Chapter 7: The Laplace Transform – Part 3

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Properties of Laplace and its Inverse Transforms so far:

1. Laplace Transform of Polynomials, Exponentials, $\sin$, $\cos$, etc.
2. Laplace Transforms of Derivatives
3. Translation in s-Axis and t-Axis
4. Scaling

End of story?

Questions:

- How to compute $\mathscr{L}\left\{t^n e^{at}\cos(kt)\right\}$?
- How to compute $\mathscr{L}^{-1}\left\{\frac{1}{((s-a)^2+k^2)^2}\right\}$?
- How to compute the Laplace transform of a periodic function?

Derivatives of Laplace Transforms

Consider taking the derivative of the Laplace transform $F(s) = \mathscr{L}\{f(t)\}$:

$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}\left(\int_0^\infty f(t)\,e^{-st}dt\right) = \int_0^\infty \frac{\partial}{\partial s}\left(f(t)\,e^{-st}\right)\,dt \\ &= \int_0^\infty -tf(t)\,e^{-st}dt = -\mathscr{L}\{tf(t)\}\,.\end{aligned}$$

Applying the calculation repetitively, we obtain the following theorem:

Theorem

Let $f(t) \xrightarrow{\mathscr{L}} F(s)$ *and* $f(t)$ *is of exponential order,*

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s), \quad \mathscr{L}^{-1}\left\{\frac{d^n}{ds^n}F(s)\right\} = (-t)^n f(t).$$

Derivatives:

$$f^{(n)}(t) \quad \xrightarrow{\mathscr{L}} \quad s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$$

$$F^{(n)}(s) \quad \xrightarrow{\mathscr{L}^{-1}} \quad (-t)^n f(t)$$

Examples

Example

Evaluate $\mathscr{L}\left\{t^2 \cos t\right\}$.

Solution 1: Since $\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}$, we have

$$\begin{aligned}\mathscr{L}\left\{t^2 \cos t\right\} &= \frac{d^2}{ds^2}\frac{s}{s^2+1} = \frac{d^2}{ds^2}\left(\frac{1/2}{s-i} + \frac{1/2}{s+i}\right) \\ &= \frac{1}{(s-i)^3} + \frac{1}{(s+i)^3} = \boxed{\frac{2s^3-6s}{\left(s^2+1\right)^3}}\end{aligned}$$

Solution 2: Since $e^{it} = \cos t + i \sin t$, we have

$$\mathscr{L}\left\{t^2 e^{it}\right\} = \mathscr{L}\left\{t^2 \cos t\right\} + i \cdot \mathscr{L}\left\{t^2 \sin t\right\} = \frac{2}{(s-i)^3}.$$

Hence, $\mathscr{L}\left\{t^2 \cos t\right\} = \operatorname{Re}\left\{\frac{2}{(s-i)^3}\right\} = \frac{2s^3-6s}{\left(s^2+1\right)^3}$.

Convolution and its Laplace Transform

We have seen the Laplace transform of derivatives. How about integrals?

Definition (Convolution)

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau$$

Note: Convolution is exchangeable: $f * g = g * f$. (why?)

Theorem (Convolution in $t \iff$ Multiplication in s)

Let $f(t) \xrightarrow{\mathscr{L}} F(s)$ *and* $g(t) \xrightarrow{\mathscr{L}} G(s)$. *Then,*

$$\mathscr{L}\{(f * g)(t)\} = F(s)G(s).$$

Proof of the Convolution Theorem

Write $F(s) = \int_0^\infty f(\tau_1)e^{-s\tau_1}\,d\tau_1$, $G(s) = \int_0^\infty g(\tau_2)e^{-s\tau_2}\,d\tau_2$. Hence,

$$\begin{aligned}
F(s)\,G(s) &= \left(\int_0^\infty f(\tau_1)e^{-s\tau_1}\,d\tau_1\right)\left(\int_0^\infty g(\tau_2)e^{-s\tau_2}\,d\tau_2\right) \\
&= \int_0^\infty \int_0^\infty f(\tau_1)g(\tau_2)e^{-s(\tau_1+\tau_2)}\ d\tau_2\ d\tau_1 \\
&= \int_0^\infty \int_{\tau_1}^\infty f(\tau_1)g(t-\tau_1)e^{-st}\ dt\ d\tau_1 \quad (t := \tau_1 + \tau_2) \\
&= \int_0^\infty \int_0^t f(\tau_1)g(t-\tau_1)e^{-st}\ d\tau_1\ dt \quad (\text{exchange the order}) \\
&= \int_0^\infty \left(\int_0^t f(\tau_1)g(t-\tau_1)\,d\tau_1\right) e^{-st}dt \\
&= \mathscr{L}\left\{(f*g)(t)\right\}
\end{aligned}$$

Examples

Example (Use Laplace Transform to Compute Convolution)

Evaluate the convolution of e^t and $\sin t$.

