Chapter 5: Modeling with Higher-Order Differential Equations

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1 Linear Models: Initial-Value Problems

2 Nonlinear Models

3 Summary

Modeling with Second Order Linear Differential Equation

We focus on two linear dynamical systems modeled by the following:

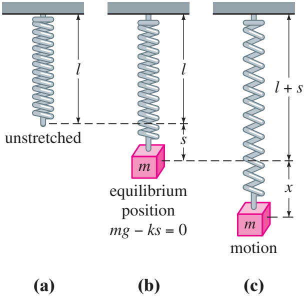
$$ay'' + by' + cy = g(t),\ \ y(0) = y_0,\ \ y'(0) = y_1,$$

where the initial conditions are at time $t = 0$.

The two systems are:

- Spring/Mass Systems
- LRC Series Circuits

Hooke's Law + Newton's Second Law



Assume that the equilibrium position is $x = 0$, and x 向下為正

Due to Hooke's Law, net force $= mg - k(s + x)$.

Note that at equilibrium 淨力為零 $\implies mg = ks$.

Hence, by Newton's Second Law,

$$mx'' = mg - k(s + x) = -kx$$
$$\implies mx'' + kx = 0$$
$$\implies x'' + \frac{k}{m}x = \boxed{x'' + \omega^2 x = 0}$$

where $\omega = \sqrt{k/m}$.

Free Undamped Motion

Solution to $x'' + \omega^2 x = 0$:

$$\boxed{x(t) = c_1 \cos \omega t + c_2 \sin \omega t}.$$

Free: No external force $\iff$ Homogeneous Equation

Undamped: Motion is periodic (period $= \frac{2\pi}{\omega}$), no loss in energy.

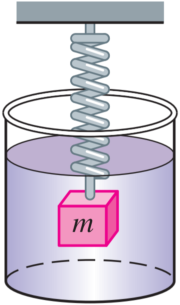
Alternative Representation of $x(t)$:

$$\boxed{x(t) = A \sin(\omega t + \phi)}$$

where

- $A := \sqrt{c_1^2 + c_2^2}$ denotes the **amplitude** of the motion
- $\phi := \tan^{-1} \frac{c_1}{c_2}$ denotes the **initial phase angle**

Free Damped Motion



Assume that the mass is in a surrounding medium with a resisting force proportional to the velocity.

Net force $= mg - k(s+x) - \beta x'$.

Hence, by Newton's Second Law,

$$mx'' = mg - k(s+x) - \beta x' = -kx - \beta x'$$
$$\Longrightarrow \quad x'' + \frac{\beta}{m}x' + \frac{k}{m}x = \boxed{x'' + 2\lambda x' + \omega^2 x = 0}$$

where $\omega = \sqrt{k/m}$ and $\lambda = \beta/2m$.

Solutions of Free Damped Motion

$D^2 + 2\lambda D + \omega^2$ has two roots $-\lambda \pm \sqrt{\lambda^2 - \omega^2}$.

Solution to $x'' + 2\lambda x' + \omega^2 x = 0$:

- **Overdamped** $\lambda^2 > \omega^2$:

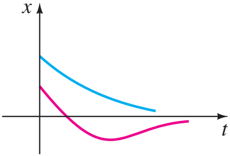
$$\boxed{x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)}$$

- **Critically damped** $\lambda^2 = \omega^2$:

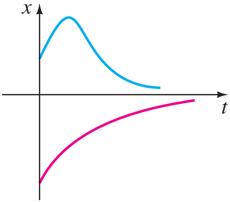
$$\boxed{x(t) = e^{-\lambda t}(c_1 + c_2 t)}$$

- **Underdamped** $\lambda^2 < \omega^2$:

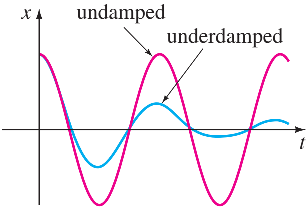
$$\boxed{x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2 - \lambda^2}\,t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\,t\right)}$$



(a) Overdamped

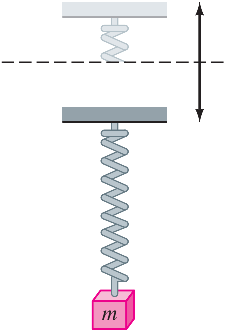


(b) Critically damped



(c) Underdamped

Driven Motion



Assume that the certain external force $f(t)$ is applied to the system. For example, the support is vertically oscillating.

