Chapter 6: Series Solutions of Linear Equations

王奕翔

Department of Electrical Engineering
National Taiwan University

ihwang@ntu.edu.tw

November 13, 2013

Solving Higher-Order Linear Equations

In Chapter 4, we learns how to **analytically** solve two special kinds of higher-order linear differential equations:

1. Linear Differential Equation with Constant Coefficients
2. Cauchy-Euler Equations

Essentially only one kind – linear DE with constant coefficients!

Because to solve Cauchy-Euler DE, we substitute $x = e^t$!

Question: Is it possible to solve other kinds, like the following?

$$(x^2 + 2x - 3)y'' - 2(x + 1)y' + 2y = 0$$

Idea: Express the solution function as a power series!

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

Focus on: Linear Second-Order Differential Equations

Throughout this lecture, we shall focus on solving homogeneous linear second order differential equations

$$a_2(x)y'' + a_1(x)y + a_0(x) = 0,$$

using the method of power series.

Standard Form: Frequently throughout the discussions in this lecture:

$$y'' + P(x)y' + Q(x)y = 0.$$

Power Series

Definition

A power series in $(x - a)$ (or a power series centered at a) is an infinite series of the following form:

$$\sum_{n=0}^{\infty} c_n (x - a)^n,$$

where $\{c_n\}_0^{\infty}$ is a sequence of real numbers.

Some Examples:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots, \quad \sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + \cdots.$$

Convergence, Divergence, Absolute Convergence

Convergence: A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges at $x = x_0$ if

$$\lim_{N\to\infty} \sum_{n=0}^{N} c_n(x_0 - a)^n \quad \text{exists.}$$

Otherwise, the power series diverges at $x = x_0$.

Absolute Convergence: A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges *absolutely* at $x = x_0$ if

$$\lim_{N\to\infty} \sum_{n=0}^{N} |c_n(x_0 - a)^n| \quad \text{exists.}$$

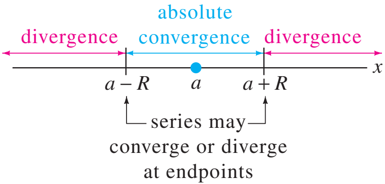
Ratio Test: Suppose $c_n \neq 0$ for all n, then the following test tells us about the convergence of the series:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x_0 - a)^{n+1}}{c_n(x_0 - a)^n} \right| = |x_0 - a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \qquad \begin{array}{ll} < 1 & \text{absolute convergence} \\ > 1 & \text{divergence} \\ = 1 & \text{not sure} \end{array}$$

Interval of Convergence

Interval of Convergence: Every power series has an interval of convergence $I = (a - R, a + R)$, in which he power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ converges absolutely.

$R > 0$ is called the **radius of convergence**.



A power series defines a function of x, $f(x) := \sum_{n=0}^{\infty} c_n(x - a)^n$ for $x \in I$.

Function Defined by a Power Series $\sum_{n=0}^{\infty} c_n(x-a)^n$

Define the function (I: interval of convergence)

$$y(x) := \sum_{n=0}^{\infty} c_n(x-a)^n, \ x \in I.$$

Differentiation

$$y'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \ x \in I$$

$$y''(x) = 2c_2 + 6c_3x + 12c_4x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}, \ x \in I$$

Taylor's Series

If a function $f(x)$ is *infinitely differentiable* at a point a, then it can be represented by **Taylor's Series** as follows, with a radius of convergence $R > 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Examples

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \;\; x \in \mathbb{R}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \;\; x \in (-1, 1)$$

Maclaurin Series	Interval of Convergence
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$[-1, 1]$
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$	$(-1, 1]$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1, 1)$

Ordinary and Singular Points

Focus on homogeneous linear 2nd order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Rewrite it into its standard form $\boxed{y'' + P(x)y' + Q(x)y = 0}$.

Definition (Ordinary and Singular Points)

$x = x_0$ is an **ordinary point** of the above DE if both $P(x)$ and $Q(x)$ are analytic at x_0. Otherwise, $x = x_0$ is a **singular** point.

