Chapter 11: Fourier Series

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Fourier Series is invented by Joseph Fourier, which basically asserts that most periodic functions can be represented by infinite sums of sine and cosine functions.

Jean Baptiste Joseph Fourier, (1768 - 1830).

Fourier's Motivation: Solving the Heat Equation

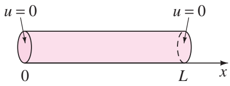
Solve $u(x,t)$: $\quad k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0$

subject to : $\quad u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$ — Boundary condition

$\quad u(x,0) = f(x), \quad 0 < x < L$ — Initial condition

The above is called the **Heat Equation**, which can be derived from heat transfer theory.

Prior to Fourier, there is no known solution to the BVP if $f(x)$ (initial temperature distribution over the space) is general.



Below, let's try to follow Fourier's steps in solving this problem and see how Fourier Series is motivated.

Fourier's Motivation: Solving the Heat Equation

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Step 1: Assume that the solution takes the form $\boxed{u(x,t) = X(x)\,T(t)}$.

(This approach was also taken by other predecessors like D. Bernoulli.)

Step 2: Convert the original PDE into the following:

$$kX''T = XT' \implies \frac{X''}{X} = \frac{T'}{kT} = -\lambda \implies \begin{cases} X'' + \lambda X &= 0 \\ T' + \lambda kT &= 0. \end{cases}$$

Boundary condition becomes $X(0)\,T(t) = X(L)\,T(t) = 0$.
Since we want non-trivial solutions, $T(t) \neq 0 \implies X(0) = X(L) = 0$.

Fourier's Motivation: Solving the Heat Equation

Solve $u(x,t) = X(x)T(t)$: $\begin{cases} X'' + \lambda X &= 0 \\ T' + \lambda k T &= 0. \end{cases}$

subject to : $X(0) = X(L) = 0,$ Boundary condition

$u(x,0) = f(x), \quad 0 < x < L$ Initial condition

Step 3: λ remains to be determined. What values should λ take?

1. $\lambda = 0$: $X(x) = c_1 + c_2 x$. $X(0) = X(L) = 0 \implies c_1 = c_2 = 0$.
2. $\lambda = -\alpha^2 < 0$: $X(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$.
   Plug in $X(0) = X(L) = 0$, we get $c_1 = c_2 = 0$.
3. $\lambda = \alpha^2 > 0$: $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.
   Plug in $X(0) = X(L) = 0$, we get $c_1 = 0$, and $c_2 \sin(\alpha L) = 0$.
   Hence, $c_2 \neq 0$ only if $\alpha L = n\pi$.

To obtain a non-trivial solution, pick $\boxed{\lambda = \frac{n^2\pi^2}{L^2}, \ n = 1, 2, \ldots}$.

Fourier's Motivation: Solving the Heat Equation

Solve $u(x,t) = X(x)\,T(t)$: $\begin{cases} X'' + \lambda X &= 0 \\ T' + \lambda k T &= 0. \end{cases}$

subject to : $X(0) = X(L) = 0,$ Boundary condition

$u(x,0) = f(x), \quad 0 < x < L$ Initial condition

Step 4: Once we fix $\lambda = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$, we obtain

$$X(x) = c_2 \sin\left(\frac{n\pi}{L}x\right), \quad T(t) = c_3 \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)$$

$$\implies u_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\exp\left(-k\frac{n^2\pi^2}{L^2}t\right), \quad (A_n := c_2 c_3)$$

Step 5: Plug in the initial condition $\implies f(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$

not true for general $f(x)$!

Fourier's Motivation: Solving the Heat Equation

Solve $u(x,t) = X(x)\,T(t)$: $\begin{cases} X'' + \lambda X &= 0 \\ T' + \lambda k T &= 0. \end{cases}$

subject to : $X(0) = X(L) = 0,$ Boundary condition

$u(x,0) = f(x), \quad 0 < x < L$ Initial condition

Step 6: By the superposition principle, below satisfies the PDE.

$$\sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-k\frac{n^2\pi^2}{L^2}t\right) \quad \text{for any } N$$

The question is, can it satisfy $u(x,0) = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$?

Not likely

Key Observation: $f(x)$ is arbitrary and hence not necessarily a finite sum of sine functions.

Fourier's Idea: How about an infinite series? If we can represent arbitrary $f(x)$ by the infinite series (for $0 < x < L$)

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right),$$

and we can find the values of $\{A_n\}$, the problem is solved.

This motivates the theory of **Fourier Series**.

Functions as Vectors: Inner Product

Definition (Inner Product of Functions)

The inner product of $f_1(x)$ and $f_2(x)$ on an interval $[a, b]$ is defined as

$$\langle f_1, f_2 \rangle := \int_a^b f_1(x) f_2(x) \, dx$$

Once inner product is defined, we can accordingly define **norm**.

Definition (Norm of a Function)

The norm of a function $f(x)$ on an interval $[a, b]$ is

$$||f(x)|| := \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b (f(x))^2 \, dx}$$

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Orthogonality of Functions

Definition (Orthogonal Functions)

$f_1(x)$ and $f_2(x)$ are **orthogonal** on an interval $[a, b]$ if $\langle f_1, f_2 \rangle = 0$.

Definition (Orthogonal Set)

$\{\phi_0(x), \phi_1(x), \cdots\}$ are **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n.$$

Definition (Orthonormal Set)

$\{\phi_0(x), \phi_1(x), \cdots\}$ are **orthonomal** on an interval $[a, b]$ if they are orthogonal and $||\phi_n(x)|| = 1$ for all n.