Net force $= mg - k(s+x) - \beta x + f(t)$.

Hence, by Newton's Second Law,

$$mx'' = mg - k(s+x) - \beta x' + f(t) = -kx - \beta x' + f(t)$$

$$\Longrightarrow \quad x'' + \frac{\beta}{m}x' + \frac{k}{m}x = \boxed{x'' + 2\lambda x' + \omega^2 x = F(t)}$$

where $\omega = \sqrt{k/m}$, $\lambda = \beta/2m$, and $F(t) = f(t)/m$.

When $F(t)$ is Periodic

Solve

$$x'' + 2\lambda x' + \omega^2 x = F_0 \sin \gamma t.$$

1. Find the complementary solution:

$$x_c(t) = \begin{cases} e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right), & \lambda^2 > \omega^2 \\ e^{-\lambda t}\left(c_1 + c_2 t\right), & \lambda^2 = \omega^2 \\ e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2-\lambda^2}\,t + c_2 \sin\sqrt{\omega^2-\lambda^2}\,t\right), & \lambda^2 < \omega^2 \end{cases}$$

2. Find a particular solution:

$$x_p(t) = \begin{cases} A\sin\gamma t + B\cos\gamma t, & \lambda \neq 0 \\ A\sin\gamma t + B\cos\gamma t, & \lambda = 0,\ \omega^2 \neq \gamma^2 \\ At\sin\gamma t + Bt\cos\gamma t, & \lambda = 0,\ \omega^2 = \gamma^2 \end{cases}$$

Driven Damped Motion: Steady-State vs. Transient

When $\lambda \neq 0$, it is a **damped** system, and the general solution is $x(t) = x_c(t) + A\sin\gamma t + B\cos\gamma t$, where

$$x_c(t) = e^{-\lambda t}\begin{cases} \left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right), & \lambda^2 > \omega^2 \\ (c_1 + c_2 t), & \lambda^2 = \omega^2 \\ \left(c_1\cos\sqrt{\omega^2-\lambda^2}\,t + c_2\sin\sqrt{\omega^2-\lambda^2}\,t\right), & \lambda^2 < \omega^2 \end{cases}$$

Note that if $\lambda > 0$, $x_c(t) \to 0$ as $t \to \infty$.

$\therefore x(t) \to A\sin\gamma t + B\cos\gamma t$ as $t \to \infty$. Decompose $x(t)$ into two parts:

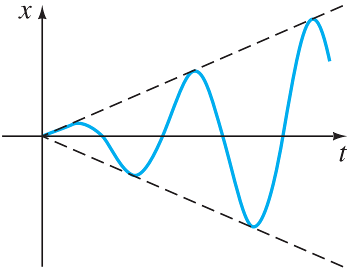
$$x(t) = \underbrace{x_c(t)}_{\textbf{transient}} + \underbrace{A\sin\gamma t + B\cos\gamma t}_{\textbf{steady-state}}$$

Pure Resonance

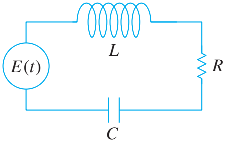
When $\lambda = 0$ and $\omega^2 = \gamma^2$, it is a **undamped** system, and the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + At \sin \omega t + Bt \cos \omega t$$

Note that $x(t) \to \infty$ as $t \to \infty$, which is because of **resonance**.



Series Circuit



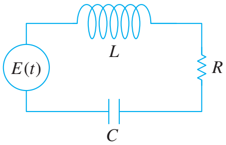
Recall from Chapter 1 that the voltage drop across the three elements are $L\frac{dI}{dt}$, IR, and $\frac{q}{C}$ respectively.

Using the fact that $I = \frac{dq}{dt}$ and Kirchhoff's Law, we have

$$Lq'' + Rq' + q/C = E(t).$$

- **Overdamped** $R^2 > 4L/C$
- **Critically damped** $R^2 = 4L/C$
- **Underdamped** $R^2 < 4L/C$

Steady-State Current



Example

For the external voltage $E(t) = E_0 \sin \gamma t$, find the steady-state current.