Analytic at a Point: a function $f(x)$ is analytic at a point $x = x_0$ if and only if $f(x)$ can be represented as a power series $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ with a positive radius of convergence.

In our lecture analytic $\equiv$ infinitely differentiable.

Examples: Ordinary and Singular Points

1. Constant coefficients: $a_2 y'' + a_1 y' + a_0 y = 0$. Every $x \in \mathbb{R}$ is ordinary.

2. Cauchy-Euler DE: $x^2 y'' + xy' + y = 0$.
   - $P(x) = \frac{1}{x}$ is analytic at $x \in \mathbb{R} \setminus \{0\}$.
   - $Q(x) = \frac{1}{x^2}$ is analytic at $x \in \mathbb{R} \setminus \{0\}$.

   Hence, $x = 0$ is the only singular point.

3. Polynomial Coefficients: $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, where $a_2(x) \neq 0, a_1(x), a_0(x)$ are all polynomials of x.
   - $P(x) = \frac{a_1(x)}{a_2(x)}$ is analytic at $x \in \mathbb{R} \setminus \{r \in \mathbb{R} : a_2(r) = 0\}$.
   - $Q(x) = \frac{a_0(x)}{a_2(x)}$ is analytic at $x \in \mathbb{R} \setminus \{r \in \mathbb{R} : a_2(r) = 0\}$.

   Hence, $\{r \in \mathbb{R} : a_2(r) = 0\}$ are singular points.

4. $y'' + xy' + (\ln x)\, y = 0$.
   - $P(x) = x$ is analytic at $x \in \mathbb{R}$.
   - $Q(x) = \ln x$ is analytic at $x \in (0, \infty)$.

   Hence, every $x \leq 0$ is singular.

Existence of Power Series Solutions about Ordinary Points

The following theorem lays the theoretical foundations of the method.

Theorem

Let $x = x_0$ be an ordinary point of a homogenous linear 2nd order DE. Then, we can find two linearly independent solutions in the form of power series centered at x_0, that is,

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

Moreover, the radius of convergence $\geq$ the distance from x_0 to the closest singular point in $\mathbb{C}$.

Example: Minimum Radius of Convergence

Example

Consider a linear second order DE $(x^2+1)y''+xy'-y=0$.
Find the minimum radius of convergence of a power series solution about the ordinary points $x=-1$ and $x=0$.

A: The singular points in the complex domain $\mathbb{C}$ is $\pm i$.

The distance between -1 and $\pm i$ is $\sqrt{1^2+1^2}=\sqrt{2}$. The distance between 0 and $\pm i$ is 1.

Based on the previous theorem, we obtain the minimum radius of convergence $R=\sqrt{2}$ and $R=1$ respectively.

In other words, for $|x+1|<\sqrt{2}$ and $|x|<1$, the power series solution of the DE exists (and converges absolutely).

Example: Finding Power Series Solutions

Example

Consider a linear second order DE $(x^2+1)y''+xy'-y=0$.
Find two linearly independent power series solution about the ordinary point $x=0$.

A: From the previous discussion, we know that the interval of definition of the solutions should be $(-1,1)$.

Plug in the power series representation $y=\sum_{n=0}^{\infty} c_n x^n$:

$$y=\sum_{n=0}^{\infty} c_n x^n,\ y'=\sum_{n=0}^{\infty} nc_n x^{n-1},\ y''=\sum_{n=0}^{\infty} c_n n(n-1)x^{n-2}$$

$$\begin{aligned}\implies 0 &= (x^2+1)y''+xy'-y \\ &= \sum_{n=0}^{\infty} \left\{(n^2-1)c_n+(n+2)(n+1)c_{n+2}\right\}x^n\end{aligned}$$

$$(x^2+1)y'' + xy' - y \qquad \sum_{n=0}^{\infty} \left\{ (n^2-1)c_n + (n+2)(n+1)c_{n+2} \right\} x^n$$

