Chapter 7: The Laplace Transform

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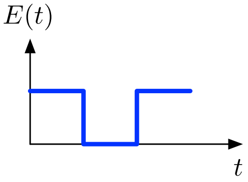
Solving an initial value problem associated with a linear differential equation:

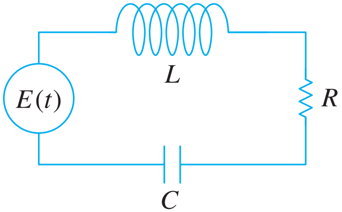
1. General solution = *complimentary* solution + *particular* solution.
2. Plug in the initial conditions to specify the undetermined coefficients.

Question: Is there a faster way?

In Chapter 4, 5, and 6, we majorly deal with linear differential equations with *continuous, differentiable, or analytic* coefficients.

But in real applications, sometimes this is not true.
For example:





Square voltage input: **Periodic, Discontinuous**.

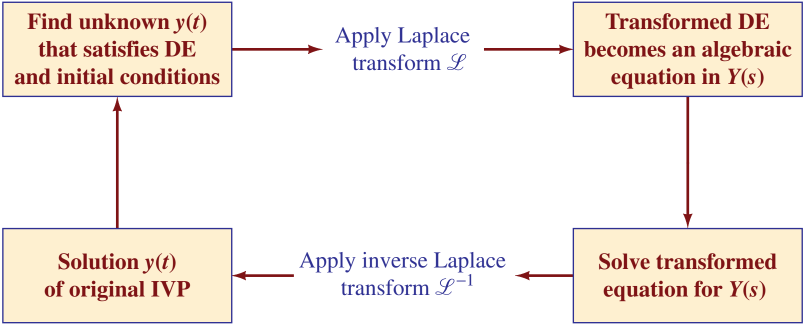
Question: How to solve the current? How to deal with discontinuity?

In this lecture we introduce a powerful tool:

Laplace Transform

Invented by Pierre-Simon Laplace (1749 - 1827).

Overview of the Method



1 Laplace and Inverse Laplace Transform: Definitions and Basics

Definition of the Laplace Transform

Definition

For a function $f(t)$ defined for $t \geq 0$, its **Laplace Transfrom** is defined as

$$F(s) := \mathscr{L}\left\{f(t)\right\} := \int_0^\infty e^{-st} f(t)\,dt,$$

given that the improper integral converges.

Note: Use capital letters to denote transforms.

$$f(t) \xrightarrow{\mathscr{L}} F(s), \quad g(t) \xrightarrow{\mathscr{L}} G(s), \quad y(t) \xrightarrow{\mathscr{L}} Y(s), \text{ etc.}$$

Note: The domain of the Laplace transform $F(s)$ (that is, where the improper integral converges) depends on the function $f(t)$

Examples of Laplace Transform

Example

Evaluate $\mathscr{L}\{1\}$.

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{T\to\infty} \int_0^T e^{-st}\,dt$$
$$= \lim_{T\to\infty} \left[\frac{-e^{-st}}{s}\right]_0^T = \lim_{T\to\infty} \frac{1-e^{-sT}}{s}.$$

When does the above converge? $s > 0$!

Hence, the domain of $\mathscr{L}\{1\}$ is $s > 0$, and $\boxed{\mathscr{L}\{1\} = \frac{1}{s}}$.

Examples of Laplace Transform

Example

Evaluate $\mathscr{L}\{t\}$.

$$\mathscr{L}\{t\} = \int_0^\infty te^{-st}dt = \lim_{T\to\infty}\int_0^T td\left(\frac{-e^{-st}}{s}\right)$$

$$= \lim_{T\to\infty}\left[\frac{-te^{-st}}{s}\right]_0^T + \int_0^T \frac{1}{s}e^{-st}dt = \lim_{T\to\infty}\frac{-Te^{-sT}}{s} + \frac{1}{s}\mathscr{L}\{1\}.$$

When does the above converge? $s > 0$!

Hence, the domain of $\mathscr{L}\{t\}$ is $s > 0$, and $\boxed{\mathscr{L}\{t\} = \frac{1}{s^2}}$.

Laplace Transform of t^n

$$\boxed{\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, \ldots, \;\; s > 0}$$

Proof: One way is to prove it by induction. We will show another proof after discussing the Laplace transform of the derivative of a function.

Laplace Transform of e^{at}

$$\boxed{\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \ s > a}$$

Proof:

$$\begin{aligned}\mathscr{L}\left\{e^{at}\right\} &= \int_0^\infty e^{at}e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt \\ &= \lim_{T\to\infty}\left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^T = \lim_{T\to\infty}\frac{1-e^{-(s-a)T}}{s-a}\end{aligned}$$

When does the above converge? $s - a > 0$!