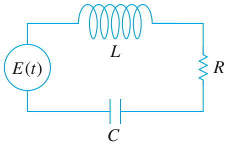
Observation:

1. $E_0 \sin \gamma t = \mathrm{Im}\left\{E_0 e^{i\gamma t}\right\} = \frac{1}{2i}\left(E_0 e^{i\gamma t} - E_0 e^{-i\gamma t}\right)$.
2. We just need to find the particular solution q_p.
3. Superposition principle of nonhomogeneous linear DE: if

   $$q_{p,1} \text{ is a particular solution of } Lq'' + Rq' + q/C = E_0 e^{i\gamma t}$$
   $$q_{p,2} \text{ is a particular solution of } Lq'' + Rq' + q/C = E_0 e^{-i\gamma t}$$

   then $q_p := \frac{1}{2i}\left(q_{p,1} - q_{p,2}\right)$ is a particular solution of the original DE.
4. $q_{p,1}^* = q_{p,2}$ and therefore $q_p := \frac{1}{2i}\left(q_{p,1} - q_{p,2}\right) = \mathrm{Im}\left\{q_{p,1}\right\}$.

Steady-State Current



Example

For the external voltage $E(t) = E_0 \sin \gamma t$, find the steady-state current.

We just need to solve the following:

$$Lq'' + Rq' + q/C = E_0 e^{i\gamma t}.$$

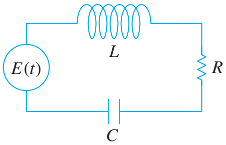
Note that the particular solution take the form $q_s e^{i\gamma t}$. Plug it in we get

$$q_s \left(L(i\gamma)^2 + R(i\gamma) + 1/C \right) = E_0 \implies q_{p,1}(t) = \frac{E_0}{\left(\frac{1}{C} - L\gamma^2 \right) + i\gamma R} e^{i\gamma t}.$$

Hence the steady-state (complex) current

$$I_{p,1}(t) = \frac{E_0}{R + i\left(\gamma L - \frac{1}{\gamma C} \right)} e^{i\gamma t}$$

Steady-State Current



Example

For the external voltage $E(t) = E_0 \sin \gamma t$, find the steady-state current.

Let's further manipulate the steady-state (complex) current

$$I_{p,1}(t) = \frac{E_0}{R + i\left(\gamma L - \frac{1}{\gamma C}\right)} e^{i\gamma t} = \frac{E_0}{R + iX} e^{i\gamma t},$$

where $X := \gamma L - \frac{1}{\gamma C}$ is called the **reactance** of the circuit.

The steady-state (real) current is just the imaginary part of the above:

$$I_p(t) = \operatorname{Im}\left\{I_{p,1}(t)\right\} = \frac{E_0}{R^2 + X^2}\left(R \sin \gamma t - X \cos \gamma t\right) = \frac{E_0}{Z} \sin\left(\gamma t - \phi\right),$$

where $Z := \sqrt{R^2 + X^2}$ is called the **impedance** of the circuit, $\phi = \tan^{-1} \frac{X}{R}$.

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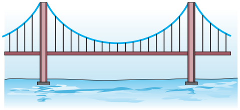
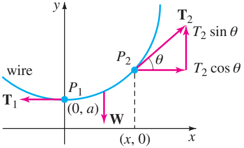
Suspended Cable





Consider a suspended cable with the weight per unit length $= \rho$. We would like to find the shape of the cable, that is, the function $y(x)$.

At a point (x, y) of the suspended cable, we have

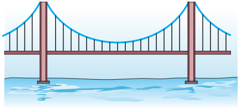
$$\begin{cases} T_1 = T_2 \cos\theta, & \text{Horizontal Net Force} = 0 \\ W = \rho s = T_2 \sin\theta, & \text{Vertical Net Force} = 0 \end{cases}.$$

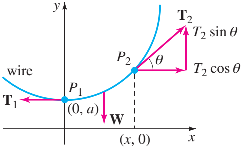
Here $s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ is the total cable length between $(0, a)$ and (x, y).

