# Chapter 7: The Laplace Transform – Part 2

王奕翔

Department of Electrical Engineering
National Taiwan University

ihwang@ntu.edu.tw

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So far we have learned

1. Basic properties of Laplace Transform
2. Inverse transform (Memorize with Laplace transform in pairs!)
3. How to use Laplace Transform to solve an IVP

End of story?

# Solving a Second-Order IVP with Laplace Transform

**Example**

Solve $y'' - 2y' + y = e^{2t}, \quad y(0) = 1,\ y'(0) = 5.$

**Step 1**: Laplace-transform both sides:

$$\mathscr{L}\left\{y''\right\} - 2\mathscr{L}\left\{y'\right\} + \mathscr{L}\left\{y\right\} = \mathscr{L}\left\{e^{2t}\right\}$$
$$\implies \left(s^2 Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s-2}$$
$$\implies (s^2 - 2s + 1)\,Y(s) = s + 3 + \frac{1}{s-2}$$

**Step 2**: Solve $Y(s)$: $Y(s) = \dfrac{s+3}{(s-1)^2} + \dfrac{1}{(s-1)^2(s-2)}$.

**Step 3**: Compute the inverse Laplace transform of $Y(s)$: How to compute?

$$Y(s) = \frac{3}{(s-1)^2} + \frac{1}{s-2} \implies y(t) = 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + e^{2t}.$$

**Partial fraction decomposition**:

$$\frac{s+3}{(s-1)^2} = \frac{(s-1)+4}{(s-1)^2} = \frac{1}{s-1} + \frac{4}{(s-1)^2}.$$

$$\frac{1}{(s-1)^2(s-2)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$A = \left[\frac{1}{(s-1)^2\cancel{(s-2)}}\right]_{s=2} = 1$$

$$C = \left[\frac{1}{\cancel{(s-1)^2}(s-2)}\right]_{s=1} = -1$$

$$1 = (B(s-1)+C)(s-2) + A(s^2-2s+1) \implies B = -A = -1.$$

We already know that $\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$.

If we know what is the inverse transform of a function $F(s)$ when it is **translated** by $1$ in the $s$-axis, that is, $\mathscr{L}^{-1}\left\{F(s-1)\right\}$, we can solve!

Need more properties of Laplace and its inverse transforms!

1 Translations and Scaling

2 Summary

# Translation on the $s$-Axis

Recall that

$$1 \xrightarrow{\mathscr{L}} \frac{1}{s}, \quad e^{at} \xrightarrow{\mathscr{L}} \frac{1}{s-a}.$$

Multiplying 1 by $e^{at}$ in $t$-domain results in right-shift of $a$ in $s$-domain.

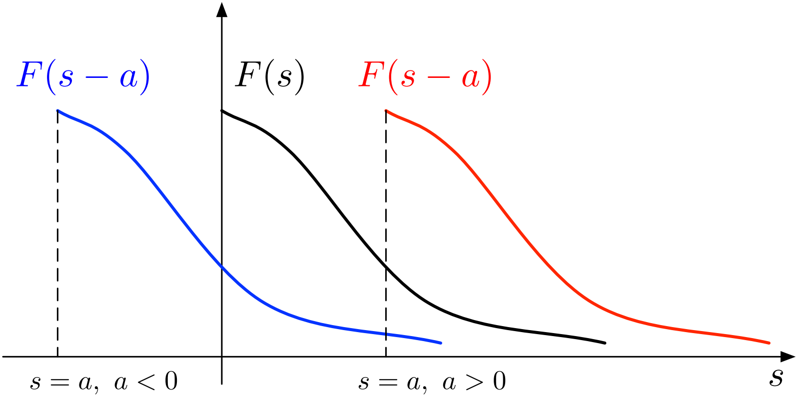
**Theorem**

*Let $f(t) \xrightarrow{\mathscr{L}} F(s)$. For any $a$,*

$$\boxed{\mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a)}.$$

**Proof**:

$$\mathscr{L}\left\{e^{at} f(t)\right\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a).$$


