Chapter 7: The Laplace Transform

王奕翔

Department of Electrical Engineering
National Taiwan University

ihwang@ntu.edu.tw

November 18, 2013

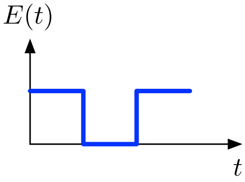
Solving an initial value problem associated with a linear differential equation:

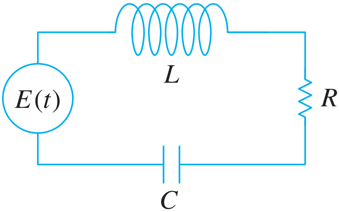
1. General solution = *complimentary* solution + *particular* solution.
2. Plug in the initial conditions to specify the undetermined coefficients.

Question: Is there a faster way?

In Chapter 4, 5, and 6, we majorly deal with linear differential equations with *continuous, differentiable, or analytic* coefficients.

But in real applications, sometimes this is not true.
For example:





Square voltage input: **Periodic, Discontinuous**.

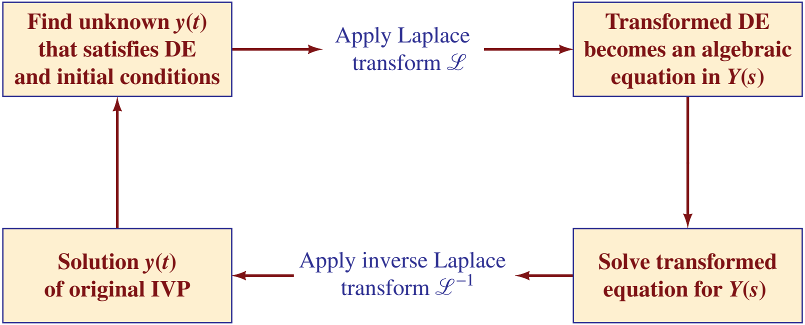
Question: How to solve the current? How to deal with discontinuity?

In this lecture we introduce a powerful tool:

Laplace Transform

Invented by Pierre-Simon Laplace (1749 - 1827).

Overview of the Method



1 Laplace and Inverse Laplace Transform: Definitions and Basics

Definition of the Laplace Transform

Definition

For a function $f(t)$ defined for $t \geq 0$, its **Laplace Transfrom** is defined as

$$F(s) := \mathscr{L}\left\{f(t)\right\} := \int_0^\infty e^{-st} f(t)\, dt,$$

given that the improper integral converges.

Note: Use capital letters to denote transforms.

$$f(t) \xrightarrow{\mathscr{L}} F(s), \quad g(t) \xrightarrow{\mathscr{L}} G(s), \quad y(t) \xrightarrow{\mathscr{L}} Y(s), \text{ etc.}$$

Note: The domain of the Laplace transform $F(s)$ (that is, where the improper integral converges) depends on the function $f(t)$

Examples of Laplace Transform

Example

Evaluate $\mathscr{L}\{1\}$.

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{T\to\infty}\int_0^T e^{-st}dt$$

$$= \lim_{T\to\infty}\left[\frac{-e^{-st}}{s}\right]_0^T = \lim_{T\to\infty}\frac{1-e^{-sT}}{s}.$$

When does the above converge? $s > 0$!

Hence, the domain of $\mathscr{L}\{1\}$ is $s > 0$, and $\boxed{\mathscr{L}\{1\} = \frac{1}{s}}$.

Examples of Laplace Transform

Example

Evaluate $\mathscr{L}\{t\}$.

$$\mathscr{L}\{t\} = \int_0^\infty te^{-st}dt = \lim_{T\to\infty}\int_0^T td\left(\frac{-e^{-st}}{s}\right)$$
$$= \lim_{T\to\infty}\left[\frac{-te^{-st}}{s}\right]_0^T + \int_0^T \frac{1}{s}e^{-st}dt = \lim_{T\to\infty}\frac{-Te^{-sT}}{s} + \frac{1}{s}\mathscr{L}\{1\}.$$

When does the above converge? $s > 0$!

Hence, the domain of $\mathscr{L}\{t\}$ is $s > 0$, and $\boxed{\mathscr{L}\{t\} = \frac{1}{s^2}}$.

Laplace Transform of t^n

$$\boxed{\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, \ldots, \; s > 0}$$

Proof: One way is to prove it by induction. We will show another proof after discussing the Laplace transform of the derivative of a function.

Laplace Transform of e^{at}

$$\boxed{\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a},\ s > a}$$

Proof:

$$\begin{aligned}\mathscr{L}\left\{e^{at}\right\} &= \int_0^\infty e^{at}e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt \\ &= \lim_{T\to\infty}\left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^T = \lim_{T\to\infty}\frac{1-e^{-(s-a)T}}{s-a}\end{aligned}$$

When does the above converge? $s-a>0$!

Hence, the domain of $\mathscr{L}\left\{e^{at}\right\}$ is $s>a$, and $\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a}$.

