Chapter 7: The Laplace Transform – Part 2

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So far we have learned

1. Basic properties of Laplace Transform
2. Inverse transform (Memorize with Laplace transform in pairs!)
3. How to use Laplace Transform to solve an IVP

End of story?

Solving a Second-Order IVP with Laplace Transform

Example

Solve $y'' - 2y' + y = e^{2t}, \quad y(0) = 1,\ y'(0) = 5.$

Step 1: Laplace-transform both sides:

$$\begin{aligned}
&\mathscr{L}\{y''\} - 2\mathscr{L}\{y'\} + \mathscr{L}\{y\} = \mathscr{L}\{e^{2t}\} \\
&\implies \left(s^2 Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s-2} \\
&\implies (s^2 - 2s + 1)\, Y(s) = s + 3 + \frac{1}{s-2}
\end{aligned}$$

Step 2: Solve $Y(s)$: $Y(s) = \dfrac{s+3}{(s-1)^2} + \dfrac{1}{(s-1)^2(s-2)}$.

Step 3: Compute the inverse Laplace transform of $Y(s)$: How to compute?

$$Y(s) = \frac{3}{(s-1)^2} + \frac{1}{s-2} \implies y(t) = 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + e^{2t}.$$

Partial fraction decomposition:

$$\frac{s+3}{(s-1)^2} = \frac{(s-1)+4}{(s-1)^2} = \frac{1}{s-1} + \frac{4}{(s-1)^2}.$$

$$\frac{1}{(s-1)^2(s-2)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$A = \left[\frac{1}{(s-1)^2\cancel{(s-2)}}\right]_{s=2} = 1$$

$$C = \left[\frac{1}{\cancel{(s-1)^2}(s-2)}\right]_{s=1} = -1$$

$$1 = (B(s-1)+C)(s-2) + A(s^2-2s+1) \implies B = -A = -1.$$

We already know that $\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$.

If we know what is the inverse transform of a function $F(s)$ when it is **translated** by 1 in the s-axis, that is, $\mathscr{L}^{-1}\left\{F(s-1)\right\}$, we can solve!

Need more properties of Laplace and its inverse transforms!

1 Properties of Translations

Translation on the s-Axis

Recall that

$$1 \xrightarrow{\mathscr{L}} \frac{1}{s}, \quad e^{at} \xrightarrow{\mathscr{L}} \frac{1}{s-a}.$$

Multiplying 1 by e^{at} in t-domain results in right-shift of a in s-domain.

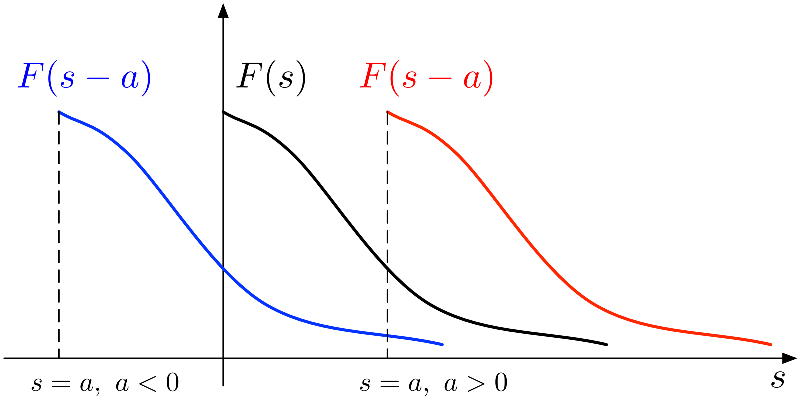
Theorem

Let $f(t) \xrightarrow{\mathscr{L}} F(s)$. For any a,

$$\boxed{\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)}.$$

Proof:

$$\mathscr{L}\left\{e^{at}f(t)\right\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a).$$


$F(s-a)$
$F(s)$
$F(s-a)$
$s=a,\ a<0$
$s=a,\ a>0$
s

Back to the Problem

Example

Solve $y'' - 2y' + y = e^{2t}, \quad y(0) = 1,\ y'(0) = 5.$

Step 3: Compute the inverse Laplace transform of $Y(s)$: How to compute?

