's try to follow Fourier's steps in solving this problem and see how Fourier Series is motivated.

# Fourier's Motivation: Solving the Heat Equation

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**Step 1**: Assume that the solution takes the form $\boxed{u(x,t) = X(x)\,T(t)}$.

(This approach was also taken by other predecessors like D. Bernoulli.)

**Step 2**: Convert the original PDE into the following:

$$kX''T = XT' \implies \frac{X''}{X} = \frac{T'}{kT} = -\lambda \implies \begin{cases} X'' + \lambda X &= 0 \\ T' + \lambda kT &= 0. \end{cases}$$

Boundary condition becomes $X(0)\,T(t) = X(L)\,T(t) = 0$.
Since we want non-trivial solutions, $T(t) \neq 0 \implies X(0) = X(L) = 0$.

# Fourier's Motivation: Solving the Heat Equation

Solve $u(x,t) = X(x)\,T(t)$ : $\begin{cases} X'' + \lambda X &= 0 \\ T' + \lambda k T &= 0. \end{cases}$

subject to : $X(0) = X(L) = 0,$ Boundary condition

$u(x,0) = f(x), \quad 0 < x < L$ Initial condition

**Step 3**: $\lambda$ remains to be determined. What values should $\lambda$ take?

1. $\lambda = 0$: $X(x) = c_1 + c_2 x$. $X(0) = X(L) = 0 \implies c_1 = c_2 = 0$.
2. $\lambda = -\alpha^2 < 0$: $X(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$.
   Plug in $X(0) = X(L) = 0$, we get $c_1 = c_2 = 0$.
3. $\lambda = \alpha^2 > 0$: $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.
   Plug in $X(0) = X(L) = 0$, we get $c_1 = 0$, and $c_2 \sin(\alpha L) = 0$.
   Hence, $c_2 \neq 0$ only if $\alpha L = n\pi$.

To obtain a non-trivial solution, pick $\boxed{\lambda = \dfrac{n^2\pi^2}{L^2}, \; n = 1, 2, \ldots}$.

# Fourier's Motivation: Solving the Heat Equation

Solve $u(x,t) = X(x)\,T(t)$ : $\begin{cases} X'' + \lambda X &= 0 \\ T' + \lambda k T &= 0. \end{cases}$

subject to : $X(0) = X(L) = 0,$ Boundary condition

$u(x,0) = f(x), \quad 0 < x < L$ Initial condition

**Step 4**: Once we fix $\lambda = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$, we obtain

$$X(x) = c_2 \sin\left(\frac{n\pi}{L}x\right), \quad T(t) = c_3 \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)$$

$$\implies u_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-k\frac{n^2\pi^2}{L^2}t\right), \quad (A_n := c_2 c_3)$$

**Step 5**: Plug in the initial condition $\implies f(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$

not true for general $f(x)$!

# Fourier's Motivation: Solving the Heat Equation

Solve $u(x,t) = X(x)\,T(t)$ : $\begin{cases} X'' + \lambda X &= 0 \\ T' + \lambda k T &= 0. \end{cases}$

subject to : $X(0) = X(L) = 0,$ Boundary condition

$u(x,0) = f(x), \quad 0 < x < L$ Initial condition

**Step 6**: By the superposition principle, below satisfies the PDE.

$$\sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-k\frac{n^2\pi^2}{L^2}t\right) \text{ for any } N$$

The question is, can it satisfy $u(x,0) = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$?

Not likely

**Key Observation**: $f(x)$ is arbitrary and hence not necessarily a finite sum of sine functions.

**Fourier's Idea**: How about an infinite series? If we can represent arbitrary $f(x)$ by the infinite series (for $0 < x < L$)

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right),$$

and we can find the values of $\{A_n\}$, the problem is solved.

This motivates the theory of **Fourier Series**.

1 Orthogonal Functions

2 Fourier Series

# Functions as Vectors: Inner Product

## Definition (Inner Product of Functions)

The inner product of $f_1(x)$ and $f_2(x)$ on an interval $[a, b]$ is defined as

$$\langle f_1, f_2 \rangle := \int_a^b f_1(x) f_2(x)\, dx$$

Once inner product is defined, we can accordingly define **norm**.

## Definition (Norm of a Function)

The norm of a function $f(x)$ on an interval $[a, b]$ is

$$||f(x)|| := \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b (f(x))^2 \ dx}$$

.

# Orthogonality of Functions

## Definition (Orthogonal Functions)

$f_1(x)$ and $f_2(x)$ are **orthogonal** on an interval $[a, b]$ if $\langle f_1, f_2 \rangle = 0$.