Since $\mathscr{L}\left\{e^t\right\} = \frac{1}{s-1}$, $\mathscr{L}\left\{\sin t\right\} = \frac{1}{s^2+1}$, we have

$$\mathscr{L}\left\{e^t * \sin t\right\} = \frac{1}{(s-1)(s^2+1)} = \frac{1/2}{s-1} - \frac{1/2s}{s^2+1} - \frac{1/2}{s^2+1}.$$

Hence,

$$e^t * \sin t = \mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\} = \boxed{\frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t}.$$

Examples

Example (Finding Inverse Transforms of Products)

Evaluate $\mathscr{L}^{-1}\left\{\frac{s}{(s^2+k^2)^2}\right\}$.

Write $\frac{s}{\left(s^2+k^2\right)^2} = \frac{s}{s^2+k^2} \cdot \frac{1}{s^2+k^2}$. Note that

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt), \quad \mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt).$$

By the convolution theorem, we have

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s}{\left(s^2+k^2\right)^2}\right\} &= \frac{1}{k}\int_0^t \cos(k\tau)\sin(k(t-\tau))\,d\tau \\ &= \frac{1}{2k}\int_0^t \{\sin(kt) - \sin(k(2\tau-t))\}\,d\tau \\ &= \frac{1}{2k}\left[\tau\sin(kt) + \frac{1}{2k}\cos(k(2\tau-t))\right]_0^t = \boxed{\frac{1}{2k}t\sin(kt)}.\end{aligned}$$

Laplace Transform of Integrals

Theorem

Let $f(t) \xrightarrow{\mathscr{L}} F(s)$. *By the convolution theorem,*

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

Example

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

We know that $\mathscr{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$. By the theorem above, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\frac{s}{(s^2+1)^2}\right\} = \int_0^t \frac{\tau\sin\tau}{2}\,d\tau = \left[\frac{\sin\tau - \tau\cos\tau}{2}\right]_0^t = \boxed{\frac{\sin t - t\cos t}{2}}.$$

Integral of Laplace Transform

Theorem

Let $f(t) \xrightarrow{\mathscr{L}} F(s)$. *If* $\mathscr{L}\left\{\frac{f(t)}{t}\right\}$ *exists, then*

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

Proof:

$$\int_s^\infty F(u)\,du = \int_s^\infty \int_0^\infty f(t)\,e^{-ut} dt\ du = \int_0^\infty f(t)\left(\int_s^\infty e^{-ut} du\right) dt$$
$$= \int_0^\infty f(t)\frac{e^{-st}}{t} dt = \mathscr{L}\left\{\frac{f(t)}{t}\right\}$$

Integral Equation

Volterra Integral Equation of $y(t)$:

$$y(t) = g(t) + (h * y)(t) = g(t) + \int_0^t y(\tau)h(t-\tau)d\tau.$$

We can efficiently solve this kind of equation using Laplace transform.

Example

Solve $y(t) = 3t^2 - e^{-t} - \int_0^t y(\tau)e^{t-\tau}\,d\tau$.

Taking Laplace transform on both sides, we get $Y(s) = \frac{6}{s^3} - \frac{1}{s+1} - \frac{Y(s)}{s-1}$. Hence,

$$Y(s) = \frac{6(s-1)}{s^4} - \frac{s-1}{s(s+1)} = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$\implies \boxed{y(t) = 3t^2 - t^3 + 1 - 2e^{-t}}.$$

Periodic Functions

A function $f(t)$ is **periodic** with period $T > 0$ if $f(t) = f(t+T)$, for all t.