Hence, the domain of $\mathscr{L}\left\{e^{at}\right\}$ is $s > a$, and $\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a}$.

Laplace Transform of $\sin(kt)$ and $\cos(kt)$

$$\boxed{\mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2},\ \mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2},\ \ s>0}$$

Proof:

$$\begin{aligned}
\mathscr{L}\left\{\sin(kt)\right\} &= \int_0^\infty \sin(kt)\, e^{-st} dt = \int_0^\infty \sin(kt)\, d\left(\frac{-e^{-st}}{s}\right) \\
&= \left[\frac{-\sin(kt)\, e^{-st}}{s}\right]_0^\infty + \frac{k}{s}\int_0^\infty \cos(kt)\, e^{-st} dt \\
&= \left[\frac{-\sin(kt)\, e^{-st}}{s}\right]_0^\infty + \frac{k}{s}\mathscr{L}\left\{\cos(kt)\right\}
\end{aligned}$$

When does the above converge? $s>0$! $\implies \left[\frac{-\sin(kt)\, e^{-st}}{s}\right]_0^\infty = 0$

Laplace Transform of $\sin(kt)$ and $\cos(kt)$

$$\boxed{\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2},\ \mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2},\quad s>0}$$

Proof:

$$\begin{aligned}\mathscr{L}\{\cos(kt)\} &= \int_0^\infty \cos(kt)\,e^{-st}dt = \int_0^\infty \cos(kt)\,d\left(\frac{-e^{-st}}{s}\right)\\ &= \left[\frac{-\cos(kt)\,e^{-st}}{s}\right]_0^\infty - \frac{k}{s}\int_0^\infty \sin(kt)\,e^{-st}dt\\ &= \left[\frac{-\cos(kt)\,e^{-st}}{s}\right]_0^\infty - \frac{k}{s}\mathscr{L}\{\sin(kt)\}\end{aligned}$$

When does the above converge? $s>0$! $\implies \left[\frac{-\cos(kt)\,e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$.

Laplace Transform of $\sin(kt)$ and $\cos(kt)$

$$\boxed{\mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2},\ \mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2},\ \ s>0}$$

Proof:

$$\begin{cases} \mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s}\mathscr{L}\left\{\cos(kt)\right\} \\ \mathscr{L}\left\{\cos(kt)\right\} = \frac{1}{s} - \frac{k}{s}\mathscr{L}\left\{\sin(kt)\right\} \end{cases}$$

Solve the above, we get the result:

$$\mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s}\mathscr{L}\left\{\cos(kt)\right\} = \frac{k}{s^2} - \frac{k^2}{s^2}\mathscr{L}\left\{\sin(kt)\right\}$$

$$\Longrightarrow\ \frac{s^2+k^2}{s^2}\mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s^2}\ \Longrightarrow\ \mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2}$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{k}\mathscr{L}\left\{\sin(kt)\right\} = \frac{s}{s^2+k^2}.$$

Laplace Transform is Linear

Theorem

For any α, β, $f(t) \xrightarrow{\mathscr{L}} F(s)$, $g(t) \xrightarrow{\mathscr{L}} G(s)$,

$$\boxed{\mathscr{L}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha F(s) + \beta G(s)}$$

Proof: It can be proved by the linearity of integral.

Example

Evaluate $\mathscr{L}\left\{\sinh(kt)\right\}$ and $\mathscr{L}\left\{\cosh(kt)\right\}$.

A: $\sinh(kt) = \frac{1}{2}\left(e^{kt} - e^{-kt}\right)$, $\cosh(kt) = \frac{1}{2}\left(e^{kt} + e^{-kt}\right)$. Hence

$$\sinh(kt) \xrightarrow{\mathscr{L}} \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \boxed{\frac{k}{s^2 - k^2}, \quad s > |k|}$$

$$\cosh(kt) \xrightarrow{\mathscr{L}} \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \boxed{\frac{s}{s^2 - k^2}, \quad s > |k|}.$$

Laplace Transforms of Some Basic Functions

$f(t)$	$F(s)$	Domain of $F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$	$s > 0$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$	$s > 0$
$\sinh(kt)$	$\dfrac{k}{s^2-k^2}$	$s > \|k\|$
$\cosh(kt)$	$\dfrac{s}{s^2-k^2}$	$s > \|k\|$

Existence of Laplace Transform

Theorem (Sufficient Conditions for the Existence of Laplace Transform)

If a function $f(t)$ is

- ***piecewise continuous*** *on $[0, \infty)$, and*
- ***of exponential order****,*

then $\mathscr{L}\{f(t)\}$ exists for $s > c$ for some constant c.