Since $\frac{dy}{dx} = \tan\theta = \frac{\rho s}{T_1}$, we get

$$\frac{dy}{dx} = \frac{W}{T_1} = \frac{\rho}{T_1} \int_0^x \sqrt{1 + \frac{dy}{dx}^2}\, dx \implies \boxed{y'' = \frac{\rho}{T_1}\sqrt{1 + (y')^2}}.$$

Suspended Cable





Solve $y'' = \frac{\rho}{T_1}\sqrt{1+(y')^2}$ (dependent variable y missing) by substituting $u := y'$:

$$u' = \frac{\rho}{T_1}\sqrt{1+u^2} \implies \int \frac{du}{\sqrt{1+u^2}} = \int \frac{\rho}{T_1}\,dx$$

$$\implies \sinh^{-1} u = \frac{\rho}{T_1}x + c_1.$$

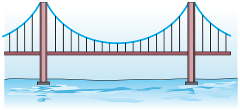
Since $u(0) = y'(0) = 0$, we have $c_1 = 0$. Therefore,

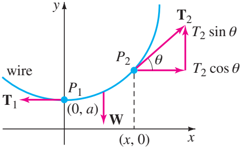
$$u = y' = \sinh\left(\frac{\rho}{T_1}x\right) \implies y = \frac{T_1}{\rho}\cosh\left(\frac{\rho}{T_1}x\right) + c_2.$$

Since $y(0) = a$, we have $c_2 = a - \frac{T_1}{\rho}$. Hence,

$$\boxed{y(x) = a + \frac{T_1}{\rho}\left\{\cosh\left(\frac{\rho}{T_1}x\right) - 1\right\}}.$$

Suspended Cable





You can also solve $y'' = \frac{\rho}{T_1}\sqrt{1+(y')^2}$ (independent variable x missing) by substituting $u := y'$:

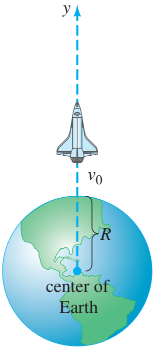
$$u\frac{du}{dy} = \frac{\rho}{T_1}\sqrt{1+u^2} \implies \int \frac{u\,du}{\sqrt{1+u^2}} = \int \frac{\rho}{T_1}\,dy$$

$$\implies \sqrt{1+u^2} = \frac{\rho}{T_1}y + c_1.$$

Since $u(0) = y'(0) = 0$, $y(0) = a$, we have $c_1 = 1 - \frac{\rho}{T_1}a$.

$$\vdots$$

Escape Velocity of a Rocket



Consider a rocket (mass $= m$) launched vertically from the ground. Ignore air resistance. When its fuel is used up, the distance from the center of Earth is $y_0 \approx R$, the radius of Earth, and the velocity is v_0.

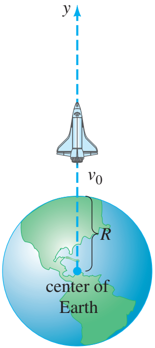
We would like to learn how large v_0 the motion of the rocket after its fuel is used up. By Newton's law of universal gravitation, we have: ($M :=$ the mass of Earth, $R :=$ the radius of Earth, $G :=$ the gravitational constant)

$$my'' = -\frac{GMm}{y^2} \implies y'' = -\frac{GM}{y^2}.$$

Since near the surface of Earth, $mg = \frac{GMm}{R^2} \implies GM = gR^2$, we have

$$\boxed{y'' = -\frac{gR^2}{y^2}}.$$

Escape Velocity of a Rocket



Again, solve $y'' = -\frac{gR^2}{y^2}$ (independent variable x missing) by substituting $v := y'$:

$$v\frac{dv}{dy} = -gR^2 y^{-2} \implies vdv = -gR^2 y^{-2}\,dy$$

$$\implies v^2 = \frac{2gR^2}{y} + c_1$$

Since $v(0) = v_0$, $y(0) = y_0 \approx R$, we have $c_1 = v_0^2 - 2gR$. Hence,

$$v^2 = v_0^2 + \frac{2gR^2}{y} - 2gR.$$

In order to reach $y = \infty$, we require $\boxed{v_0 > \sqrt{2gR}}$.

Short Recap

- Free vs. Driven Motion $\Longleftrightarrow$ Homogeneous vs. Nonhomogeneous Linear DE
- Overdamped, Critically Damped, Underdamped, Undamped
- Transient vs. Steady State
- Nonlinear Models

Self-Practice Exercises

5-1: 1, 7, 13, 21, 35, 49, 57

5-3: 7, 15, 17, 19