## Definition (Orthogonal Set)

$\{\phi_0(x), \phi_1(x), \cdots\}$ are **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n.$$

## Definition (Orthonormal Set)

$\{\phi_0(x), \phi_1(x), \cdots\}$ are **orthonomal** on an interval $[a, b]$ if they are orthogonal and $||\phi_n(x)|| = 1$ for all $n$.

# Examples

**Example (Orthogonal or Not Depends on the Inverval)**

The functions $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal on the interval $[a, b]$, $a < b$, only if $a = -b$.

**Proof**: When $a < b$,

$$\langle x, x^2 \rangle = \int_a^b x^3 \, dx = \left[ \frac{1}{4} x^4 \right]_a^b = \frac{1}{4} \left( a^4 - b^4 \right) = 0 \iff a + b = 0$$

# Examples

**Example (Exponential Functions are Not Orthogonal)**

For $\lambda_1, \lambda_2 \in \mathbb{R}$, $f_1(x) = e^{\lambda_1 x}$ and $f_2(x) = e^{\lambda_2 x}$ are not orthogonal on any interval $[a, b]$, $a < b$.

**Proof**: If $\lambda_1 = -\lambda_2$,

$$\langle e^{\lambda_1 x}, e^{\lambda_2 x} \rangle = \int_a^b e^{(\lambda_1 + \lambda_2)x}\, dx = b - a \neq 0.$$

If $\lambda_1 \neq -\lambda_2$,

$$\langle e^{\lambda_1 x}, e^{\lambda_2 x} \rangle = \int_a^b e^{(\lambda_1 + \lambda_2)x}\, dx = \frac{e^{(\lambda_1+\lambda_2)b} - e^{(\lambda_1+\lambda_2)a}}{\lambda_1 + \lambda_2} \neq 0,$$

since an exponential function is strictly monotone.

# Examples

## Example

The set of functions $\left\{\sin\left(\frac{n\pi}{L}x\right) \mid n = 1, 2, \ldots\right\}$ are orthogonal on $[0, L]$.

**Proof**: Let $\phi_n(x) := \sin\left(\frac{n\pi}{L}x\right)$. For $m \neq n$,

$$\begin{aligned}
\langle \phi_m, \phi_n \rangle &= \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\, dx \\
&= \int_0^L \frac{1}{2}\left\{\cos\left(\frac{(m-n)\pi}{L}x\right) - \cos\left(\frac{(m+n)\pi}{L}x\right)\right\}\, dx \\
&= \frac{L}{2(m-n)\pi}\left[\sin\left(\frac{(m-n)\pi}{L}x\right)\right]_0^L \\
&\quad - \frac{L}{2(m+n)\pi}\left[\sin\left(\frac{(m+n)\pi}{L}x\right)\right]_0^L \\
&= 0 - 0 = 0.
\end{aligned}$$

# Orthogonal Series Expansion

**Question**: For a infinite orthogonal set $\{\phi_n(x) \mid n = 0, 1, \ldots\}$ on some interval $[a, b]$, can we expand an arbitrary function $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \ ?$$

If so, how to find the coefficients $\{c_n\}$?

We answer the former question later with a particular set of orthogonal functions.

For the latter, simply take the inner product $\langle f, \phi_m \rangle$ to find the coefficient $c_m$!

$$\langle f, \phi_m \rangle = \sum_{n=0}^{\infty} c_n \langle \phi_n, \phi_m \rangle = c_m ||\phi_m||^2 \implies c_m = \frac{\langle f, \phi_m \rangle}{||\phi_m||^2}.$$

# Coefficients in the Solution of the Heat Equation

Recall in solving the Heat equation, the last step is to determine $\{A_n \mid n = 1, 2, \ldots\}$ such that $f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$.

Based on the principle developed above, we obtain $A_n = \frac{\langle f, \phi_n \rangle}{||\phi_n||^2}$, where $\phi_n(x) := \sin\left(\frac{n\pi}{L}x\right)$.

$$||\phi_n||^2 = \int_0^L \left(\sin\left(\frac{n\pi}{L}x\right)\right)^2 \, dx = \frac{1}{2}\int_0^L \left\{1 - \cos\left(\frac{2n\pi}{L}x\right)\right\} \, dx = \frac{L}{2}.$$

Hence, $A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) \, dx$, and

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)$$

## Remaining question:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

Will the infinite series converge for $x \in [0, L]$?

Does it converge to the function $f(x)$ for $x \in [0, L]$?

# Complete Set

For an arbitrary (infinite) set of orthogonal functions $\{\phi_n(x)\}$, it is **not** true that any function $f(x)$ in a *space* $S$ of functions, can be **truthfully** represented by its orthogonal series expansion.

