Chapter 4: Higher-Order Differential Equations – Part 2

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Solving Linear Higher-Order Differential Equations

The steps to find the general solution of a linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

1. Find the general solution of its homogeneous counterpart ($g(x) = 0$):

   $$y_c = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \;\; c_i \in \mathbb{R}, \;\forall\; i = 1, 2, \ldots, n.$$

   Here $\{f_1, f_2, \ldots, f_n\}$ is a fundamental set of solutions.

2. Find a particular solution y_p such that it satisfies (1).

3. The general solution of (1) is

   $$y = y_c + y_p = y_p + c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \;\; c_i \in \mathbb{R}, \;\forall\; i.$$

Types of Equations and Methods to be Covered

Regarding how to find the general solutions of homogeneous linear DE, we have discussed two types of equations so far:

- Linear equations with constant coefficients
- Cauchy-Euler equation

Regarding how to find a particular solution of a nonhomogeneous linear DE, we shall focus on these two kinds as well in this lecture.

We will present two methods of finding a particular solution

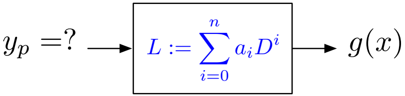
- Undetermined Coefficients (4-4, 4-5)
- Variation of Parameters (4-6)

A System-Level Picture

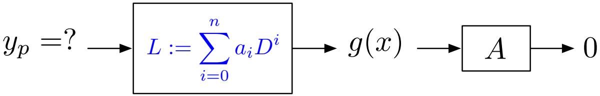
Consider the equation with constant coefficients

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x). \qquad (2)$$

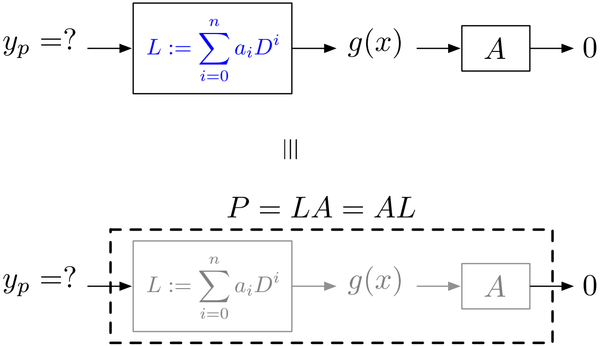
Pictorially sketch the problem of finding a particular solution y_p:



If there exists $A(D)$ (polynomial of D) which is a differential operator such that $A\{g(x)\} = 0$, let us concatenate it as follows:



A System-Level Picture

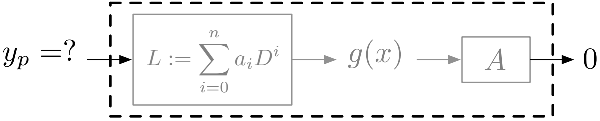


Hence, a particular solution y_p of (2), that is, $L\{y_p\} = g(x)$, must also be a solution of the homogeneous linear equation with constant coefficients

$$P\{y\} = 0.$$

The High-Level Idea

$$P = LA = AL$$



Suppose the degree of the polynomial $A(D)$ is m, while the degree of the polynomial $L(D)$ is n.

Let $\mathcal{P}$ be a fundamental set of solutions of $P\{y\} = 0$, and a subset $\mathcal{L} \subseteq \mathcal{P}$ be a fundamental set of solutions of $L\{y\} = 0$.

Since y_p is a solution of $P\{y\} = 0$, it can be written as follows:

$$y_p = \overbrace{\sum_{i:f_i \in \mathcal{L}} c_i f_i(x)}^{\text{solution of } L\{y\}=0} + \sum_{i:f_i \in \mathcal{P}\setminus\mathcal{L}} c_i f_i(x) = y_c + \boxed{\sum_{i:f_i \in \mathcal{P}\setminus\mathcal{L}} c_i f_i(x)}$$

The High-Level Idea

$\mathcal{P}$: fundamental set of solutions of $P\{y\} = 0$.
$\mathcal{L} \subseteq \mathcal{P}$: fundamental set of solutions of $L\{y\} = 0$.