$F(s-a)$
$F(s)$
$F(s-a)$
$s=a,\ a<0$
$s=a,\ a>0$
$s$

# Back to the Problem

**Example**

Solve $y'' - 2y' + y = e^{2t}, \quad y(0) = 1,\ y'(0) = 5.$

**Step 3**: Compute the inverse Laplace transform of $Y(s)$: How to compute?

$$Y(s) = \frac{3}{(s-1)^2} + \frac{1}{s-2} \implies y(t) = 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + e^{2t}.$$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^t \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = te^t.$$

Hence,

$$Y(s) = \frac{3}{(s-1)^2} + \frac{1}{s-2} \implies \boxed{y(t) = 3te^t + e^{2t}}.$$

# Laplace Transform of $t^n e^{at}$

$$\boxed{\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}, \quad n = 0, 1, 2, \ldots, \ s > a}$$

We can obtain the inverse Laplace transform of $\frac{1}{(s-a)^n}$:

$$\boxed{\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{t^{n-1}}{(n-1)!}e^{at}, \quad n = 1, 2, \ldots}$$

# Laplace Transform of $e^{at}\sin(kt)$ and $e^{at}\cos(kt)$

$$\mathscr{L}\left\{e^{at}\sin(kt)\right\} = \frac{k}{(s-a)^2+k^2}, \quad s > a$$

$$\mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{(s-a)}{(s-a)^2+k^2}, \quad s > a$$

We can obtain the corresponding inverse Laplace transforms:

$$\mathscr{L}^{-1}\left\{\frac{k}{(s-a)^2+k^2}\right\} = e^{at}\sin(kt)$$

$$\mathscr{L}^{-1}\left\{\frac{(s-a)}{(s-a)^2+k^2}\right\} = e^{at}\cos(kt)$$

# Solving a Second-Order IVP with Laplace Transform

**Example**

Solve $y'' - 4y' + 5y = t^2 e^{2t}, \quad y(0) = 2, \; y'(0) = 6.$

**Step 1**: Laplace-transform both sides:

$$\mathscr{L}\{y''\} - 4\mathscr{L}\{y'\} + 5\mathscr{L}\{y\} = \mathscr{L}\{t^2 e^{2t}\}$$

$$\Longrightarrow \left(s^2 Y(s) - sy(0) - y'(0)\right) - 4\left(sY(s) - y(0)\right) + 5Y(s) = \frac{2}{(s-2)^3}$$

$$\Longrightarrow (s^2 - 4s + 5)\,Y(s) = 2s - 2 + \frac{2}{(s-2)^3}$$

**Step 2**: Solve $Y(s)$: $Y(s) = \dfrac{2s-2}{s^2 - 4s + 5} + \dfrac{2}{(s^2 - 4s + 5)(s-2)^3}$.

**Step 3**: Compute the inverse Laplace transform of $Y(s)$:

$$Y(s) = \frac{4(s-2)}{(s-2)^2 + 1} + \frac{2}{(s-2)^2 + 1} + \frac{-2}{s-2} + \frac{2}{(s-2)^3}$$

$$\Longrightarrow \boxed{y(t) = 4e^{2t}\cos t + 2e^{2t}\sin t - 2e^{2t} + t^2 e^{2t}}.$$

**Partial fraction decomposition**:

$$\frac{2s-2}{s^2-4s+5} = \frac{2(s-2)}{(s-2)^2+1} + \frac{2}{(s-2)^2+1}$$

$$\frac{2}{(s^2-4s+5)(s-2)^3} = \frac{A(s-2)+B}{(s-2)^2+1} + \frac{C}{s-2} + \frac{D}{(s-2)^2} + \frac{E}{(s-2)^3}$$

Tedious to calculate ...

