Chapter 4: Higher-Order Differential Equations – Part 3

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1 Solving Systems of Linear Differential Equations

2 Summary

Systems of Linear Differential Equations

So far we focus on solving an ODE with only **one** dependent variable.

Ordinary Differential Equation $\implies$ 1 independent variable.

Partial Differential Equation $\implies$ > 1 independent variables.

Here we look at a system of linear ODE (> 1 dependent variables), and see how to solve it

Example: Let t be the independent variable, x, y be dependent variables.

$$\begin{cases} x'' + 2x' + y'' = x + 3y + \sin t \\ x' + y' = -4x + 2y + e^{-t} \end{cases}.$$

How to Solve?

$$\begin{cases} x'' + 2x' + y'' = x + 3y + \sin t \\ x' + y' = -4x + 2y + e^{-t} \end{cases} .$$

Let us use the differential operator to rewrite it as follows:

$$\begin{cases} D^2\{x\} + 2D\{x\} + D^2\{y\} = x + 3y + \sin t \\ D\{x\} + D\{y\} = -4x + 2y + e^{-t} \end{cases}$$

$$\Longrightarrow \begin{cases} \left(D^2 + 2D - 1\right)\{x\} + \left(D^2 - 3\right)\{y\} & = \sin t \\ (D+4)\{x\} + (D-2)\{y\} & = e^{-t} \end{cases}$$

Idea:

- Eliminate y and get a linear DE of x to solve $x(t)$.
- Eliminate x and get a linear DE of y to solve $y(t)$.

Elimination

$$\begin{cases} L_{11}\{x\} + L_{12}\{y\} = g_1(t) & \text{(1a)} \\ L_{21}\{x\} + L_{22}\{y\} = g_2(t) & \text{(1b)} \end{cases}$$

where $L_{11} = D^2 + 2D - 1,\ L_{12} = D^2 - 3,\ L_{21} = D + 4,\ L_{22} = D - 2.$

To eliminate y:

$$\text{(1a)} \times L_{22} - \text{(1b)} \times L_{12} \implies (L_{11}L_{22} - L_{21}L_{12})\{x\} = L_{22}\{g_1\} - L_{12}\{g_2\}$$

To eliminate x:

$$\text{(1a)} \times L_{21} - \text{(1b)} \times L_{11} \implies (L_{12}L_{21} - L_{22}L_{11})\{x\} = L_{21}\{g_1\} - L_{11}\{g_2\}$$

$$\begin{cases} L_{11}\{x\} + L_{12}\{y\} = g_1(t) & \text{(1a)} \\ L_{21}\{x\} + L_{22}\{y\} = g_2(t) & \text{(1b)} \end{cases}$$

Similar to the Cramer's Rule, after the elimination we get

$$\begin{cases} L\{x\} = \widetilde{g}_1(t) & \text{(2a)} \\ L\{y\} = \widetilde{g}_2(t) & \text{(2b)} \end{cases}$$

where $L = \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix}, \quad \widetilde{g}_1(t) = \begin{vmatrix} g_1(t) & L_{12} \\ g_2(t) & L_{22} \end{vmatrix}, \quad \widetilde{g}_2(t) = \begin{vmatrix} L_{11} & g_1(t) \\ L_{21} & g_2(t) \end{vmatrix}.$

$$\text{Solutions of (1)} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \text{Solutions of (2)}$$

Note: We should always plug the solutions found in solving (2) back to (1) and find out all additional constraints.

Solving a System of Linear DE with Constant Coefficients

1. Convert it into the following form:

$$\begin{cases} L_{11}\{y_1\} + L_{12}\{y_2\} + \cdots + L_{1k}\{y_k\} = g_1(t) \\ L_{21}\{y_1\} + L_{22}\{y_2\} + \cdots + L_{2k}\{y_k\} = g_2(t) \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ L_{k1}\{y_1\} + L_{k2}\{y_2\} + \cdots + L_{kk}\{y_k\} = g_k(t) \end{cases}$$

2. Use Cramer's rule to get $L\{y_j\} = \widetilde{g}_j(t)$, $j = 1, \ldots, k$, where

$$L = \begin{vmatrix} L_{11} & L_{12} & \cdots & L_{1k} \\ L_{21} & L_{22} & \cdots & L_{2k} \\ \vdots & \vdots & & \vdots \\ L_{k1} & L_{k2} & \cdots & L_{kk} \end{vmatrix}, \quad \widetilde{g}_j(t) = L\big|_{j\text{-th column replaced by } \begin{bmatrix} g_1 & \cdots & g_k \end{bmatrix}^T}$$

3. Solve each $y_j(t)$, $j = 1, \ldots, k$.
4. Plug into the initial system, find additional constraints on the coefficients in the complimentary solutions $\{y_{1c}, y_{2c}, \ldots, y_{kc}\}$, and finalize.

Notes and Tips

- It is very important to plug the general solutions found for $\{y_1, y_2, \ldots, y_k\}$ back to the original system of equations to find additional constraints on the coefficients in the complimentary solutions $\{y_{1c}, y_{2c}, \ldots, y_{kc}\}$ (see example).
- When solving $L\{y_j\} = \widetilde{g}_j(t)$, sometimes we can eliminate redundant operators (see example).
- When $k = 2$, that is, only two dependent variables to be solved, after solving one dependent variable, it may save some time if we plug the general solution we found back to the original system and find the solution of the other dependent variable (see example).