$$Y(s) = \frac{3}{(s-1)^2} + \frac{1}{s-2} \implies y(t) = 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + e^{2t}.$$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^t\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = te^t.$$

Hence,

$$Y(s) = \frac{3}{(s-1)^2} + \frac{1}{s-2} \implies \boxed{y(t) = 3te^t + e^{2t}}.$$

Laplace Transform of $t^n e^{at}$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}, \quad n = 0, 1, 2, \ldots, \ s > a$$

We can obtain the inverse Laplace transform of $\frac{1}{(s-a)^n}$:

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{t^{n-1}}{(n-1)!}e^{at}, \quad n = 1, 2, \ldots$$

Laplace Transform of $e^{at}\sin(kt)$ and $e^{at}\cos(kt)$

$$\mathscr{L}\left\{e^{at}\sin(kt)\right\} = \frac{k}{(s-a)^2+k^2}, \quad s > a$$

$$\mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{(s-a)}{(s-a)^2+k^2}, \quad s > a$$

We can obtain the corresponding inverse Laplace transforms:

$$\mathscr{L}^{-1}\left\{\frac{k}{(s-a)^2+k^2}\right\} = e^{at}\sin(kt)$$

$$\mathscr{L}^{-1}\left\{\frac{(s-a)}{(s-a)^2+k^2}\right\} = e^{at}\cos(kt)$$

Solving a Second-Order IVP with Laplace Transform

Example

Solve $y'' - 4y' + 5y = t^2 e^{2t}, \quad y(0) = 2,\ y'(0) = 6.$

Step 1: Laplace-transform both sides:

$$\begin{aligned}
&\mathscr{L}\{y''\} - 4\mathscr{L}\{y'\} + 5\mathscr{L}\{y\} = \mathscr{L}\{t^2 e^{2t}\} \\
&\Longrightarrow \left(s^2 Y(s) - sy(0) - y'(0)\right) - 4\,(s\,Y(s) - y(0)) + 5\,Y(s) = \frac{2}{(s-2)^3} \\
&\Longrightarrow (s^2 - 4s + 5)\,Y(s) = 2s - 2 + \frac{2}{(s-2)^3}
\end{aligned}$$

Step 2: Solve $Y(s)$: $Y(s) = \dfrac{2s-2}{s^2-4s+5} + \dfrac{2}{(s^2-4s+5)(s-2)^3}$.

Step 3: Compute the inverse Laplace transform of $Y(s)$:

$$\begin{aligned}
Y(s) &= \frac{4(s-2)}{(s-2)^2+1} + \frac{2}{(s-2)^2+1} + \frac{-2}{s-2} + \frac{2}{(s-2)^3} \\
\Longrightarrow &\ \boxed{y(t) = 4e^{2t}\cos t + 2e^{2t}\sin t - 2e^{2t} + t^2 e^{2t}}.
\end{aligned}$$

Partial fraction decomposition:

$$\frac{2s-2}{s^2-4s+5} = \frac{2(s-2)}{(s-2)^2+1} + \frac{2}{(s-2)^2+1}$$

$$\frac{2}{(s^2-4s+5)(s-2)^3} = \frac{A(s-2)+B}{(s-2)^2+1} + \frac{C}{s-2} + \frac{D}{(s-2)^2} + \frac{E}{(s-2)^3}$$

Tedious to calculate ...

Tip 1: Working in $\mathbb{C}$ makes life easier!

$$\frac{2}{(s^2-4s+5)(s-2)^3} = \frac{A(s-2)+B}{(s-2)^2+1} + \frac{C}{s-2} + \frac{D}{(s-2)^2} + \frac{E}{(s-2)^3}$$
$$= \frac{F}{s-2-i} + \frac{F^*}{s-2+i} + \frac{C}{s-2} + \frac{D}{(s-2)^2} + \frac{E}{(s-2)^3}$$

$$F = \left[\frac{2}{\cancel{((s-2-i)}(s-2+i)(s-2)^3}\right]_{s=2+i} = \frac{2}{2i \cdot i^3} = 1$$

$$A = F + F^* = 2\mathrm{Re}\{F\} = 2,$$
$$B = i(F - F^*) = -2\mathrm{Im}\{F\} = 0$$

