Chapter 4: Higher-Order Differential Equations – Part 3

王奕翔

Department of Electrical Engineering
National Taiwan University

ihwang@ntu.edu.tw

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Systems of Linear Differential Equations

So far we focus on solving an ODE with only **one** dependent variable.

Ordinary Differential Equation $\implies$ 1 independent variable.

Partial Differential Equation $\implies$ > 1 independent variables.

Here we look at a system of linear ODE (> 1 dependent variables), and see how to solve it

Example: Let t be the independent variable, x, y be dependent variables.

$$\begin{cases} x'' + 2x' + y'' = x + 3y + \sin t \\ x' + y' = -4x + 2y + e^{-t} \end{cases} .$$

How to Solve?

$$\begin{cases} x'' + 2x' + y'' = x + 3y + \sin t \\ x' + y' = -4x + 2y + e^{-t} \end{cases}.$$

Let us use the differential operator to rewrite it as follows:

$$\begin{cases} D^2\{x\} + 2D\{x\} + D^2\{y\} = x + 3y + \sin t \\ D\{x\} + D\{y\} = -4x + 2y + e^{-t} \end{cases}$$

$$\implies \begin{cases} \left(D^2 + 2D - 1\right)\{x\} + \left(D^2 - 3\right)\{y\} & = \sin t \\ (D + 4)\{x\} + (D - 2)\{y\} & = e^{-t} \end{cases}$$

Idea:

- Eliminate y and get a linear DE of x to solve $x(t)$.
- Eliminate x and get a linear DE of y to solve $y(t)$.

Elimination

$$\begin{cases} L_{11}\{x\} + L_{12}\{y\} = g_1(t) & \text{(1a)} \\ L_{21}\{x\} + L_{22}\{y\} = g_2(t) & \text{(1b)} \end{cases}$$

where $L_{11} = D^2 + 2D - 1,\ L_{12} = D^2 - 3,\ L_{21} = D + 4,\ L_{22} = D - 2.$

To eliminate y:

$$\text{(1a)} \times L_{22} - \text{(1b)} \times L_{12} \implies (L_{11}L_{22} - L_{21}L_{12})\{x\} = L_{22}\{g_1\} - L_{12}\{g_2\}$$

To eliminate x:

$$\text{(1a)} \times L_{21} - \text{(1b)} \times L_{11} \implies (L_{12}L_{21} - L_{22}L_{11})\{y\} = L_{21}\{g_1\} - L_{11}\{g_2\}$$

$$\begin{cases} L_{11}\{x\} + L_{12}\{y\} = g_1(t) & \text{(1a)} \\ L_{21}\{x\} + L_{22}\{y\} = g_2(t) & \text{(1b)} \end{cases}$$

Similar to the Cramer's Rule, after the elimination we get

$$\begin{cases} L\{x\} = \widetilde{g}_1(t) & \text{(2a)} \\ L\{y\} = \widetilde{g}_2(t) & \text{(2b)} \end{cases}$$

where $L = \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix}$, $\widetilde{g}_1(t) = \begin{vmatrix} g_1(t) & L_{12} \\ g_2(t) & L_{22} \end{vmatrix}$, $\widetilde{g}_2(t) = \begin{vmatrix} L_{11} & g_1(t) \\ L_{21} & g_2(t) \end{vmatrix}$.

$$\text{Solutions of (1)} \underset{\not\Longleftarrow}{\Longrightarrow} \text{Solutions of (2)}$$

Note: We should always plug the solutions found in solving (2) back to (1) and find out all additional constraints.

Solving a System of Linear DE with Constant Coefficients

1. Convert it into the following form:

$$\begin{cases} L_{11}\{y_1\} & + & L_{12}\{y_2\} & + & \cdots & + & L_{1k}\{y_k\} & = & g_1(t) \\ L_{21}\{y_1\} & + & L_{22}\{y_2\} & + & \cdots & + & L_{2k}\{y_k\} & = & g_2(t) \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ L_{k1}\{y_1\} & + & L_{k2}\{y_2\} & + & \cdots & + & L_{kk}\{y_k\} & = & g_k(t) \end{cases}$$

2. Use Cramer's rule to get $L\{y_j\} = \widetilde{g}_j(t)$, $j = 1, \ldots, k$, where

$$L = \begin{vmatrix} L_{11} & L_{12} & \cdots & L_{1k} \\ L_{21} & L_{22} & \cdots & L_{2k} \\ \vdots & \vdots & & \vdots \\ L_{k1} & L_{k2} & \cdots & L_{kk} \end{vmatrix}, \quad \widetilde{g}_j(t) = L\Big|_{j\text{-th column replaced by } \begin{bmatrix} g_1 & \cdots & g_k \end{bmatrix}^T}$$

3. Solve each $y_j(t)$, $j = 1, \ldots, k$.
4. Plug into the initial system, find additional constraints on the coefficients in the complimentary solutions $\{y_{1c}, y_{2c}, \ldots, y_{kc}\}$, and finalize.