Only when the set of orthogonal functions is **complete** in $S$, the orthogonal series expansion will (essentially) converge to any $f(x)$ in $S$.

## Example

$\{\sin(nx) \mid n = 1, 2, \ldots\}$ is orthogonal on $[-\pi, \pi]$ but not complete in the set of all continuous functions defined on $[-\pi, \pi]$.

It is quite straightforward to show that $\langle \sin(mx), \sin(nx) \rangle = 0$ for any $n \neq m$ on $[-\pi, \pi]$.

To show that it is not complete, note that any even function (like $1$, $x^2$, $\cos x$) cannot be represented by $\sum c_n \sin(nx)$ when $x < 0$, because the series is an odd function.

# A Orthogonal Set of Functions

**Lemma**

*The following set of functions are orthogonal on* $[-p, p]$ (in fact, $[a, a+2p]$ for any $a \in \mathbb{R}$).

$$\left\{1, \cos\left(\frac{n\pi}{p}x\right), \sin\left(\frac{n\pi}{p}x\right) \middle| n = 1, 2, \ldots\right\}.$$

If we expand a function using the above orthogonal set of functions, we obtain the **Fourier series** of the function.

Later we will see, this set is complete in the set of all continuous functions with continuous derivatives defined on $[a, a+2\pi]$.

# Definition of Fourier Series

## Definition

The **Fourier series** of a function $f(x)$ defined on the interval $(-p,p)$ is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi}{p}x\right)+b_n\sin\left(\frac{n\pi}{p}x\right)\right\},$$

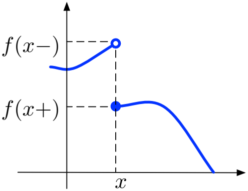
$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)\,dx,\ a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\left(\frac{n\pi}{p}x\right)\,dx,\ b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\left(\frac{n\pi}{p}x\right)\,dx.$$

These coefficients are called **Fourier coefficients**.

**Note**: In the textbook, Fourier series is defined over the interval $(-p,p)$. In fact, we can also define it over the interval $(a,a+2p)$ for any $a\in\mathbb{R}$.

The formulas for the Fourier series and Fourier coefficients are the same except that the integral is taken from $a$ to $a+2p$.

# Convergence of Fourier Series




**Question**:
How about the end points $\pm p$ ?

Answered later through periodic extension.

## Theorem

*Let $f$ and $f'$ be* ***piecewise continuous*** *on $[-p, p]$.*

*On $(-p, p)$, its Fourier series converges to*

- *$f(x)$ at a point where $f(x)$ is continuous*
- *$\frac{1}{2}(f(x+) + f(x-))$ where $f(x)$ is discontinuous.*

*Here*

$$f(x+) := \lim_{h \downarrow 0} f(x+h), \quad f(x-) := \lim_{h \downarrow 0} f(x-h).$$

**Note**: Again, the interval of interest can be changed from $[-p, p]$ to $[a, a+2p]$ for any $a \in \mathbb{R}$.

# Periodic Extension

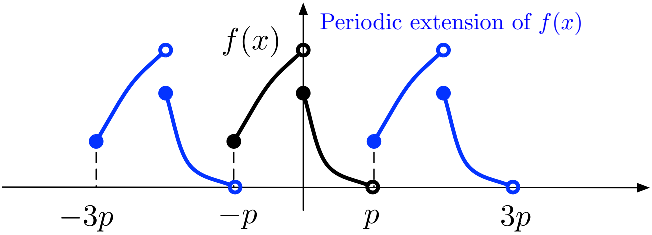
Note that a Fourier series consists of periodic functions:

| Function | Fundamental Period |
|---|---|
| $\cos\left(\frac{n\pi}{p}x\right)$ | $\frac{2p}{n}$ |
| $\sin\left(\frac{n\pi}{p}x\right)$ | $\frac{2p}{n}$ |

Hence, if a Fourier series converges for $x \in [-p, p]$ (or $[a, a + 2p]$), it also converges for any $x \in \mathbb{R}$.

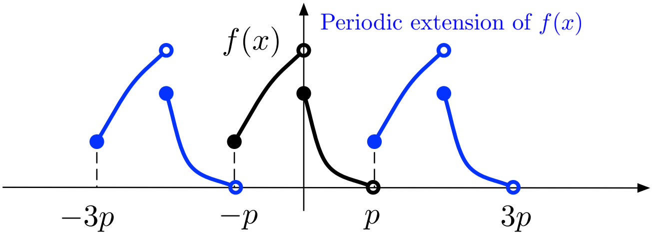
Moreover, it is a periodic function with fundamental period $2p$ (the largest fundamental period of its components).