To find a particular solution y_p of $L\{y\} = g(x)$, we can simply plug in

$$L\left\{ \sum_{i: f_i \in \mathcal{P} \setminus \mathcal{L}} c_i f_i(x) \right\} = g(x)$$

and find the values of the **undetermined coefficients**

$$\{c_i \mid i : f_i \in \mathcal{P} \setminus \mathcal{L}\}.$$

The High-Level Idea: Summary

Let $L(D) := \sum_{i=0}^{n} a_i D^i$.

Goal: Find a particular solution y_p such that $L\{y_p\} = g(x)$

Assumption: $\exists$ a polynomial of D, $A(D)$, such that $A\{g(x)\} = 0$.

Procedure:

1. Find a fundamental set of solutions of $P\{y\} = 0$, $\mathcal{P}$, where $P(D) := L(D)A(D)$.
2. Find a fundamental set of solutions of $L\{y\} = 0$, $\mathcal{L}$, and $\mathcal{L} \subseteq \mathcal{P}$.
3. Plug in $L\{y_p\} = g(x)$ with $y_p =$ a linear combination of the functions in $\mathcal{P} \setminus \mathcal{L}$.
4. Solve the undetermined coefficients in the linear combination.

Example

Example

Derive the general solution of $y'' + 3y' + 2y = 4x^2$.

A: This is a nonhomogeneous linear DE. Rewrite it as $L\{y\} = 4x^2$, where

$$L(D) = D^2 + 3D + 2 = (D+1)(D+2).\ \text{Two roots: } -1, -2.$$

1. Find the general solution of $L\{y\} = 0$: $\boxed{y_c = c_1 e^{-x} + c_2 e^{-2x}}$.
2. Find that $A\{4x^2\} = 0$, where $A(D) = D^3$.
3. Find the general solution of $P\{y\} = 0$, where $P(D) = L(D)A(D)$:

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 + c_4 x + c_5 x^2 = y_c + \boxed{c_3 + c_4 x + c_5 x^2}.$$

4. Let $y_p = A + Bx + Cx^2$: $L\{y_p\} = 4x^2 \implies A = 7, B = -6, C = 2$.
5. General Solution: $\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2}$.

$$
\begin{aligned}
&{y_p}' = B + 2Cx,\ \ {y_p}'' = 2C \\
&\Longrightarrow\ 4x^2 = L\{y_p\} = 2C + 6Cx + 3B + 2A + 2Bx + 2Cx^2 \\
&\qquad\qquad\qquad\qquad = 2Cx^2 + 2(B + 3C)x + (2A + 3B + 2C) \\
&\Longrightarrow\ \begin{cases} 2C = 4 \\ B + 3C = 0 \\ 2A + 3B + 2C = 0 \end{cases} \\
&\Longrightarrow\ C = 2, B = -6, A = 7.
\end{aligned}
$$

For What Kinds of $g(x)$ will this Work?

$$g(x) \longrightarrow \boxed{A} \longrightarrow 0$$

A : polynomial of D with constant coefficients

In other words, $g(x)$ is a solution to the homogeneous linear DE with constant coefficients $A\{y\} = 0$.

Recall:

$A(D)$	$g(x)$ (solution of $A\{y\} = 0$)
$(D-m)^k$	$e^{mx}, xe^{mx}, \ldots, x^{k-1}e^{mx}$
$\left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^k$	$e^{\alpha x}\sin\beta x, \ldots, x^{k-1}e^{\alpha x}\sin\beta x$
	$e^{\alpha x}\cos\beta x, \ldots, x^{k-1}e^{\alpha x}\cos\beta x$

If $g(x)$ is a linear combination of the above functions, we can use the method of undetermined coefficients to find a particular solution.