Example

$$\boxed{\text{Solve} \quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}}$$

A: Rewrite it as

$$\begin{cases} \left(D^2 - 4\right)\{x\} + D^2\{y\} = t^2 & \text{(3a)} \\ (D+1)\{x\} + D\{y\} = 0 & \text{(3b)} \end{cases}$$

Based on the method mentioned above, we compute

$$L = \begin{vmatrix} D^2 - 4 & D^2 \\ D+1 & D \end{vmatrix} = -(D^2 + 4D) = -D(D+4)$$

$$\widetilde{g}_1(t) = \begin{vmatrix} t^2 & D^2 \\ 0 & D \end{vmatrix} = D\left\{t^2\right\}, \quad \widetilde{g}_2(t) = \begin{vmatrix} D^2 - 4 & t^2 \\ D+1 & 0 \end{vmatrix} = -(D+1)\left\{t^2\right\}$$

Example: Convert into two separate linear equations

$$\boxed{\text{Solve} \quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}}$$

Hence, solutions of the original system of equations must be solutions of the the following:

$$\begin{cases} L\{x\} = \widetilde{g}_1(t) \iff \left(-\cancel{D}(D+4)\right)\{x\} = \cancel{D}\left\{t^2\right\} \\ L\{y\} = \widetilde{g}_2(t) \iff (-D(D+4))\{y\} = -(D+1)\left\{t^2\right\} \end{cases}$$

Question: why can we cancel the repeated operators on both sides?

Ans: because instead of eliminating y by $D\{(3a)\} - D^2\{(3b)\}$, we can simply eliminate y by $(3a) - D\{(3b)\}$.

Solution 1: Solving x and y separately

$$\boxed{\text{Solve} \quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}}$$

We convert the above into

$$\begin{cases} (D+4)\{x\} = -t^2 & \text{(4a)} \\ (D(D+4))\{y\} = t^2 + 2t & \text{(4b)} \end{cases}$$

Step 1. Solve (5a): $x_c = c_1 e^{-4t}$, $x_p = A + Bt + Ct^2$, and

$$-t^2 = (D+4)\{x_p\} = (4A+B) + (4B+2C)t + 4Ct^2$$
$$\implies C = \frac{-1}{4}, B = \frac{-1}{2}C = \frac{1}{8}, A = \frac{-1}{4}B = \frac{-1}{32}.$$

Hence $\boxed{x(t) = c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2}$.

Solution 1: Solving x and y separately

$$\boxed{\text{Solve} \quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}}$$

We convert the above into

$$\begin{cases} (D+4)\{x\} = -t^2 & \text{(5a)} \\ (D(D+4))\{y\} = t^2 + 2t & \text{(5b)} \end{cases}$$

Step 2. Solve (5b): $y_c = c_2 + c_3 e^{-4t}$, $y_p = At + Bt^2 + Ct^3$, and

$$t^2 + 2t = (D^2 + 4D)\{y_p\} = (4A + 2B) + (8B + 6C)t + 12Ct^2$$

$$\implies C = \frac{1}{12}, B = \frac{1}{4} - \frac{3}{4}C = \frac{3}{16}, A = \frac{-1}{2}B = \frac{-3}{32}.$$

Hence $\boxed{y(t) = c_2 + c_3 e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3}$.

Solution 1: Solving x and y separately

$$\boxed{\text{Solve}\quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}}$$

We find that for some c_1, c_2, c_3,

$$\begin{cases} x(t) = & c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2 \\ y(t) = & c_2 + c_3 e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3 \end{cases}$$

Final Step. Plug them back to find the constraints on $\{c_1, c_2, c_3\}$.

$$\begin{cases} (12c_1 + 16c_3)e^{-4t} + t^2 = t^2 \\ -(3c_1 + 4c_3)e^{-4t} = 0 \end{cases}$$

which implies $\boxed{c_3 = -3/4c_1}$. Hence, the final solution is

$$\begin{cases} x(t) = & c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2 \\ y(t) = & c_2 - \frac{3}{4}c_1 e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3 \end{cases}$$

Solution 2: Solving x first and then plugging in to find y

$$\boxed{\text{Solve} \quad \begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}}$$

After Step 1., we find that for some c_1, $x(t) = c_1 e^{-4t} - \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2$.

Plug this back to the second equation, we get

$$\begin{aligned} y' &= -x' - x = 3c_1 e^{-4t} - \frac{3}{32} + \frac{3}{8}t + \frac{1}{4}t^2 \\ \implies y &= c_2 - \frac{3}{4}c_1 e^{-4t} - \frac{3}{32}t + \frac{3}{16}t^2 + \frac{1}{12}t^3 \end{aligned}$$

Plug it back to the first equation, it is also satisfied. Done!

Short Recap

- Solving Systems of Linear DEs by Systematic Elimination
- 代回原式以找出complimentary solutions中係數的關係

Self-Practice Exercises

4-9: 1, 7, 11, 15, 19, 21