Tip 2: Taking derivatives

$$\frac{2}{(s^2-4s+5)(s-2)^3} = \frac{2(s-2)}{(s-2)^2+1} + \frac{C}{s-2} + \frac{D}{(s-2)^2} + \frac{E}{(s-2)^3}$$

$$E = \left[\frac{2}{s^2-4s+5}\right]_{s=2} = 2$$

$$D = \left[\frac{d}{d(s-2)}\frac{2}{s^2-4s+5}\right]_{s=2} = \left[\frac{-2}{\left[(s-2)^2+1\right]^2}2(s-2)\right]_{s=2} = 0$$

$$C = \left[\frac{1}{2!}\frac{d^2}{d(s-2)^2}\frac{2}{s^2-4s+5}\right]_{s=2}$$

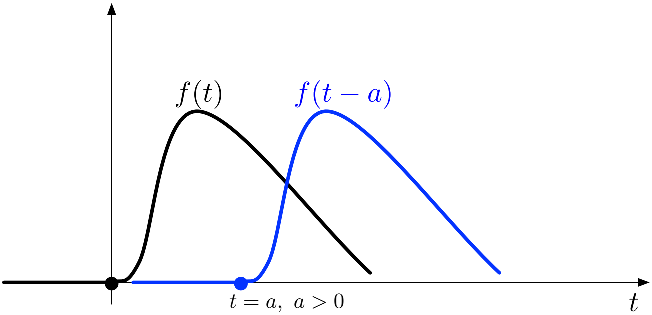
$$= -2\left[\frac{\left[(s-2)^2+1\right]^2 - 2(s-2)^2 2\left[(s-2)^2+1\right]}{\left[(s-2)^2+1\right]^4}\right]_{s=2} = -2$$

Translation on the t-Axis

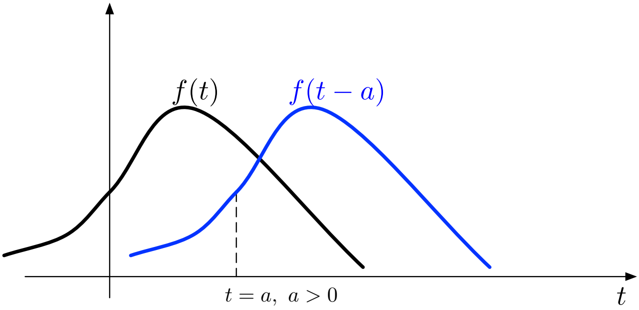
Let's compute $\mathscr{L}\{f(t-a)\}$, $a>0$, given that $\mathscr{L}\{f(t)\}=F(s)$:

$$\begin{aligned}\mathscr{L}\{f(t-a)\} &= \int_0^\infty f(t-a)e^{-st}\,dt \overset{\tau:=t-a}{=} \int_{-a}^\infty f(\tau)e^{-s(\tau+a)}\,d\tau \\ &= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau + e^{-as}\int_{-a}^0 f(\tau)e^{-s\tau}\,d\tau \\ &= e^{-as}F(s) + e^{-as}\int_{-a}^0 f(\tau)e^{-s\tau}\,d\tau \\ &= \boxed{e^{-as}F(s)}, \quad \text{if } f(t)=0 \text{ when } t<0\end{aligned}$$

If $f(t) = 0$ for $t < 0$, then $\boxed{\mathscr{L}\{f(t-a)\} = e^{-as}\mathscr{L}\{f(t)\}}$, for $a > 0$.



How about functions $f(t)$ that is non-zero for $t < 0$?