Notes and Tips

- It is very important to plug the general solutions found for $\{y_1, y_2, \ldots, y_k\}$ back to the original system of equations to find additional constraints on the coefficients in the complimentary solutions $\{y_{1c}, y_{2c}, \ldots, y_{kc}\}$ (see example).
- When solving $L\{y_j\} = \widetilde{g}_j(t)$, sometimes we can eliminate redundant operators (see example).
- When $k = 2$, that is, only two dependent variables to be solved, after solving one dependent variable, it may save some time if we plug the general solution we found back to the original system and find the solution of the other dependent variable (see example).

Example

Solve $\begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$

A: Rewrite it as

$$\begin{cases} \left(D^2 - 4\right)\{x\} + D^2\{y\} = t^2 & \text{(3a)} \\ (D+1)\{x\} + D\{y\} = 0 & \text{(3b)} \end{cases}$$

Based on the method mentioned above, we compute

$$L = \begin{vmatrix} D^2 - 4 & D^2 \\ D+1 & D \end{vmatrix} = -(D^2 + 4D) = -D(D+4)$$

$$\widetilde{g}_1(t) = \begin{vmatrix} t^2 & D^2 \\ 0 & D \end{vmatrix} = D\left\{t^2\right\}, \quad \widetilde{g}_2(t) = \begin{vmatrix} D^2 - 4 & t^2 \\ D+1 & 0 \end{vmatrix} = -(D+1)\left\{t^2\right\}$$

Example: Convert into two separate linear equations

$$\text{Solve} \quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$$

Hence, solutions of the original system of equations must be solutions of the the following:

$$\begin{cases} L\{x\} = \widetilde{g}_1(t) \iff (-\cancel{D}(D+4))\{x\} = \cancel{D}\{t^2\} \\ L\{y\} = \widetilde{g}_2(t) \iff (-D(D+4))\{y\} = -(D+1)\{t^2\} \end{cases}$$

Question: why can we cancel the repeated operators on both sides?

Ans: because instead of eliminating y by $D\{(3a)\} - D^2\{(3b)\}$, we can simply eliminate y by $(3a) - D\{(3b)\}$.

Solution 1: Solving x and y separately

Solve $\begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$

We convert the above into

$$\begin{cases} (D+4)\{x\} = -t^2 & \text{(4a)} \\ (D(D+4))\{y\} = t^2 + 2t & \text{(4b)} \end{cases}$$

Step 1. Solve (5a): $x_c = c_1 e^{-4t}$, $x_p = A + Bt + Ct^2$, and

$$-t^2 = (D+4)\{x_p\} = (4A+B) + (4B+2C)t + 4Ct^2$$
$$\implies C = \frac{-1}{4}, B = \frac{-1}{2}C = \frac{1}{8}, A = \frac{-1}{4}B = \frac{-1}{32}.$$

Hence $\boxed{x(t) = c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2}$.

Solution 1: Solving x and y separately

$$\text{Solve} \quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$$

We convert the above into

$$\begin{cases} (D+4)\{x\} = -t^2 & \text{(5a)} \\ (D(D+4))\{y\} = t^2 + 2t & \text{(5b)} \end{cases}$$

Step 2. Solve (5b): $y_c = c_2 + c_3 e^{-4t}$, $y_p = At + Bt^2 + Ct^3$, and

$$t^2 + 2t = (D^2 + 4D)\{y_p\} = (4A + 2B) + (8B + 6C)t + 12Ct^2$$

$$\implies C = \frac{1}{12}, B = \frac{1}{4} - \frac{3}{4}C = \frac{3}{16}, A = \frac{-1}{2}B = \frac{-3}{32}.$$

Hence $\boxed{y(t) = c_2 + c_3 e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3}$.

Solution 1: Solving x and y separately

Solve $\begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$

We find that for some c_1, c_2, c_3,

$$\begin{cases} x(t) = & c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8} t - \frac{1}{4} t^2 \\ y(t) = & c_2 + c_3 e^{-4t} - \frac{3}{32} t + \frac{3}{16} t^2 + \frac{1}{12} t^3 \end{cases}$$

Final Step. Plug them back to find the constraints on $\{c_1, c_2, c_3\}$.

$$\begin{cases} (12c_1 + 16c_3)e^{-4t} + t^2 = t^2 \\ -(3c_1 + 4c_3)e^{-4t} = 0 \end{cases}$$

which implies $\boxed{c_3 = -3/4c_1}$. Hence, the final solution is

$$\begin{cases} x(t) = & c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8} t - \frac{1}{4} t^2 \\ y(t) = & c_2 - \frac{3}{4} c_1 e^{-4t} - \frac{3}{32} t + \frac{3}{16} t^2 + \frac{1}{12} t^3 \end{cases}$$

Solution 2: Solving x first and then plugging in to find y

Solve $\begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$

After Step 1., we find that for some c_1, $x(t) = c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2$.