$g(x)$ and its Annihilator A

$$g(x) \longrightarrow \boxed{A} \longrightarrow 0$$

A : polynomial of D with constant coefficients

For $k = 0, 1, \ldots,$ and $m, \alpha, \beta \in \mathbb{R}$,

$g(x)$	A
x^k	D^{k+1}
$x^k e^{mx}$	$(D - m)^{k+1}$
$x^k \sin \beta x$	$\left(D^2 + \beta^2\right)^{k+1}$
$x^k \cos \beta x$	$\left(D^2 + \beta^2\right)^{k+1}$
$x^k e^{\alpha x} \sin \beta x$	$\left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^{k+1}$
$x^k e^{\alpha x} \cos \beta x$	$\left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^{k+1}$

More Examples

Example

Derive the general solution of $y'' + y = x\cos x - \cos x$.

Example

Derive the general solution of $y'' - 3y' = 8e^{3x} + 4\sin x$.

Example

Derive the general solution of $y'' - 2y' + y = 10e^{-2x}\cos x$.

Example

Derive the general solution of $y'' - 2y' + y = x$.

Nonhomogeneous Cauchy-Euler Equation

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

With the substitution $x = e^t$, convert the Cauchy-Euler Equation to a linear DE with constant coefficients.

Note: To use the method of undetermined coefficients, we have to make sure that $g(e^t)$ takes the form in the table on the previous slide.

Nonhomogeneous Cauchy-Euler Equation

Example

Derive the general solution of $x^2y'' - xy' + y = \ln x$, $x > 0$.

A: With $x = e^t$, we have $\ln x = t$ and

$$x^2D_x^2 - xD_x + 1 = D_t(D_t - 1) - D_t + 1 = (D_t - 1)^2 .$$

Hence the original DE becomes $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t$, and the general solution is

$$y = c_1e^t + c_2te^t + 2 + t = \boxed{c_1x + c_2x\ln x + 2 + \ln x}.$$

1 Methods of Undetermined Coefficients

2 Variation of Parameters

3 Summary

Overview

Variation of parameters is a powerful method to find a particular solution y_p of **any** linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

given that the general solution of the corresponding homogeneous DE

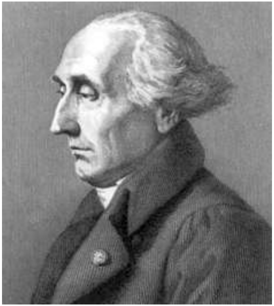
$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (3)$$

can be found.

No restrictions on $g(x)$! $g(x)$ can be $1/x$, $\csc x$, $\ln x$, etc.

This method is due to Joseph-Louis Lagrange.

Joseph-Louis Lagrange
(born as Giuseppe Luigi Lagrancia)



First Order DE

In Chapter 2 we use the method of **integrating factors** to solve

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Here we use a different method. Let $f_1(x)$ be a solution of the homogeneous linear DE (can be found using separation of variables)

$$a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

To find a particular solution, let $\boxed{y_p = f_1(x)u_1(x)}$, and plug it back:

$$y_p{}' = \frac{df_1}{dx}u_1 + f_1\frac{du_1}{dx} \implies g(x) = a_1(x)f_1(x)\frac{du_1}{dx}.$$

Then a $u_1(x)$ can be found by integrating $\frac{g(x)}{a_1(x)f_1(x)}$.

$$
{y_p}' = \frac{df}{dx}u + f\frac{du}{dx} \implies g(x) = a_1(x)\,{y_p}' + a_0(x)\,y_p
$$

$$
\begin{aligned}
&= a_1(x)\left(\frac{df}{dx}u + f\frac{du}{dx}\right) + a_0(x)fu \\
&= a_1(x)f(x)\frac{du}{dx} + u\left(a_1(x)\frac{df}{dx} + a_0(x)f\right) \\
&= a_1(x)f(x)\frac{du}{dx}
\end{aligned}
$$

Second Order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x).$$

Let f_1, f_2 be two **linearly independent** solutions of the homogeneous linear DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

To find a particular solution, let $\boxed{y_p = u_1 f_1 + u_2 f_2}$, and plug it back.
The remaining task is to find $u_1(x)$ and $u_2(x)$.