**Tip 1**: Working in $\mathbb{C}$ makes life easier!

$$\begin{aligned}\frac{2}{(s^2-4s+5)(s-2)^3} &= \frac{A(s-2)+B}{(s-2)^2+1} + \frac{C}{s-2} + \frac{D}{(s-2)^2} + \frac{E}{(s-2)^3} \\ &= \frac{F}{s-2-i} + \frac{F^*}{s-2+i} + \frac{C}{s-2} + \frac{D}{(s-2)^2} + \frac{E}{(s-2)^3}\end{aligned}$$

$$F = \left[\frac{2}{(\cancel{(s-2-i)})(s-2+i)(s-2)^3}\right]_{s=2+i} = \frac{2}{2i\cdot i^3} = 1$$

$$\begin{aligned}A &= F + F^* = 2\mathrm{Re}\left\{F\right\} = 2, \\ B &= i(F - F^*) = -2\mathrm{Im}\left\{F\right\} = 0\end{aligned}$$

**Tip 2**: Taking derivatives

$$\frac{2}{(s^2-4s+5)(s-2)^3} = \frac{2(s-2)}{(s-2)^2+1} + \frac{C}{s-2} + \frac{D}{(s-2)^2} + \frac{E}{(s-2)^3}$$

$$E = \left[\frac{2}{s^2-4s+5}\right]_{s=2} = 2$$

$$D = \left[\frac{d}{d(s-2)}\frac{2}{s^2-4s+5}\right]_{s=2} = \left[\frac{-2}{\left[(s-2)^2+1\right]^2}2(s-2)\right]_{s=2} = 0$$

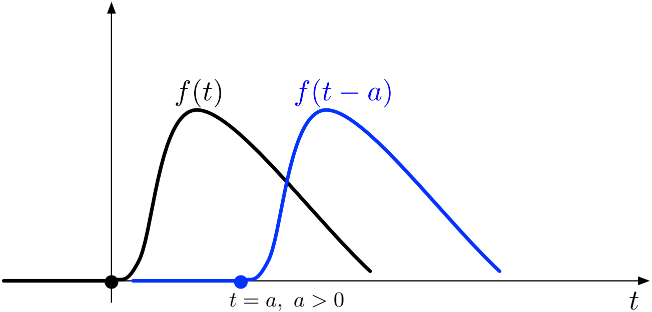
$$C = \left[\frac{1}{2!}\frac{d^2}{d(s-2)^2}\frac{2}{s^2-4s+5}\right]_{s=2}$$
$$= -2\left[\frac{\left[(s-2)^2+1\right]^2 - 2(s-2)^2 2\left[(s-2)^2+1\right]}{\left[(s-2)^2+1\right]^4}\right]_{s=2} = -2$$

# Translation on the $t$-Axis

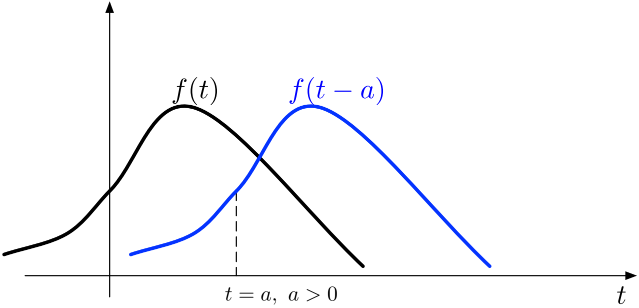
Let's compute $\mathscr{L}\{f(t-a)\}$, $a>0$, given that $\mathscr{L}\{f(t)\}=F(s)$:

$$\begin{aligned}
\mathscr{L}\{f(t-a)\} &= \int_0^\infty f(t-a)e^{-st}\,dt \overset{\tau:=t-a}{=} \int_{-a}^\infty f(\tau)e^{-s(\tau+a)}\,d\tau \\
&= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau + e^{-as}\int_{-a}^0 f(\tau)e^{-s\tau}\,d\tau \\
&= e^{-as}F(s) + e^{-as}\int_{-a}^0 f(\tau)e^{-s\tau}\,d\tau \\
&= \boxed{e^{-as}F(s)}, \quad \text{if } f(t)=0 \text{ when } t<0
\end{aligned}$$

If $f(t) = 0$ for $t < 0$, then $\boxed{\mathscr{L}\{f(t-a)\} = e^{-as}\mathscr{L}\{f(t)\}}$, for $a > 0$.



How about functions $f(t)$ that is non-zero for $t < 0$?