Definition

A function $f(t)$ is of exponential order if $\exists\ c \in \mathbb{R}, M > 0, \tau > 0$ such that

$$|f(t)| \leq Me^{ct},\ \forall\ t > \tau.$$

Note: If $f(t)$ is of exponential order, then $\exists\ c \in \mathbb{R}$ such that for $s > c$,

$$\lim_{t \to \infty} f(t)e^{-st} = 0.$$

Existence of Laplace Transform

Theorem (Sufficient Conditions for the Existence of Laplace Transform)

If a function $f(t)$ is

- ***piecewise continuous*** *on $[0, \infty)$, and*
- ***of exponential order***,

then $\mathscr{L}\{f(t)\}$ exists for $s > c$ for some constant c.

Proof: For sufficiently large $T > \tau$, we split the following integral:

$$\int_0^T f(t)\,dt = \underbrace{\int_0^\tau f(t)\,e^{-st}dt}_{I_1} + \underbrace{\int_\tau^T f(t)\,e^{-st}dt}_{I_2}.$$

We only need to prove that I_2 converges as $T \to \infty$:

$$|I_2| \leq \int_\tau^T |f(t)\,e^{-st}|\,dt = \int_\tau^T |f(t)|\,e^{-st}dt \leq \int_\tau^T Me^{ct}e^{-st}dt,$$

which converges as $T \to \infty$ for $s > c$ since $\mathscr{L}\left\{e^{ct}\right\}$ exists.

In this lecture, we focus on functions that are

- **piecewise continuous** on $[0, \infty)$, and
- **of exponential order**

Laplace Transform of Derivatives

Suppose $f(t)$ is continuous on $[0,\infty)$ and of exponential order, and $f'(t)$ is also continuous on $[0,\infty)$, then the Laplace transform of $f'(t)$ can be found as follows:

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty e^{-st} d\left(f(t)\right) \\ &= \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt = \boxed{s\mathscr{L}\{f(t)\} - f(0)},\quad s > c\end{aligned}$$

Note: since $f(t)$ is of exponential order, $f(t)e^{-st} \to 0$ as $t \to \infty$ for $s > c$ for some constant c.

Similarly, if $f'(t)$ is also of exponential order, we can find

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0) = \boxed{s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)}.$$

Laplace Transform of Derivatives

Theorem

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order, and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) := \mathscr{L}\{f(t)\}$.

Example

Evaluate $\mathscr{L}\{t^n\}$.

A: Let $f(t) = t^n$. Since $f^{(n)}(t) = n!$, $f^{(k)}(0) = 0$ for any $0 \le k \le n-1$, using the above theorem we get

$$\mathscr{L}\{n!\} = s^n F(s) = \frac{n!}{s} \implies F(s) = \frac{n!}{s^{n+1}}.$$

Inverse Laplace Transform

$$\mathscr{L}\{f(t)\} = F(s) \iff \mathscr{L}^{-1}\{F(s)\} = f(t)$$

$F(s)$: Laplace transform of $f(t)$ $\iff$ $f(t)$: inverse Laplace transform of $F(s)$

Note: Inverse Laplace transform is also linear:

$$\boxed{\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)}$$

Some Inverse Laplace Transforms

$F(s)$	$f(t)$	Domain of $F(s)$
$\dfrac{n!}{s^{n+1}}$	t^n	$s > 0$
$\dfrac{1}{s-a}$	e^{at}	$s > a$
$\dfrac{k}{s^2+k^2}$	$\sin(kt)$	$s > 0$
$\dfrac{s}{s^2+k^2}$	$\cos(kt)$	$s > 0$
$\dfrac{k}{s^2-k^2}$	$\sinh(kt)$	$s > \|k\|$
$\dfrac{s}{s^2-k^2}$	$\cosh(kt)$	$s > \|k\|$

Examples

Example

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

Step 1: Decompose

$$\frac{-2s+6}{s^2+4} = -2\frac{s}{s^2+4} + 3\frac{2}{s^2+4}.$$

Step 2: By the linearity of inverse Laplace transform,

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = -2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \boxed{-2\cos 2t + 3\sin 2t}.$$

Examples

Example

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{(s+3)^2}{(s-1)(s-2)(s+4)}\right\}$.

Step 1: Decompose into partial fractions:

$$F(s) := \frac{(s+3)^2}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}.$$

We can find A, B, C by the following:

$$A = \left[\frac{(s+3)^2}{\cancel{(s-1)}(s-2)(s+4)}\right]_{s=1} = \frac{16}{-5}, \quad B = \left[\frac{(s+3)^2}{(s-1)\cancel{(s-2)}(s+4)}\right]_{s=2} = \frac{25}{6}$$

$$C = \left[\frac{(s+3)^2}{(s-1)(s-2)\cancel{(s+4)}}\right]_{s=-4} = \frac{1}{30}$$

Step 2: Linearity $\implies f(t) = \boxed{-\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}}$.