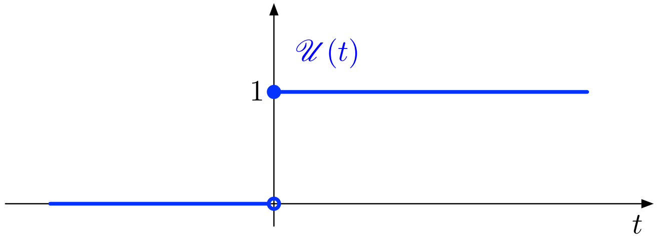


Unit Step Function

Definition (Unit Step Function)

$$\mathcal{U}(t) := \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

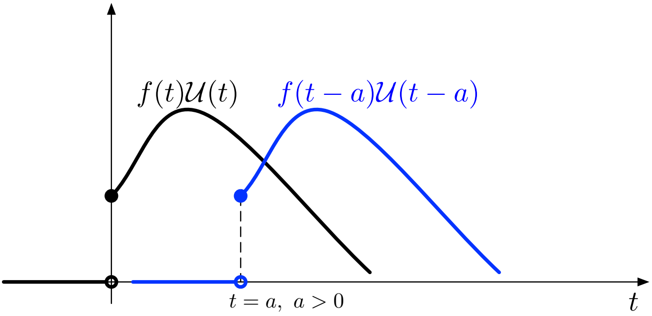
Note: $\mathscr{L}\{f(t)\mathcal{U}(t)\} = \mathscr{L}\{f(t)\}$.



Theorem (Translation on the t-Axis)

For $a > 0$,

$$\mathscr{L}\left\{f(t-a)\mathcal{U}(t-a)\right\} = e^{-as}\mathscr{L}\left\{f(t)\mathcal{U}(t)\right\} = e^{-as}\mathscr{L}\left\{f(t)\right\}.$$



Examples: Laplace Transforms

Example

Calculate $\mathscr{L}\left\{\mathcal{U}(t-a)\right\}$.

$$\mathscr{L}\left\{\mathcal{U}(t-a)\right\} = \mathscr{L}\left\{1\cdot\mathcal{U}(t-a)\right\} = e^{-as}\mathscr{L}\left\{1\right\} = \frac{e^{-as}}{s}.$$

Example

Calculate $\mathscr{L}\left\{\cos t\,\mathcal{U}(t-\pi)\right\}$.

$$\begin{aligned}\mathscr{L}\left\{\cos t\,\mathcal{U}(t-\pi)\right\} &= \mathscr{L}\left\{\cos(t+\pi-\pi)\mathcal{U}(t-\pi)\right\}\\ &= e^{-\pi s}\mathscr{L}\left\{\cos(t+\pi)\right\} = e^{-\pi s}\mathscr{L}\left\{-\cos t\right\}\\ &= -e^{-\pi s}\frac{s}{s^2+1}.\end{aligned}$$

Examples: Inverse Laplace Transforms

Example

Calculate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-4)e^s}\right\}$.

Since $\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$, according to the translation property:

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-4)e^s}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-4}e^{-s}\right\} = \boxed{e^{4(t-1)}\mathcal{U}(t-1)}.$$

Examples: Inverse Laplace Transforms

Example

Calculate $\mathscr{L}^{-1}\left\{\dfrac{se^{-2s}+e^{-s}}{(s-1)(s-2)}\right\}$.

A: First we organize the term as follows:

$$\begin{aligned}\frac{se^{-2s}+e^{-s}}{(s-1)(s-2)} &= \frac{s}{(s-1)(s-2)}e^{-2s}+\frac{1}{(s-1)(s-2)}e^{-s}\\ &= \left\{\frac{-1}{s-1}+\frac{2}{s-2}\right\}e^{-2s}+\left\{\frac{-1}{s-1}+\frac{1}{s-2}\right\}e^{-s}.\end{aligned}$$

Due to linearity and the translation property, the inverse transform is

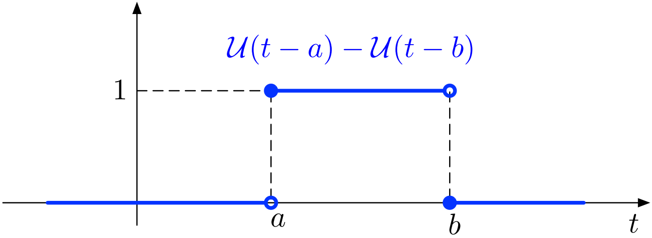
$$\boxed{\left\{-e^{(t-2)}+2e^{2(t-2)}\right\}\mathcal{U}(t-2)+\left\{-e^{(t-1)}+e^{2(t-1)}\right\}\mathcal{U}(t-1)}.$$

Piecewise-Defined Function

Unit step function is useful in representing **piecewise-defined** functions.