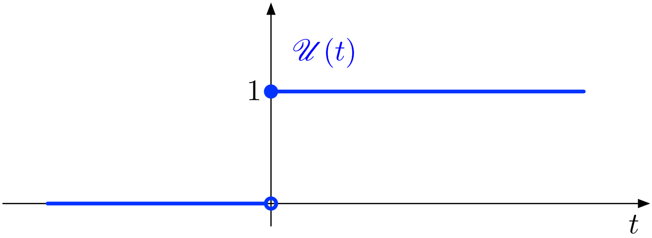


# Unit Step Function

## Definition (Unit Step Function)

$$\mathcal{U}(t) := \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

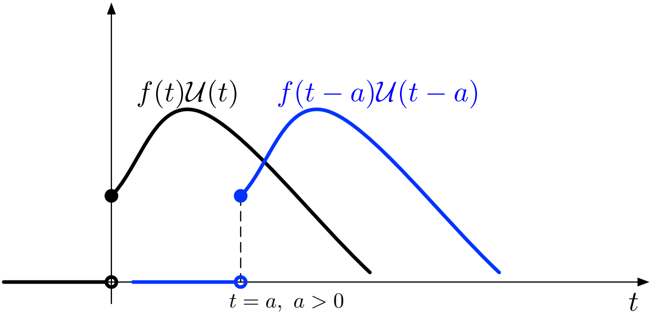
**Note**: $\mathscr{L}\{f(t)\mathcal{U}(t)\} = \mathscr{L}\{f(t)\}$.



## Theorem (Translation on the $t$-Axis)

*For* $a > 0$,

$$\mathscr{L}\left\{f(t-a)\mathcal{U}(t-a)\right\} = e^{-as}\mathscr{L}\left\{f(t)\mathcal{U}(t)\right\} = e^{-as}\mathscr{L}\left\{f(t)\right\}.$$



# Examples: Laplace Transforms

**Example**

Calculate $\mathscr{L}\left\{\mathcal{U}(t-a)\right\}$.

$$\mathscr{L}\left\{\mathcal{U}(t-a)\right\} = \mathscr{L}\left\{1 \cdot \mathcal{U}(t-a)\right\} = e^{-as}\mathscr{L}\left\{1\right\} = \frac{e^{-as}}{s}.$$

**Example**

Calculate $\mathscr{L}\left\{\cos t\, \mathcal{U}(t-\pi)\right\}$.

$$\begin{aligned}\mathscr{L}\left\{\cos t\, \mathcal{U}(t-\pi)\right\} &= \mathscr{L}\left\{\cos(t+\pi-\pi)\mathcal{U}(t-\pi)\right\} \\ &= e^{-\pi s}\mathscr{L}\left\{\cos(t+\pi)\right\} = e^{-\pi s}\mathscr{L}\left\{-\cos t\right\} \\ &= -e^{-\pi s}\frac{s}{s^2+1}.\end{aligned}$$

# Examples: Inverse Laplace Transforms

## Example

Calculate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-4)e^s}\right\}$.

Since $\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$, according to the translation property:

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-4)e^s}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-4}e^{-s}\right\} = \boxed{e^{4(t-1)}\mathcal{U}(t-1)}.$$

# Examples: Inverse Laplace Transforms

**Example**

Calculate $\mathscr{L}^{-1}\left\{\dfrac{se^{-2s}+e^{-s}}{(s-1)(s-2)}\right\}$.

A: First we organize the term as follows:

$$\begin{aligned}\frac{se^{-2s}+e^{-s}}{(s-1)(s-2)} &= \frac{s}{(s-1)(s-2)}e^{-2s}+\frac{1}{(s-1)(s-2)}e^{-s}\\ &= \left\{\frac{-1}{s-1}+\frac{2}{s-2}\right\}e^{-2s}+\left\{\frac{-1}{s-1}+\frac{1}{s-2}\right\}e^{-s}.\end{aligned}$$