Plug this back to the second equation, we get

$$y' = -x' - x = 3c_1 e^{-4t} - \frac{3}{32} + \frac{3}{8}t + \frac{1}{4}t^2$$
$$\Longrightarrow y = c_2 - \frac{3}{4}c_1 e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3$$

Plug it back to the first equation, it is also satisfied. Done!

Linear vs. Nonlinear Differential Equations

Differences:

1. Superposition principle does not hold for nonlinear DE.
2. Nonlinear DE can have singular solutions, while linear DE will not have singular solutions.
3. Usually there is no analytical tool to solve a nonlinear DE of higher order.

We will present some special kinds of nonlinear second order DE that can be solved analytically.

Second Order DE: Reduction of Order

A second order DE can be written in the general form $F(x, y, y', y'') = 0$, where the dependent variable is y and the independent variable is x.

In the following two cases, we are able to use the substitution $u := y'$ to reduce the order of the equation, and get a first order DE of u.

1. Dependent variable missing: since $y' = u$, $y'' = \frac{du}{dx}$

$$F\left(x, y', y''\right) = 0 \implies \boxed{F\left(x, u, \frac{du}{dx}\right) = 0}.$$

   The dependent variable is u and the independent variable is x.

2. Independent variable missing: since $y' = u$, $y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$

$$F\left(y, y', y''\right) = 0 \implies \boxed{F\left(y, u, u\frac{du}{dy}\right) = 0}.$$

   The dependent variable is u and the independent variable is y.

Example: Dependent Variable Missing

Example

Solve $y'' = 2x(y')^2$.

A: Set $u = \frac{dy}{dx}$, and we get a new equation of u:

$$\frac{du}{dx} = 2xu^2.$$

The above nonlinear first order DE can be solved by separation of variables:

$$\frac{du}{u^2} = 2xdx \implies -\frac{1}{u} = x^2 + c \implies u = \frac{-1}{x^2 + c}.$$

Besides, $u = 0$ is a singular solution.

Example: Dependent Variable Missing

Example

Solve $y'' = 2x(y')^2$.

Case 0: $u = \frac{dy}{dx} = 0 \implies \boxed{y = c_2}$, $c_2 \in \mathbb{R}$.

Case 1: $u = \frac{dy}{dx} = \frac{-1}{x^2+c}$ where $c > 0$. Let $c := c_1^2$, $c_1 > 0$, then

$$\frac{dy}{dx} = \frac{-1}{x^2 + c_1^2} \implies \boxed{y = c_2 - \frac{1}{c_1}\tan^{-1}\frac{x}{c_1}}.$$

Case 2: $u = \frac{dy}{dx} = \frac{-1}{x^2+c}$ where $c < 0$. Let $c := -c_1^2$, $c_1 > 0$, then

$$\frac{dy}{dx} = \frac{-1}{x^2 - c_1^2} = \frac{1}{2c_1}\left(\frac{1}{x+c_1} - \frac{1}{x-c_1}\right) \implies \boxed{y = c_2 + \frac{1}{2c_1}\ln\left|\frac{x+c_1}{x-c_1}\right|}.$$

Case 3: $u = \frac{dy}{dx} = \frac{-1}{x^2+c}$ where $c = 0$. Then $\frac{dy}{dx} = \frac{-1}{x^2} \implies \boxed{y = c_2 + x^{-1}}$.

Example: Independent Variable Missing

Example

Solve $yy'' = (y')^2$.

A: Set $u = \frac{dy}{dx}$, and we get a new equation of u using $\frac{d^2y}{dx^2} = u\frac{du}{dy}$:

$$yu\frac{du}{dy} = u^2.$$

The above nonlinear first order DE can be solved by separation of variables:

$$\frac{du}{u} = \frac{dy}{y} \implies \ln|u| = \ln|y| + c \implies u = (\pm e^c)y.$$

Incorporating the singular solution $u = 0$, we have $u = c_1 y$, $c_1 \in \mathbb{R}$.

Example: Independent Variable Missing

Example

Solve $yy'' = (y')^2$.

Case 1: $c_1 = 0$:

$$u = \frac{dy}{dx} = 0 \implies y = c_2, \ \ c_2 \in \mathbb{R}.$$

Case 2: $c_1 \neq 0$:

$$u = \frac{dy}{dx} = c_1 y \implies \frac{dy}{y} = c_1 \, dx \implies y = c_2 e^{c_1 x}, \ \ c_2 \in \mathbb{R}.$$

Incorporating Case 1 and Case 2, we get $\boxed{y = c_2 e^{c_1 x}}$, $c_1, c_2 \in \mathbb{R}$.

Short Recap

- Solving systems of linear DEs by systematic elimination
- In solving systems of linear DEs, remember:
  代回原式找出complimentary solutions中係數的關係
- Solving a nonlinear second order DE missing the dependent or the independent variable by reduction of order with the substitution $u = y'$.

Self-Practice Exercises

4-9: 1, 7, 11, 15, 19, 21

4-10: 3, 5, 7, 9, 13, 19