$$y''$$

$$(x^2 + 1)y'' + xy' - y$$

$$x^2y'' + xy' - y$$

$$\sum_{n=0}^{\infty} \left\{ (n^2 - 1)c_n + (n+2)(n+1)c_{n+2} \right\} x^n$$

$$y''$$

$$(x^2 + 1)y'' + xy' - y \qquad \sum_{n=0}^{\infty} \left\{ (n^2 - 1)c_n + (n+2)(n+1)c_{n+2} \right\} x^n$$

$$x^2y'' + xy' - y$$

$$x^2y'' = x^2 \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} c_n n(n-1)x^n$$

$$xy' = x \sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_n n x^n$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y'' = \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k$$

y''

$$(x^2+1)y'' + xy' - y \qquad \sum_{n=0}^{\infty} \left\{ (n^2-1)c_n + (n+2)(n+1)c_{n+2} \right\} x^n$$

$$x^2y'' + xy' - y$$

$$x^2y'' = x^2 \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} c_n n(n-1)x^n$$

$$xy' = x \sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_n n x^n$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y''$$

$$y'' = \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2}$$
$$= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$$
$$= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k$$

$$(x^2+1)y'' + xy' - y$$

$$\sum_{n=0}^{\infty} \left\{ (n^2-1)c_n + (n+2)(n+1)c_{n+2} \right\} x^n$$

$$x^2y'' + xy' - y$$

$$x^2y'' = x^2 \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} c_n n(n-1)x^n$$

$$xy' = x \sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_n n x^n$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Recursive Formula of Coefficients in Power Series Solution

Example

Consider a linear second order DE $(x^2+1)y''+xy'-y=0$.
Find two linearly independent power series solution about the ordinary point $x=0$.

Plug in the power series representation $y=\sum_{n=0}^{\infty} c_n x^n$, we get

$$0=\sum_{n=0}^{\infty}\left\{(n^2-1)c_n+(n+2)(n+1)c_{n+2}\right\}x^n$$

$$\Longrightarrow (n^2-1)c_n+(n+2)(n+1)c_{n+2}=0,\ \ n=0,1,2,\ldots$$

$$\Longrightarrow c_2=\frac{1}{2}c_0,\ \ c_3=0,\ \ c_{n+2}=\frac{1-n}{2+n}c_n,\ \ n=2,3,4,\ldots$$

$$\Longrightarrow c_2=\frac{1}{2}c_0,\, c_4=\frac{-1}{2\cdot 4}c_0,\, c_6=\frac{1\cdot 3}{2\cdot 4\cdot 6}c_0,\cdots$$

$$c_3=c_5=c_7=\cdots=0.$$

$$(n^2-1)c_n+(n+2)(n+1)c_{n+2}=0,\ \ n\geq 0$$

- $n=0$

$$-c_0+2c_2=0 \implies c_2=\frac{1}{2}c_0$$

- $n=1$

$$0+6c_3=0 \implies c_3=0$$

- $n\geq 2$

$$(n-1)(n+1)c_n+(n+2)(n+1)c_{n+2}=0$$
$$\implies c_{n+2}=\frac{1-n}{2+n}c_n$$

Wrapping Up

Example

Consider a linear second order DE $(x^2+1)y''+xy'-y=0$.
Find two linearly independent power series solution about the ordinary point $x=0$.

Therefore

$$y = c_0\left\{1+\frac{1}{2}x^2+\frac{-1}{2\cdot 4}x^4+\frac{1\cdot 3}{2\cdot 4\cdot 6}x^6+\cdots\right\}+c_1x$$

Thus we obtain two linearly independent solutions: $y_1(x)=x$, and

$$y_2(x) = 1+\frac{1}{2}x^2+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{2n},\ |x|<1.$$

Regular and Irregular Singular Points

Focus on homogeneous linear 2nd order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Rewrite it into its standard form $\boxed{y'' + P(x)y' + Q(x)y = 0}$.

Definition (Regular and Irregular Singular Points)

A singular point $x = x_0$ of the above DE is **regular** if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at x_0. Otherwise, $x = x_0$ is an **irregular** singular point.

Note: There may not be power series solutions about a singular point $x = x_0$. However, it is possible to obtain a generalized power series solution

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

In other words, at a regular singular point $x = x_0$, we can convert the standard form

$$y'' + P(x)y' + Q(x)y = 0,$$

into

$$\boxed{(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0}.$$

where $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at $x = x_0$, that is,

$$\boxed{p(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \;\; q(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n, \;\; |x - x_0| < R}$$

for some $R > 0$.