Laplace Transform of $\sin(kt)$ and $\cos(kt)$

$$\boxed{\mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2},\ \mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2},\ \ s > 0}$$

Proof:

$$\begin{aligned}
\mathscr{L}\left\{\sin(kt)\right\} &= \int_0^\infty \sin(kt)\,e^{-st}dt = \int_0^\infty \sin(kt)\,d\left(\frac{-e^{-st}}{s}\right) \\
&= \left[\frac{-\sin(kt)\,e^{-st}}{s}\right]_0^\infty + \frac{k}{s}\int_0^\infty \cos(kt)\,e^{-st}dt \\
&= \left[\frac{-\sin(kt)\,e^{-st}}{s}\right]_0^\infty + \frac{k}{s}\mathscr{L}\left\{\cos(kt)\right\}
\end{aligned}$$

When does the above converge? $s > 0$! $\implies \left[\frac{-\sin(kt)\,e^{-st}}{s}\right]_0^\infty = 0$

Laplace Transform of $\sin(kt)$ and $\cos(kt)$

$$\boxed{\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2},\ \mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2},\ \ s>0}$$

Proof:

$$\begin{aligned}
\mathscr{L}\{\cos(kt)\} &= \int_0^\infty \cos(kt)\,e^{-st}dt = \int_0^\infty \cos(kt)\,d\left(\frac{-e^{-st}}{s}\right) \\
&= \left[\frac{-\cos(kt)\,e^{-st}}{s}\right]_0^\infty - \frac{k}{s}\int_0^\infty \sin(kt)\,e^{-st}dt \\
&= \left[\frac{-\cos(kt)\,e^{-st}}{s}\right]_0^\infty - \frac{k}{s}\mathscr{L}\{\sin(kt)\}
\end{aligned}$$

When does the above converge? $s>0$! $\implies \left[\frac{-\cos(kt)\,e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$.

Laplace Transform of $\sin(kt)$ and $\cos(kt)$

$$\boxed{\mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2},\ \mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2},\ \ s>0}$$

Proof:

$$\begin{cases} \mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s}\mathscr{L}\left\{\cos(kt)\right\} \\ \mathscr{L}\left\{\cos(kt)\right\} = \frac{1}{s} - \frac{k}{s}\mathscr{L}\left\{\sin(kt)\right\} \end{cases}$$

Solve the above, we get the result:

$$\mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s}\mathscr{L}\left\{\cos(kt)\right\} = \frac{k}{s^2} - \frac{k^2}{s^2}\mathscr{L}\left\{\sin(kt)\right\}$$

$$\implies \frac{s^2+k^2}{s^2}\mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s^2} \implies \mathscr{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2}$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{k}\mathscr{L}\left\{\sin(kt)\right\} = \frac{s}{s^2+k^2}.$$

Laplace Transform is Linear

Theorem

For any α, β, $f(t) \xrightarrow{\mathscr{L}} F(s)$, $g(t) \xrightarrow{\mathscr{L}} G(s)$,

$$\boxed{\mathscr{L}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha F(s) + \beta G(s)}$$

Proof: It can be proved by the linearity of integral.

Example

Evaluate $\mathscr{L}\left\{\sinh(kt)\right\}$ and $\mathscr{L}\left\{\cosh(kt)\right\}$.

A: $\sinh(kt) = \frac{1}{2}\left(e^{kt} - e^{-kt}\right)$, $\cosh(kt) = \frac{1}{2}\left(e^{kt} + e^{-kt}\right)$. Hence

$$\sinh(kt) \xrightarrow{\mathscr{L}} \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \boxed{\frac{k}{s^2-k^2}, \quad s > |k|}$$

$$\cosh(kt) \xrightarrow{\mathscr{L}} \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \boxed{\frac{s}{s^2-k^2}, \quad s > |k|}.$$

Laplace Transforms of Some Basic Functions

$$t^n \xrightarrow{\mathscr{L}} \frac{n!}{s^{n+1}} \qquad s > 0$$

$$e^{at} \xrightarrow{\mathscr{L}} \frac{1}{s-a} \qquad s > a$$

$$\sin(kt) \xrightarrow{\mathscr{L}} \frac{k}{s^2+k^2} \qquad s > 0$$

$$\cos(kt) \xrightarrow{\mathscr{L}} \frac{s}{s^2+k^2} \qquad s > 0$$

$$\sinh(kt) \xrightarrow{\mathscr{L}} \frac{k}{s^2-k^2} \qquad s > |k|$$

$$\cosh(kt) \xrightarrow{\mathscr{L}} \frac{s}{s^2-k^2} \qquad s > |k|$$

Existence of Laplace Transform

Theorem (Sufficient Conditions for the Existence of Laplace Transform)

If a function $f(t)$ is

- ***piecewise continuous*** *on $[0, \infty)$, and*
- ***of exponential order****,*

then $\mathscr{L}\{f(t)\}$ exists for $s > c$ for some constant c.

Definition

A function $f(t)$ is of exponential order if $\exists\ c \in \mathbb{R}, M > 0, \tau > 0$ such that

$$|f(t)| < Me^{ct},\ \forall\ t > \tau.$$

Note: If $f(t)$ is of exponential order, then $\exists\ c \in \mathbb{R}$ such that for $s > c$,

$$\lim_{t \to \infty} f(t) e^{-st} = 0.$$

Existence of Laplace Transform

Theorem (Sufficient Conditions for the Existence of Laplace Transform)

If a function $f(t)$ is

- ***piecewise continuous** on $[0, \infty)$, and*
- ***of exponential order**,*

then $\mathscr{L}\{f(t)\}$ exists for $s > c$ for some constant c.

Proof: For sufficiently large $T > \tau$, we split the following integral:

$$\int_0^T f(t)\,dt = \underbrace{\int_0^\tau f(t)\,e^{-st}\,dt}_{I_1} + \underbrace{\int_\tau^T f(t)\,e^{-st}\,dt}_{I_2}.$$

We only need to prove that I_2 converges as $T \to \infty$:

$$|I_2| \le \int_\tau^T |f(t)\,e^{-st}|\,dt = \int_\tau^T |f(t)|\,e^{-st}\,dt \le \int_\tau^T Me^{ct}e^{-st}\,dt,$$

which converges as $T \to \infty$ for $s > c$ since $\mathscr{L}\left\{e^{ct}\right\}$ exists.

In this lecture, we focus on functions that are

- **piecewise continuous** on $[0, \infty)$, and
- **of exponential order**