Examples

Example (Orthogonal or Not Depends on the Inverval)

The functions $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal on the interval $[a, b]$, $a < b$, only if $a = -b$.

Proof: When $a < b$,

$$\langle x, x^2 \rangle = \int_a^b x^3 \, dx = \left[\frac{1}{4} x^4 \right]_a^b = \frac{1}{4} \left(a^4 - b^4 \right) = 0 \iff a + b = 0$$

Examples

Example (Exponential Functions are Not Orthogonal)

For $\lambda_1, \lambda_2 \in \mathbb{R}$, $f_1(x) = e^{\lambda_1 x}$ and $f_2(x) = e^{\lambda_2 x}$ are not orthogonal on any interval $[a, b]$, $a < b$.

Proof: If $\lambda_1 = -\lambda_2$,

$$\langle e^{\lambda_1 x}, e^{\lambda_2 x} \rangle = \int_a^b e^{(\lambda_1+\lambda_2)x}\, dx = b - a \neq 0.$$

If $\lambda_1 \neq -\lambda_2$,

$$\langle e^{\lambda_1 x}, e^{\lambda_2 x} \rangle = \int_a^b e^{(\lambda_1+\lambda_2)x}\, dx = \frac{e^{(\lambda_1+\lambda_2)b} - e^{(\lambda_1+\lambda_2)a}}{\lambda_1+\lambda_2} \neq 0,$$

since an exponential function is strictly monotone.

Examples

Example

The set of functions $\left\{\sin\left(\frac{n\pi}{L}x\right) \mid n = 1, 2, \ldots\right\}$ are orthogonal on $[0, L]$.

Proof: Let $\phi_n(x) := \sin\left(\frac{n\pi}{L}x\right)$. For $m \neq n$,

$$\begin{aligned}
\langle \phi_m, \phi_n \rangle &= \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\, dx \\
&= \int_0^L \frac{1}{2}\left\{\cos\left(\frac{(m-n)\pi}{L}x\right) - \cos\left(\frac{(m+n)\pi}{L}x\right)\right\}\, dx \\
&= \frac{L}{2(m-n)\pi}\left[\sin\left(\frac{(m-n)\pi}{L}x\right)\right]_0^L \\
&\quad - \frac{L}{2(m+n)\pi}\left[\sin\left(\frac{(m+n)\pi}{L}x\right)\right]_0^L \\
&= 0 - 0 = 0.
\end{aligned}$$

Orthogonal Series Expansion

Question: For a infinite orthogonal set $\{\phi_n(x) \mid n = 0, 1, \ldots\}$ on some interval $[a, b]$, can we expand an arbitrary function $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \ ?$$

If so, how to find the coefficients $\{c_n\}$?

We answer the former question later with a particular set of orthogonal functions.

For the latter, simply take the inner product $\langle f, \phi_m \rangle$ to find the coefficient c_m!

$$\langle f, \phi_m \rangle = \sum_{n=0}^{\infty} c_n \langle \phi_n, \phi_m \rangle = c_m ||\phi_m||^2 \implies c_m = \frac{\langle f, \phi_m \rangle}{||\phi_m||^2}.$$

Coefficients in the Solution of the Heat Equation

Recall in solving the Heat equation, the last step is to determine $\{A_n \mid n = 1, 2, \ldots\}$ such that $f(x) = \displaystyle\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$.

Based on the principle developed above, we obtain $A_n = \frac{\langle f, \phi_n \rangle}{||\phi_n||^2}$, where $\phi_n(x) := \sin\left(\frac{n\pi}{L}x\right)$.

$$||\phi_n||^2 = \int_0^L \left(\sin\left(\frac{n\pi}{L}x\right)\right)^2 \, dx = \frac{1}{2}\int_0^L \left\{1 - \cos\left(\frac{2n\pi}{L}x\right)\right\} \, dx = \frac{L}{2}.$$

Hence, $A_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right) \, dx$, and

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)$$

Remaining question:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

Will the infinite series converge for $x \in [0, L]$?

Does it converge to the function $f(x)$ for $x \in [0, L]$?

A Orthogonal Set of Functions

Lemma

The following set of functions are orthogonal on $[-p, p]$.

$$\left\{ \frac{1}{2}, \cos\left(\frac{n\pi}{p}x\right), \sin\left(\frac{n\pi}{p}x\right) \middle| n = 1, 2, \dots \right\}.$$

Furthermore, the norm of each function is equal to p.

Proof: Exercise!

If we expand a function using the above orthogonal set of functions, we obtain the **Fourier series** of the function.

Definition of Fourier Series

Definition

The Fourier series of a function $f(x)$ defined on the interval $(-p, p)$ is

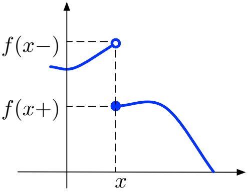
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right) \right\},$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)\,dx, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)\,dx.$$

Convergence of Fourier Series



Theorem

*Let f and f' be **piecewise continuous** on $[-p, p]$.*

- *At a point where $f(x)$ is continuous, its Fourier series converges to $f(x)$.*
- *At a point where $f(x)$ is discontinuous, its Fourier series converges to $\frac{1}{2}(f(x+) + f(x-))$.*

Here

$$f(x+) := \lim_{h \downarrow 0} f(x+h), \quad f(x-) := \lim_{h \downarrow 0} f(x-h).$$

Examples

Example

Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases}$ into a Fourier series. What does the Fourier series converge to at $x = 0$?