Examples: Classification of Singular Points

1. Cauchy-Euler DE: $x^2y'' + xy' + y = 0$. Its has one singular point $x = 0$.
   - $xP(x) = x\frac{1}{x} = 1$ is analytic at $x = 0$.
   - $x^2Q(x) = x^2\frac{1}{x^2} = 1$ is analytic at $x = 0$.

   Hence, $x = 0$ is a regular singular point.

2. Polynomial Coefficients: $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, where $a_2(x) \neq 0, a_1(x), a_0(x)$ are all polynomials of x. Let $x = x_0$ be a root of $a_2(x) = 0$. Hence $x = x_0$ is a singular point.
   - If in the denominator of the rational function $P(x) = \frac{a_1(x)}{a_2(x)}$ (after reduction), the factor $(x - x_0)$ appears at most to the first power, then $(x - x_0)P(x)$ is analytic at $x = x_0$.
   - If in the denominator of the rational function $Q(x) = \frac{a_0(x)}{a_2(x)}$ (after reduction), the factor $(x - x_0)$ appears at most to the second power, then $(x - x_0)^2Q(x)$ is analytic at $x = x_0$.

Examples: Classification of Singular Points

Example

For the second order DE $(x^2-4)^2y''+3(x-2)y'+5y=0$, find the singular points and classify them into regular and irregular ones.

A: First rewrite the DE into the standard form:

$$y''+3\frac{x-2}{(x^2-4)^2}y'+\frac{5}{(x^2-4)^2}y=y''+P(x)y'+Q(x)y=0.$$

Since $P(x)=3\frac{x-2}{(x^2-4)^2}=\frac{3}{(x-2)(x+2)^2}$ and $Q(x)=\frac{5}{(x^2-4)^2}=\frac{5}{(x-2)^2(x+2)^2}$, we have two singular points $x=2,-2$ for this DE.

$x=2$: **regular** singular point, because $(x-2)P(x)=\frac{3}{(x+2)^2}$ and $(x-2)^2Q(x)=\frac{5}{(x+2)^2}$ are both analytic at $x=2$.

$x=-2$: **irregular** singular point because $(x+2)P(x)=\frac{3}{(x-2)(x+2)}$ is not analytic at $x=-2$.

Method of Frobenius

Theorem

Let $x = x_0$ be a regular singular point of a homogenous linear 2nd order DE. Then, we can find at least one solutions in the following form:

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r},$$

where r is a constant (not necessarily an integer) to be determined. The series will converge on some interval $0 < x - x_0 < R$.

Note 1: Without loss of generality we assume that $c_0 \neq 0$.

Note 2: We have to determine

- The exponent r first,
- and then the sequence $\{c_n,\ n = 1, 2, \ldots\}$.

Method of Frobenius: Calculation

Without loss of generality, assume that the regular singular point is $x = 0$. We convert the standard form into

$$x^2 y'' + xp(x)y' + q(x)y = 0$$

where $p(x) = xP(x) = \sum_{n=0}^{\infty} a_n x^n$ and $q(x) = x^2 Q(x) = \sum_{n=0}^{\infty} b_n x^n$.

Plug in $y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$, we get

$$\begin{aligned} x^2 y'' + xp(x)y' + q(x)y = {} & x^2 \left(\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} \right) \\ & + x \left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \right) \\ & + \left(\sum_{n=0}^{\infty} b_n x^n \right) \left(\sum_{n=0}^{\infty} c_n x^{n+r} \right) \end{aligned}$$

Method of Frobenius: Calculation

Without loss of generality, assume that the regular singular point is $x = 0$. We convert the standard form into

$$x^2 y'' + xp(x)y' + q(x)y = 0$$

where $p(x) = xP(x) = \sum_{n=0}^{\infty} a_n x^n$ and $q(x) = x^2 Q(x) = \sum_{n=0}^{\infty} b_n x^n$.