Theorem

If a function $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\left\{f(t)\right\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)\,e^{-st}dt$$

For example,

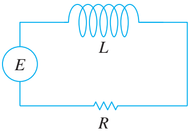
$$\begin{aligned}
\mathscr{L}\left\{\sin t\right\} &= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} \sin te^{-st}dt \\
&= \frac{1}{1-e^{-2\pi s}}\left[\frac{-\cos te^{-st} - s\sin te^{-st}}{s^2+1}\right]_0^{2\pi} \\
&= \frac{1}{1-e^{-2\pi s}}\frac{1-e^{-2\pi s}}{s^2+1} = \frac{1}{s^2+1}
\end{aligned}$$

Proof:

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^\infty f(t)e^{-st}dt \\
&= \int_0^T f(t)e^{-st}dt + \int_0^\infty f(\tau+T)e^{-s(\tau+T)}d\tau \quad (\tau := t-T) \\
&= \int_0^T f(t)e^{-st}dt + e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}d\tau \\
&= \int_0^T f(t)e^{-st}dt + e^{-sT}\mathscr{L}\{f(t)\}
\end{aligned}$$

Hence, $\left(1-e^{-sT}\right)\mathscr{L}\{f(t)\} = \int_0^T f(t)e^{-st}dt$.

LR-Circuit with Square-Wave Driving Voltage



Consider an LR-circuit with $E(t)$ being a unit square wave, period of which is $2T$, and

$$E(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & T \le t < 2T \end{cases}$$

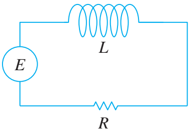
To determine its current $i(t)$ with $i(0) = 0$, we solve the following IVP:

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = 0.$$

Taking the Laplace transform on both sides, we get

$$\begin{aligned} (Ls + R)\, I(s) = \mathscr{L}\{E(t)\} &= \frac{1}{1 - e^{-2sT}} \int_0^{2T} E(t) e^{-st} dt \\ &= \frac{1}{1 - e^{-2sT}} \int_0^{T} e^{-st} dt = \frac{1 - e^{-sT}}{s\,(1 - e^{-2sT})} \end{aligned}$$

LR-Circuit with Square-Wave Driving Voltage



Consider an LR-circuit with $E(t)$ being a unit square wave, period of which is $2T$, and

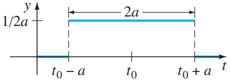
$$E(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & T \le t < 2T \end{cases}$$

$$I(s) = \frac{1 - e^{-sT}}{s(Ls+R)\left(1 - e^{-2sT}\right)} = \frac{1}{s(Ls+R)\left(1 + e^{-sT}\right)}$$
$$= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right)\left(1 - e^{-sT} + e^{-2sT} - e^{-3sT} + \cdots\right)$$

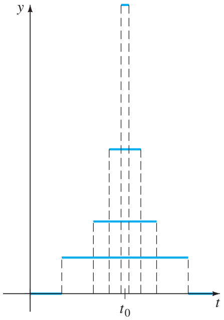
Hence,

$$\boxed{i(t) = \frac{1}{R}\sum_{k=0}^{\infty}(-1)^k\left(1 - e^{-\frac{R}{L}(t-kT)}\right)\mathcal{U}(t-kT)}$$

Unit Impulse Function



(a) graph of $\delta_a(t - t_0)$



(b) behavior of δ_a as $a \to 0$

Consider the following unit impulse function

$$\delta_a(t) := \begin{cases} \frac{1}{2a}, & -a \leq t < a \\ 0, & \text{otherwise} \end{cases}$$

For any translation $t_0 > a$, $\int_0^\infty \delta_a(t - t_0)\,dt = 1$.

As $a \to 0$, the duration of the impulse becomes shorter and shorter, and the magnitude of the impulse becomes larger and larger.

$\because \delta_a(t - t_0) = \frac{1}{2a}\{\mathcal{U}(t - (t_0 - a)) - \mathcal{U}(t - (t_0 + a))\}$, for $t_0 > a$,

$$\therefore \mathscr{L}\{\delta_a(t - t_0)\} = \frac{1}{2a}\left\{\frac{e^{-s(t_0 - a)}}{s} - \frac{e^{-s(t_0 + a)}}{s}\right\}$$

Dirac Delta Function

Definition (Dirac Delta Function)

$\delta(t-t_0) := \lim_{a\to 0} \delta_a(t-t_0)$.

$\delta(t-t_0) = \infty$ when $t = t_0$ but 0 otherwise, and $\int_0^\infty \delta(t-t_0)\,dt = 1$.