Due to linearity and the translation property, the inverse transform is

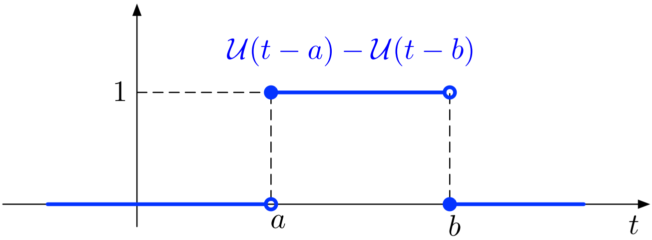
$$\boxed{\left\{-e^{(t-2)}+2e^{2(t-2)}\right\}\mathcal{U}(t-2)+\left\{-e^{(t-1)}+e^{2(t-1)}\right\}\mathcal{U}(t-1)}.$$

# Piecewise-Defined Function

Unit step function is useful in representing **piecewise-defined** functions.

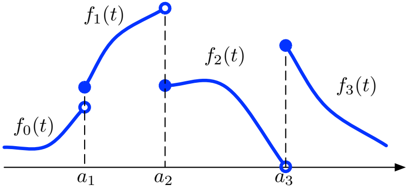
**Example**: the following function can be rewritten in terms of $\mathcal{U}$:

$$f(t) = \begin{cases} 1, & a \leq t < b \\ 0, & \text{otherwise} \end{cases} = \boxed{\mathcal{U}(t-a) - \mathcal{U}(t-b)}.$$



# Piecewise-Defined Function

$$f(t) = \begin{cases} f_0(t), & t < a_1 \\ f_1(t), & a_1 \leq t < a_2 \\ \vdots & \vdots \\ f_n(t), & t \geq a_n \end{cases}$$




$$\begin{aligned} &= f_0(t)\{1 - \mathcal{U}(t - a_1)\} \\ &\quad + f_1(t)\{\mathcal{U}(t - a_1) - \mathcal{U}(t - a_2)\} \\ &\quad + f_2(t)\{\mathcal{U}(t - a_2) - \mathcal{U}(t - a_3)\} \\ &\quad \ \vdots \qquad \vdots \\ &\quad + f_n(t)\mathcal{U}(t - a_n). \end{aligned}$$

# Solving IVP with Piecewise Defined External Drive

**Example**

Solve $y'' + 3y' + 2y = g(t),\ \ y(0) = 1,\ \ y'(0) = 5$, where $g(t) = 10\cos t$ for $\pi \le t < 3\pi$ and $g(t) = 0$ otherwise.

$$g(t) = 10\cos t\,\{\mathcal{U}(t-\pi) - \mathcal{U}(t-3\pi)\}$$

**Step 1**: Laplace-transform both sides:

$$\begin{aligned}
&\mathscr{L}\left\{y''\right\} + 3\mathscr{L}\left\{y'\right\} + 2\mathscr{L}\left\{y\right\} = \mathscr{L}\left\{10\cos t\,\{\mathcal{U}(t-\pi) - \mathcal{U}(t-3\pi)\}\right\} \\
&\implies \left(s^2\,Y(s) - sy(0) - y'(0)\right) + 3\left(s\,Y(s) - y(0)\right) + 2\,Y(s) \\
&\qquad = -\frac{10s}{s^2+1}\left(e^{-\pi s} - e^{-3\pi s}\right) \\
&\implies \left(s^2 + 3s + 2\right)Y(s) = s + 8 - \frac{10s}{s^2+1}\left(e^{-\pi s} - e^{-3\pi s}\right)
\end{aligned}$$

# Solving IVP with Piecewise Defined External Drive

## Example

Solve $y'' + 3y' + 2y = g(t),\ y(0) = 1,\ y'(0) = 5$, where $g(t) = 10\cos t$ for $\pi \le t < 3\pi$ and $g(t) = 0$ otherwise.