Plug in $y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$, we get

$$\begin{aligned} x^2 y'' + xp(x)y' + q(x)y = x^r & \left(\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^n \right) \\ & + x^r \left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} c_n (n+r) x^n \right) \\ & + x^r \left(\sum_{n=0}^{\infty} b_n x^n \right) \left(\sum_{n=0}^{\infty} c_n x^n \right) \end{aligned}$$

Method of Frobenius: Calculation

Without loss of generality, assume that the regular singular point is $x = 0$. We convert the standard form into

$$x^2 y'' + xp(x)y' + q(x)y = 0$$

where $p(x) = xP(x) = \sum_{n=0}^{\infty} a_n x^n$ and $q(x) = x^2 Q(x) = \sum_{n=0}^{\infty} b_n x^n$.

Plug in $y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$, we get

$$\begin{aligned} x^2 y'' + xp(x)y' + q(x)y = x^r &\left(\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^n \right) \\ &+ x^r \left(\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} a_{n-k} c_k (k+r) \right\} x^n \right) \\ &+ x^r \left(\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} b_{n-k} c_k \right\} x^n \right) \end{aligned}$$

Method of Frobenius: Calculation

Without loss of generality, assume that the regular singular point is $x = 0$. We convert the standard form into

$$x^2 y'' + xp(x)y' + q(x)y = 0$$

where $p(x) = xP(x) = \sum_{n=0}^{\infty} a_n x^n$ and $q(x) = x^2 Q(x) = \sum_{n=0}^{\infty} b_n x^n$.

Plug in $y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$, we get

$$x^2 y'' + xp(x)y' + q(x)y = x^r \left(\sum_{n=0}^{\infty} L_n x^n \right)$$

where

$$\boxed{L_n := c_n(n+r)(n+r-1) + \sum_{k=0}^{n} c_k \left\{ a_{n-k}(k+r) + b_{n-k} \right\} = 0, \ \forall \ n = 0, 1, 2, \ldots}$$

Indicial Equation (Index → Indices → Indicial)

Further manipulate the conditions:

$$L_n = c_n(n+r)(n+r-1) + \sum_{k=0}^{n} c_k \left\{a_{n-k}(k+r) + b_{n-k}\right\}$$

$$= c_n\{(n+r)(n+r-1) + a_0(n+r) + b_0\} + \sum_{k=0}^{n-1} c_k \left\{a_{n-k}(k+r) + b_{n-k}\right\}$$

$$= c_n I(n+r) + \sum_{k=0}^{n-1} c_k \left\{a_{n-k}(k+r) + b_{n-k}\right\} = 0.$$

For $n = 0$, the condition reduces to

$$\boxed{I(r) = r(r-1) + a_0 r + b_0 = 0}.$$

This is called the **indicial equation** of the problem, and the two roots are called **indicial roots/exponents**.

Roots of the Indicial Equation

Let the two real roots of $I(r) = r(r-1) + a_0 r + b_0 = 0$ be r_1, r_2 and $r_1 \geq r_2$.

P.S. *We do not consider the case when r_1, r_2 are complex conjugate roots.*

1. $r_1 > r_2$ and $r_1 - r_2 \notin \mathbb{Z}$: Two linearly independent solutions can be found:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \ c_0 \neq 0, \quad y_2(x) = \sum_{n=0}^{\infty} d_n x^{n+r_2}, \ d_0 \neq 0$$

2. $r_1 > r_2$ and $r_1 - r_2 \in \mathbb{Z}$: Two linearly independent solutions can be found:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \ c_0 \neq 0, \quad y_2(x) = \underbrace{C}_{\text{can be } 0} y_1(x) \ln x + \sum_{n=0}^{\infty} d_n x^{n+r_2}, \ d_0 \neq 0.$$

3. $r_1 = r_2$: Two linearly independent solutions can be found:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \ c_0 \neq 0, \quad y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} d_n x^{n+r_2}.$$