Theorem

For $t_0 > 0$, any continuous function $f(t)$, $\displaystyle\int_0^\infty \delta\,(t-t_0)\,f(t)\,dt = f(t_0)$.

Corollary

For $t_0 > 0$, $\mathscr{L}\left\{\delta(t-t_0)\right\} = e^{-st_0}$.

Proof

$$\int_0^\infty \delta(t-t_0)\, f(t)\, dt = \int_0^\infty \lim_{a\to 0} \delta_a(t-t_0)\, f(t)\, dt$$
$$= \lim_{a\to 0} \int_0^\infty \delta(t-t_0)\, f(t)\, dt = \lim_{a\to 0} \frac{\int_{t_0-a}^{t_0+a} f(t)\, dt}{2a}$$

In the limit of the last expression, we see that both the numerator and the denominator tend to 0 as $a \to 0$.

Hence, by L'Hôpital's Rule, we have:

$$\int_0^\infty \delta(t-t_0)\, f(t)\, dt = \lim_{a\to 0} \frac{\int_{t_0-a}^{t_0+a} f(t)\, dt}{2a} = \lim_{a\to 0} \frac{\frac{d}{da}\int_{t_0-a}^{t_0+a} f(t)\, dt}{2}$$
$$= \lim_{a\to 0} \frac{f(t_0+a) - (-f(t_0-a))}{2} = f(t_0).$$

IVP with Impulse External Drive

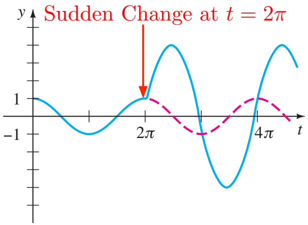
Example

Solve $y'' + y = 4\delta(t - 2\pi)$ subject to $y(0) = 1,\ y'(0) = 0$.

After taking the Laplace transform on both sides, we get

$$s^2\, Y(s) - s + Y(s) = 4e^{-2\pi s} \implies Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}$$

Hence, $y(t) = \cos t + 4\sin(t - 2\pi)\mathcal{U}(t - 2\pi) = \cos t + 4\sin t\, \mathcal{U}(t - 2\pi)$.



$$y(t) = \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \ge 2\pi \end{cases}$$

Initial Value Problem: System of Linear DE's

Idea: With Laplace Transform,

System of Linear DE's $\longrightarrow$ System of Linear Algebraic Equation

Advantage:

1. No need to worry about "implicit conditions" among undetermined coefficients
2. No need to worry about finding undetermined coefficients using initial conditions

Solve $\begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$

Step 1: Laplace Transform! $(x(0) = x_1, x'(0) = x_2, y(0) = y_1, y'(0) = y_2)$

$$\begin{cases} \left(s^2 X(s) - x_1 s - x_2\right) - 4X(s) + \left(s^2 Y(s) - y_1 s - y_2\right) = \dfrac{2}{s^3} \\ (sX(s) - x_1) + X(s) + (s Y(s) - y_1) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \left(s^2 - 4\right) X + s^2 Y = (x_1 + y_1)s + (x_2 + y_2) + \dfrac{2}{s^3} \\ (s+1) X + s Y = (x_1 + y_1) \end{cases}$$

Step 2: Solve $X(s)$, $Y(s)$: Let $a_1 := x_1 + y_1$, $a_2 = x_2 + y_2$:

$$X(s) = -\frac{a_2}{s+4} - \frac{2}{s^3(s+4)}, \quad Y(s) = \frac{a_1}{s} + \frac{a_2(s+1)}{s(s+4)} + \frac{2(s+1)}{s^4(s+4)}$$

Solve $\begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$

Step 3: Inverse Laplace transform!

$$X(s) = -\frac{a_2}{s+4} - \frac{2}{s^3(s+4)} = \frac{-a_2 + \frac{1}{32}}{s+4} - \frac{s^2 - 4s + 16}{32s^3}$$

$$\Longrightarrow \ x(t) = \left(-a_2 + \frac{1}{32}\right) e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2$$

$$Y(s) = \frac{a_1}{s} + \frac{a_2(s+1)}{s(s+4)} + \frac{2(s+1)}{s^4(s+4)}$$

$$= \frac{a_1 + \frac{a_2}{4}}{s} + \frac{\frac{3}{4}a_2 - \frac{3}{128}}{s+4} + \frac{\frac{3}{128}s^3 - \frac{3}{32}s^2 + \frac{3}{8}s + \frac{1}{2}}{s^4}$$

$$\Longrightarrow \ y(t) = a_1 + \frac{a_2}{4} + \frac{3}{128} + \left(\frac{3}{4}a_2 - \frac{3}{128}\right) e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3$$

