Chapter 4: Higher-Order Differential Equations – Part 3

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Systems of Linear Differential Equations

So far we focus on solving an ODE with only **one** dependent variable.

Ordinary Differential Equation $\implies$ 1 independent variable.

Partial Differential Equation $\implies$ > 1 independent variables.

Here we look at a system of linear ODE (> 1 dependent variables), and see how to solve it

Example: Let t be the independent variable, x, y be dependent variables.

$$\begin{cases} x'' + 2x' + y'' = x + 3y + \sin t \\ x' + y' = -4x + 2y + e^{-t} \end{cases}.$$

How to Solve?

$$\begin{cases} x'' + 2x' + y'' = x + 3y + \sin t \\ x' + y' = -4x + 2y + e^{-t} \end{cases}.$$

Let us use the differential operator to rewrite it as follows:

$$\begin{cases} D^2\{x\} + 2D\{x\} + D^2\{y\} = x + 3y + \sin t \\ D\{x\} + D\{y\} = -4x + 2y + e^{-t} \end{cases}$$

$$\implies \begin{cases} \left(D^2 + 2D - 1\right)\{x\} + \left(D^2 - 3\right)\{y\} & = \sin t \\ (D+4)\{x\} + (D-2)\{y\} & = e^{-t} \end{cases}$$

Idea:

- Eliminate y and get a linear DE of x to solve $x(t)$.
- Eliminate x and get a linear DE of y to solve $y(t)$.

Elimination

$$\begin{cases} L_{11}\{x\} + L_{12}\{y\} = g_1(t) & (1) \\ L_{21}\{x\} + L_{22}\{y\} = g_2(t) & (2) \end{cases}$$

where $L_{11} = D^2 + 2D - 1,\ L_{12} = D^2 - 3,\ L_{21} = D + 4,\ L_{22} = D - 2.$

To eliminate y:

$$(1) \times L_{22} - (2) \times L_{12} \implies (L_{11}L_{22} - L_{21}L_{12})\{x\} = L_{22}\{g_1\} - L_{12}\{g_2\}$$

To eliminate x:

$$(1) \times L_{21} - (2) \times L_{11} \implies (L_{12}L_{21} - L_{22}L_{11})\{x\} = L_{21}\{g_1\} - L_{11}\{g_2\}$$

A Simple Formula

$$\begin{cases} L_{11}\{x\} + L_{12}\{y\} = g_1(t) & (1) \\ L_{21}\{x\} + L_{22}\{y\} = g_2(t) & (2) \end{cases}$$

Similar to the Cramer's Rule, after the elimination we get

$$\begin{cases} L\{x\} = \widetilde{g}_1(t) \\ L\{y\} = \widetilde{g}_2(t) \end{cases}$$

where

$$L = \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix}, \quad \widetilde{g}_1(t) = \begin{vmatrix} g_1(t) & L_{12} \\ g_2(t) & L_{22} \end{vmatrix}, \quad \widetilde{g}_2(t) = \begin{vmatrix} L_{11} & g_1(t) \\ L_{21} & g_2(t) \end{vmatrix}.$$

General Procedure

To solve a system of linear differential equations with constant coefficients (k dependent variables $y_1, y_2, \ldots, y_k$, k equations):

1. Convert it into the following form:

$$\begin{cases} L_{11}\{y_1\} + L_{12}\{y_2\} + \cdots + L_{1k}\{y_k\} = g_1(t) \\ L_{21}\{y_1\} + L_{22}\{y_2\} + \cdots + L_{2k}\{y_k\} = g_2(t) \\ \vdots \\ L_{k1}\{y_1\} + L_{k2}\{y_2\} + \cdots + L_{kk}\{y_k\} = g_k(t) \end{cases}$$

2. Use Cramer's rule to get $L\{y_j\} = \widetilde{g}_j(t)$, $j = 1, \ldots, k$, where

$$L = \begin{vmatrix} L_{11} & L_{12} & \cdots & L_{1k} \\ L_{21} & L_{22} & \cdots & L_{2k} \\ \vdots & \vdots & & \vdots \\ L_{k1} & L_{k2} & \cdots & L_{kk} \end{vmatrix}, \quad \widetilde{g}_j(t) = L\Big|_{j\text{-th column replaced by } \begin{bmatrix} g_1 & \cdots & g_k \end{bmatrix}^T}$$

3. Solve each $y_j(t)$, $j = 1, \ldots, k$.

Some Properties

- For finding the complimentary solutions of all the dependent variables $y_1, \ldots, y_k$, only need to solve the following once:

$$L\{y\} = 0.$$

- Need k equations to uniquely solve k dependent variables $y_1, \ldots, y_k$.
- The degree of L is typically higher than the degree in each of the original system of equations.

Example

Example

Solve

$$\begin{cases} x'' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$$