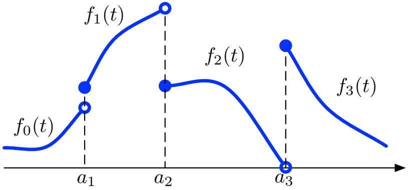
Example: the following function can be rewritten in terms of $\mathcal{U}$:

$$f(t) = \begin{cases} 1, & a \le t < b \\ 0, & \text{otherwise} \end{cases} = \boxed{\mathcal{U}(t-a) - \mathcal{U}(t-b)}.$$



Piecewise-Defined Function

$$f(t) = \begin{cases} f_0(t), & t < a_1 \\ f_1(t), & a_1 \le t < a_2 \\ \vdots & \vdots \\ f_n(t), & t \ge a_n \end{cases}$$



$$\begin{aligned} &= f_0(t)\{1 - \mathcal{U}(t-a_1)\} \\ &\quad + f_1(t)\{\mathcal{U}(t-a_1) - \mathcal{U}(t-a_2)\} \\ &\quad + f_2(t)\{\mathcal{U}(t-a_2) - \mathcal{U}(t-a_3)\} \\ &\quad\ \vdots \qquad \vdots \\ &\quad + f_n(t)\mathcal{U}(t-a_n). \end{aligned}$$

Solving IVP with Piecewise Defined External Drive

Example

Solve $y'' + 3y' + 2y = g(t),\ \ y(0) = 1,\ \ y'(0) = 5$, where $g(t) = 10\cos t$ for $\pi \le t < 3\pi$ and $g(t) = 0$ otherwise.

$$g(t) = 10\cos t\,\{\mathcal{U}(t-\pi) - \mathcal{U}(t-3\pi)\}$$

Step 1: Laplace-transform both sides:

$$\begin{aligned}
&\mathscr{L}\left\{y''\right\} + 3\mathscr{L}\left\{y'\right\} + 2\mathscr{L}\left\{y\right\} = \mathscr{L}\left\{10\cos t\,\{\mathcal{U}(t-\pi) - \mathcal{U}(t-3\pi)\}\right\} \\
&\implies \left(s^2\,Y(s) - sy(0) - y'(0)\right) + 3\left(s\,Y(s) - y(0)\right) + 2\,Y(s) \\
&\qquad = -\frac{10s}{s^2+1}\left(e^{-\pi s} - e^{-3\pi s}\right) \\
&\implies \left(s^2 + 3s + 2\right)Y(s) = s + 8 - \frac{10s}{s^2+1}\left(e^{-\pi s} - e^{-3\pi s}\right)
\end{aligned}$$

Solving IVP with Piecewise Defined External Drive

Example

Solve $y'' + 3y' + 2y = g(t),\ \ y(0) = 1,\ \ y'(0) = 5$, where $g(t) = 10\cos t$ for $\pi \le t < 3\pi$ and $g(t) = 0$ otherwise.

Step 2: Solve $Y(s)$:

$$Y(s) = \frac{s+8}{(s+1)(s+2)} - \frac{10s}{(s+1)(s+2)(s^2+1)}\left(e^{-\pi s} - e^{-3\pi s}\right).$$

Step 3: Compute the inverse Laplace transform of $Y(s)$:

$$Y(s) = \frac{7}{s+1} + \frac{-6}{s+2} + \left(\frac{5}{s+1} + \frac{-4}{s+2} + \frac{-s-3}{s^2+1}\right)\left(e^{-\pi s} - e^{-3\pi s}\right)$$

$$\begin{aligned} \implies y(t) = 7e^{-t} - 6e^{-2t} & \\ & + \Big(5e^{-(t-\pi)} - 4e^{-2(t-\pi)} - \cos(t-\pi) - 3\sin(t-\pi)\Big)\mathcal{U}(t-\pi) \\ & - \Big(5e^{-(t-3\pi)} - 4e^{-2(t-3\pi)} - \cos(t-3\pi) - 3\sin(t-3\pi)\Big)\mathcal{U}(t-3\pi) \end{aligned}$$