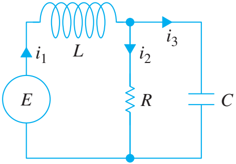
$$\boxed{\text{Solve} \quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}}$$

Step 4: Simplification!

$$\begin{aligned} x(t) &= \overbrace{\left(-a_2 + \frac{1}{32}\right)}^{c_1} e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2 \\ &= \boxed{c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2} \end{aligned}$$

$$\begin{aligned} y(t) &= \overbrace{a_1 + \frac{a_2}{4} + \frac{3}{128}}^{c_2} + \overbrace{\left(\frac{3}{4}a_2 - \frac{3}{128}\right)}^{-\frac{3c_1}{4}} e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3 \\ &= \boxed{c_2 - \frac{3}{4}c_1 e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3} \end{aligned}$$

Example: Series-Shunt Circuit



Consider an LRC-circuit with R and C shunt.

1. Voltage drop $E \to L \to R = E(t)$
2. Identical voltage drop across R and across C
3. $i_1 = i_2 + i_3$

Hence,

$$L\frac{di_1}{dt} + Ri_2 = E(t), \quad R\frac{di_2}{dt} = \frac{i_3}{C}, \quad i_3 = i_1 - i_2.$$

$$\Longrightarrow \begin{cases} L\dfrac{di_1}{dt} + Ri_2 = E(t) \\ RC\dfrac{di_2}{dt} + i_2 - i_1 = 0 \end{cases}$$

Solve $\begin{cases} L\dfrac{di_1}{dt} + Ri_2 = E \\ RC\dfrac{di_2}{dt} + i_2 = i_1 \end{cases}, \quad i_1(0) = i_2(0) = 0.$

Step 1: Laplace Transform!

$$\begin{cases} LsI_1(s) + RI_2(s) = \dfrac{E}{s} \\ (RCs+1)I_2(s) = I_1(s) \end{cases}$$

Step 2: Solve $I_1(s), I_2(s)$:

$$I_2(s) = \frac{E}{s\left(LRCs^2 + Ls + R\right)}, \quad I_1(s) = \frac{(RCs+1)\,E}{s\left(LRCs^2 + Ls + R\right)}$$

Step 3: Inverse Laplace transform! $i_2(t) = \cdots, \quad i_1(t) = \cdots$

Derivatives:

$$f^{(n)}(t) \quad \xrightarrow{\mathscr{L}} \quad s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$$

$$F^{(n)}(s) \quad \xrightarrow{\mathscr{L}^{-1}} \quad (-t)^n f(t)$$

Integrals:

$$\int_0^t f(\tau)g(t-\tau)\,d\tau \quad \xrightarrow{\mathscr{L}} \quad F(s)G(s)$$

$$\int_0^t f(\tau)\,d\tau \quad \xrightarrow{\mathscr{L}} \quad \frac{F(s)}{s}$$

$$\int_s^\infty F(u)\,du \quad \xrightarrow{\mathscr{L}^{-1}} \quad \frac{f(t)}{t}$$

Periodic Function:

$f(t)$, period T $\quad \xrightarrow{\mathscr{L}} \quad \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T f(t)e^{-st}\,dt$

Dirac Delta Function:

$\delta(t-t_0),\ t_0 \geq 0 \quad \xrightarrow{\mathscr{L}} \quad e^{-st_0}$

Short Recap

- Multiplication by $(-t)^n \iff n$-th Order Derivative in s
- Convolution in $t \iff$ Multiplication in s
- Laplace Transform of Periodic Functions: Compute the Integral within a Period
- Impulse and Dirac Delta Function
- $\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$
- Solving System of Linear DE with Laplace Transform

Self-Practice Exercises

7-4: 5, 13, 17, 23, 29, 39, 49, 51, 53, 59, 63, 66, 67

7-5: 5, 11, 13

7.6: 7, 11, 15, 17