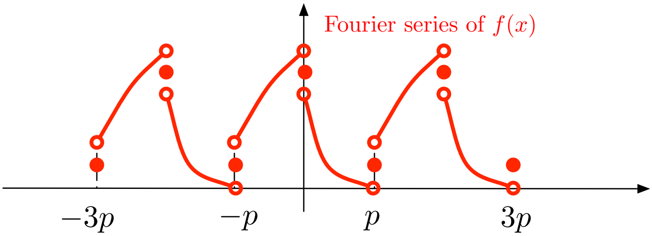
What does it converge to? It converges to the $2p$-periodic extension of $f(x)$, except the discontinuities.




At $x = \pm p, \pm 3p, \pm 5p, \ldots$, the Foruier series of $f(x)$ converges to

$$\frac{f(-p+) + f(p-)}{2}, \quad \text{where } f(-p+) := \lim_{x \downarrow -p} f(x),\ f(p-) := \lim_{x \uparrow p} f(x)$$


Periodic extension of $f(x)$
$f(x)$
$-3p$
$-p$
$p$
$3p$


Fourier series of $f(x)$
$-3p$
$-p$
$p$
$3p$

In other words,

The Fourier series of a piecewise continuous periodic function $f(x)$ with fundamental period $2p$ that has piecewise continuous $f'(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right) \right\},$$

where

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)\, dx, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)\, dx,$$

and on $\mathbb{R}$ it converges to

- $f(x)$ at a point where $f(x)$ is continuous
- $\frac{1}{2}\left(f(x+) + f(x-)\right)$ at a point where $f(x)$ is discontinuous.

# Examples

## Example

Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases}$ into a Fourier series. What does the Fourier series converge to at $x = 0$ and $x = \pi$?

# Complex Form

For a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right) \right\}$, since

$$\cos\left(\frac{n\pi}{p}x\right) = \frac{1}{2}\left(e^{i\frac{n\pi}{p}x} + e^{-i\frac{n\pi}{p}x}\right), \ \sin\left(\frac{n\pi}{p}x\right) = \frac{1}{2i}\left(e^{i\frac{n\pi}{p}x} - e^{-i\frac{n\pi}{p}x}\right)$$

it can be rewritten as follows:

$$\begin{aligned}
&\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \frac{1}{2}\left(e^{i\frac{n\pi}{p}x} + e^{-i\frac{n\pi}{p}x}\right) + b_n \frac{1}{2i}\left(e^{i\frac{n\pi}{p}x} - e^{-i\frac{n\pi}{p}x}\right) \right\} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ \frac{a_n - ib_n}{2} e^{i\frac{n\pi}{p}x} + \frac{a_n + ib_n}{2} e^{\frac{-in\pi}{p}x} \right\} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{p}x}, \quad c_0 = \frac{a_0}{2}, \ c_n = \frac{a_n - ib_n}{2}, \ c_{-n} = \frac{a_n + ib_n}{2}
\end{aligned}$$

# Complex Form

From the fact that $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \frac{a_n + ib_n}{2}$, and

$$a_n = \frac{1}{p}\int_a^{a+2p} f(x)\cos\left(\frac{n\pi}{p}x\right)\,dx, \quad b_n = \frac{1}{p}\int_a^{a+2p} f(x)\sin\left(\frac{n\pi}{p}x\right)\,dx,$$

one can verify that the Fourier series of a function $f(x)$ can be represented in the complex form

$$\boxed{\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{p}x}, \quad \text{where } c_n = \frac{1}{2p}\int_a^{a+2p} f(x)e^{-\frac{in\pi}{p}x}}.$$

# Complex Form

On the other hand, if we extend the definition of inner product to complex-valued functions:

**Definition (Inner Product of Complex-Valued Functions)**

The inner product of $f_1(x)$ and $f_2(x)$ on an interval $[a, b]$ is defined as

$$\langle f_1, f_2 \rangle := \int_a^b f_1(x) f_2^*(x)\, dx$$

Then, it is easy to verify that $\left\{ e^{\frac{in\pi}{p}x} \mid n \in \mathbb{Z} \right\}$ is an orthogonal set on any $[a, a+2p]$, and the coefficients in the expansion are exactly the same as above.

# Even and Odd Functions

- $f(x)$ is an **Odd Function** if $f(x) = -f(-x)$.
- $f(x)$ is an **Even Function** if $f(x) = f(-x)$.