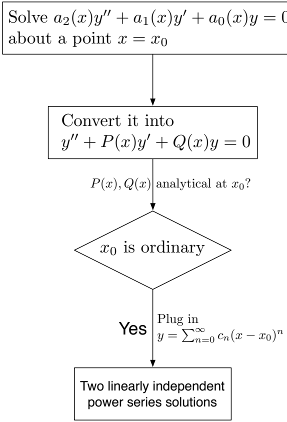
Examples

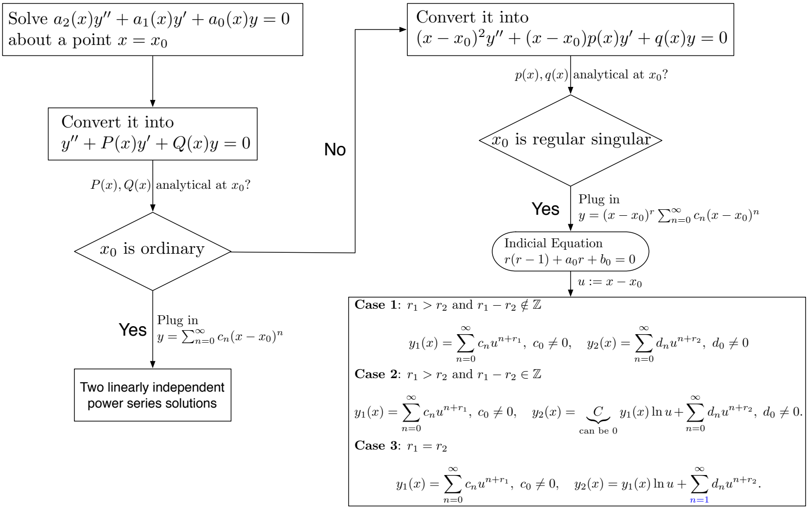
Example

Solve $2xy'' + (1+x)y' + y = 0$.

Example

Solve $xy'' + y = 0$.


Solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ about a point $x = x_0$
Convert it into $y'' + P(x)y' + Q(x)y = 0$
$P(x), Q(x)$ analytical at x_0?
x_0 is ordinary
Yes
Plug in $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$
Two linearly independent power series solutions


Solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ about a point $x = x_0$
Convert it into $y'' + P(x)y' + Q(x)y = 0$
$P(x), Q(x)$ analytical at x_0?
x_0 is ordinary
Yes
Plug in $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$
Two linearly independent power series solutions
No
Convert it into $(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$
$p(x), q(x)$ analytical at x_0?
x_0 is regular singular
Yes
Plug in $y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n$
Indicial Equation $r(r-1) + a_0 r + b_0 = 0$
$u := x - x_0$
Case 1: $r_1 > r_2$ and $r_1 - r_2 \notin \mathbb{Z}$
$y_1(x) = \sum_{n=0}^{\infty} c_n u^{n+r_1},\ c_0 \neq 0, \quad y_2(x) = \sum_{n=0}^{\infty} d_n u^{n+r_2},\ d_0 \neq 0$
Case 2: $r_1 > r_2$ and $r_1 - r_2 \in \mathbb{Z}$
$y_1(x) = \sum_{n=0}^{\infty} c_n u^{n+r_1},\ c_0 \neq 0, \quad y_2(x) = \underbrace{C}_{\text{can be } 0} y_1(x) \ln u + \sum_{n=0}^{\infty} d_n u^{n+r_2},\ d_0 \neq 0.$
Case 3: $r_1 = r_2$
$y_1(x) = \sum_{n=0}^{\infty} c_n u^{n+r_1},\ c_0 \neq 0, \quad y_2(x) = y_1(x) \ln u + \sum_{n=1}^{\infty} d_n u^{n+r_2}.$

Short Recap

- Power Series, Radius of Convergence, Analyticity, Taylor's Series
- Ordinary Points vs. Singular Points
- Power Series Solution, Recursive Formula
- Regular Singular Point vs. Irregular Singular Point
- Generalized Power Series
- Method of Frobenius, Indicial Equation

Self-Practice Exercises

6-1: 1, 7, 13, 15, 19, 23, 25, 29, 35

6-2: 1, 3, 13, 15, 19, 21, 23

6-3: 1, 3, 5, 9, 11, 13, 17, 25, 27, 29, 33