**Step 2**: Solve $Y(s)$:

$$Y(s) = \frac{s+8}{(s+1)(s+2)} - \frac{10s}{(s+1)(s+2)(s^2+1)}\left(e^{-\pi s} - e^{-3\pi s}\right).$$

**Step 3**: Compute the inverse Laplace transform of $Y(s)$:

$$Y(s) = \frac{7}{s+1} + \frac{-6}{s+2} + \left(\frac{5}{s+1} + \frac{-4}{s+2} + \frac{-s-3}{s^2+1}\right)\left(e^{-\pi s} - e^{-3\pi s}\right)$$

$$\implies y(t) = 7e^{-t} - 6e^{-2t}$$
$$+ \left(5e^{-(t-\pi)} - 4e^{-2(t-\pi)} - \cos(t-\pi) - 3\sin(t-\pi)\right)\mathcal{U}(t-\pi)$$
$$- \left(5e^{-(t-3\pi)} - 4e^{-2(t-3\pi)} - \cos(t-3\pi) - 3\sin(t-3\pi)\right)\mathcal{U}(t-3\pi)$$

# Scaling

## Theorem

*Let* $f(t) \xrightarrow{\mathscr{L}} F(s)$ *for* $s > c$. *For any* $a > 0$,

$$\boxed{\mathscr{L}\left\{f(at)\right\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad s > ac}. \quad \textit{Conversely, } \boxed{\mathscr{L}^{-1}\left\{F(as)\right\} = \frac{1}{a}f\left(\frac{t}{a}\right)}.$$

**Proof**: By definition,

$$\mathscr{L}\left\{f(at)\right\} = \int_0^\infty f(at)e^{-st}dt \overset{\tau:=at}{=} \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}F\left(\frac{s}{a}\right).$$

# Examples: Scaling and Translation

## Example

Given that $f(t) \xrightarrow{\mathscr{L}} F(s)$ for $s > c$. Evaluate $\mathscr{L}\left\{f(at-b)\mathcal{U}(at-b)\right\}$.

**Solution 1**: Scale first, then shift.

$$f(at-b)\mathcal{U}(at-b) = f\left(a\left(t-\frac{b}{a}\right)\right)\mathcal{U}\left(a\left(t-\frac{b}{a}\right)\right)$$

$$f(t) \xrightarrow{\text{scale by } a} f(at) \xrightarrow{\text{shift by } \frac{b}{a}} f\left(a\left(t-\frac{b}{a}\right)\right)\mathcal{U}\left(a\left(t-\frac{b}{a}\right)\right)$$

$$F(s) \longrightarrow \frac{1}{a}F\left(\frac{s}{a}\right) \xrightarrow{\text{multiply by } e^{-\frac{b}{a}s}} \frac{e^{-\frac{b}{a}s}}{a}F\left(\frac{s}{a}\right)$$

# Examples: Scaling and Translation

## Example

Given that $f(t) \xrightarrow{\mathscr{L}} F(s)$ for $s > c$. Evaluate $\mathscr{L}\left\{f(at-b)\,\mathcal{U}\,(at-b)\right\}$.

**Solution 2**: Shift first, then scale.

$$f(t) \xrightarrow{\text{shift by } b} f(t-b)\,\mathcal{U}\,(t-b) \xrightarrow{\text{scale by } a} f(at-b)\,\mathcal{U}\,(at-b)$$

$$F(s) \xrightarrow{\text{multiply by } e^{-bs}} e^{-bs}F(s) \longrightarrow \frac{e^{-\frac{b}{a}s}}{a}F\left(\frac{s}{a}\right)$$

**Translation:**

$$e^{at}f(t) \xrightarrow{\mathscr{L}} F(s-a)$$

$$e^{-as}F(s) \xrightarrow{\mathscr{L}^{-1}} f(t-a)\mathcal{U}(t-a)$$

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**Scaling:**

$$f(at) \xrightarrow{\mathscr{L}} \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$F(as) \xrightarrow{\mathscr{L}^{-1}} \frac{1}{a}f\left(\frac{t}{a}\right)$$

# Short Recap

- Translation in $s$ $\iff$ Multiplication by Exponential in $t$
- Unit Step Function and Piecewise-Defined Functions
- Scaling
- Tips for Partial Fraction Decomposition

# Self-Practice Exercises

7-3: 9, 17, 19, 25, 31, 39, 47, 49, 51, 55, 59, 65, 69, 83