## Property

- Both $f_1(x)$ and $f_2(x)$ are even (odd) $\implies$ $f_1(x)f_2(x)$ is even.
- $f_1(x)$ is odd but $f_2(x)$ is even $\implies$ $f_1(x)f_2(x)$ is odd.
- Both $f_1(x)$ and $f_2(x)$ are even (odd) $\implies$ $f_1(x) \pm f_2(x)$ is even (odd).
- $f(x)$ is even $\implies \displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$
- $f(x)$ is odd $\implies \displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

# Fourier Series of Even and Odd Functions

Recall: Fourier series of a function $f(x)$ on $(-p,p)$ is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi}{p}x\right)+b_n\sin\left(\frac{n\pi}{p}x\right)\right\},$$

with Fourier coefficients

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)\,dx=\begin{cases}0 & f\text{ is odd}\\ \frac{2}{p}\int_0^p f(x)\,dx & f\text{ is even}\end{cases}$$

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\left(\frac{n\pi}{p}x\right)\,dx=\begin{cases}0 & f\text{ is odd}\\ \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)\,dx & f\text{ is even}\end{cases}$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\left(\frac{n\pi}{p}x\right)\,dx=\begin{cases}\frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)\,dx & f\text{ is odd}\\ 0 & f\text{ is even}\end{cases}$$

# Fourier Series of Even and Odd Functions

**Fourier Series of an Even Function** $f(x)$ **on** $(-p, p)$:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right), \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)\, dx$$

Constant + a Series of Cosine Functions

**Fourier Series of an Odd Function** $f(x)$ **on** $(-p, p)$:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right), \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)\, dx$$

a Series of Sine Functions

# Fourier Cosine and Sine Series

## Definition

The **Fourier cosine series** of a function $f(x)$ defined on $(0, p)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right), \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)\, dx.$$

## Definition

The **Fourier sine series** of a function $f(x)$ defined on $(0, p)$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right), \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)\, dx.$$

# Half-Range Expansions

3 options to expand a function $f(x)$ defined on the interval $(0, L)$:

1. **Fourier Cosine Series**: Take $p := L$, and expand it as

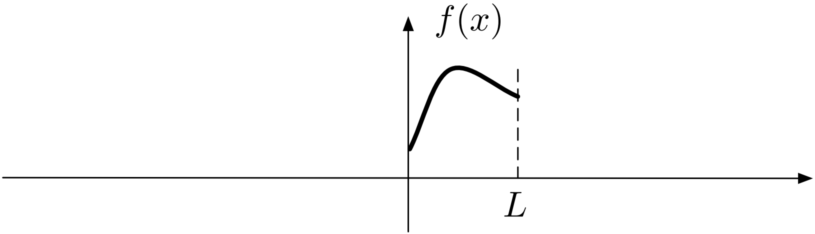
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx.$$

2. **Fourier Sine Series**: Take $p := L$, and expand it as

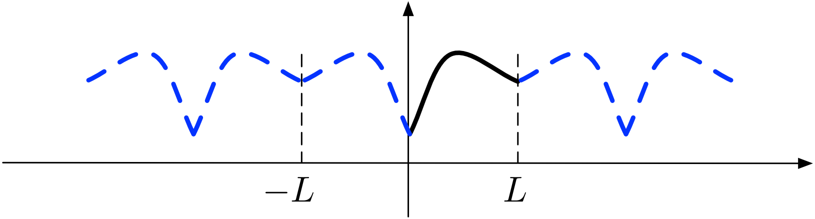
$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

3. **Fourier Series**: Take $a := 0,\ 2p := L$, and expand it as

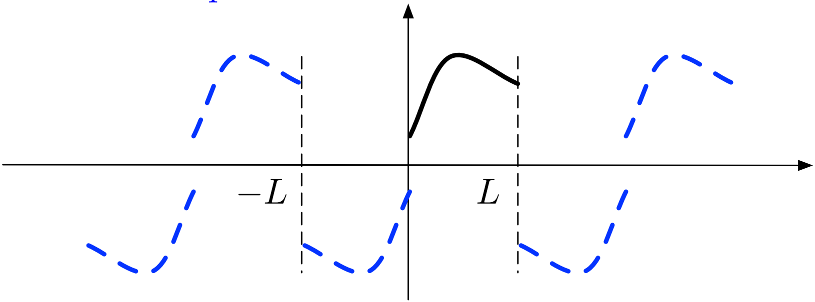
$$\sum_{n=-\infty}^{\infty} c_n e^{i\frac{2n\pi}{L}x}, \quad \text{where } c_n = \frac{1}{L}\int_0^L f(x)\,e^{-i\frac{2n\pi}{L}x}.$$


$f(x)$
$L$

## Expansion in Fourier cosine series



## Expansion in Fourier sine series



# Expansion in Fourier series