Observation: for $i = 1, 2$,

$$\begin{aligned}(u_i f_i)' &= u_i' f_i + u_i f_i' = \boxed{u_i' f_i + u_i D\{f_i\}}, \\ (u_i f_i)'' &= u_i'' f_i + 2u_i' f_i' + u_i f_i'' = (u_i'' f_i + u_i' f_i') + u_i' f_i' + u_i f_i'' \\ &= (u_i' f_i)' + u_i' f_i' + u_i f_i'' = \boxed{D\{u_i' f_i\} + u_i' f_i' + u_i D^2\{f_i\}}\end{aligned}$$

Find $u_1(x)$ and $u_2(x)$

Let $L := a_2(x)D^2 + a_1(x)D + a_0(x)$. L is a linear operator.
Hence, with $y_p = u_1 f_1 + u_2 f_2$, $L\{y_p\} = L\{u_1 f_1\} + L\{u_2 f_2\}$.

Using the fact that for $i = 1, 2$,

$$D\{u_i f_i\} = u_i' f_i + u_i D\{f_i\},\ D^2\{u_i f_i\} = D\{u_i' f_i\} + u_i' f_i' + u_i D^2\{f_i\},$$

we have

$$\begin{aligned} L\{u_i f_i\} &= a_2(x)D^2\{u_i f_i\} + a_1(x)D\{u_i f_i\} + a_0(x)u_i f_i \\ &= a_2(x)\left(D\{u_i' f_i\} + u_i' f_i' + u_i D^2\{f_i\}\right) \\ &\quad + a_1(x)\left(u_i' f_i + u_i D\{f_i\}\right) + a_0(x)u_i f_i \\ &= u_i L\{f_i\} + (a_2(x)D + a_1(x))\{u_i' f_i\} + a_2(x)u_i' f_i' \\ &= \cancel{u_i L\{f_i\}} + L_1\{u_i' f_i\} + a_2(x)u_i' f_i' \end{aligned}$$

Find $u_1(x)$ and $u_2(x)$

$L := a_2(x)D^2 + a_1(x)D + a_0(x)$. With $y_p = u_1 f_1 + u_2 f_2$, where f_1 and f_2 are two linearly independent solutions of $L\{y\} = 0$, we have

$$\begin{aligned} L\{y_p\} &= L\{u_1 f_1\} + L\{u_2 f_2\} \\ &= L_1\{{u_1}' f_1 + {u_2}' f_2\} + a_2(x)\left({u_1}' {f_1}' + {u_2}' {f_2}'\right) \\ &= g(x). \end{aligned}$$

Here $L_1 := a_2(x)D + a_1(x)$.

If u_1, u_2 satisfy the following, then $L\{y_p\} = g(x)$:

$$\begin{cases} {u_1}' f_1 + {u_2}' f_2 = 0 \\ {u_1}' {f_1}' + {u_2}' {f_2}' = \frac{g(x)}{a_2(x)} \end{cases} \implies \begin{cases} {u_1}' = \frac{W_1}{W} \\ {u_2}' = \frac{W_2}{W} \end{cases},$$

$W = \begin{vmatrix} f_1 & f_2 \\ {f_1}' & {f_2}' \end{vmatrix} \neq 0$: Wronskian of f_1, f_2, $W_1 = \begin{vmatrix} 0 & f_2 \\ \frac{g}{a_2} & {f_2}' \end{vmatrix}$, $W_2 = \begin{vmatrix} f_1 & 0 \\ {f_1}' & \frac{g}{a_2} \end{vmatrix}$

n-th Order DE $L\{y\} = g(x)$, $L := \sum_{i=0}^{n} a_i(x) D^i$

Suppose $\{f_1, f_2, \ldots, f_n\}$ form a fundamental set of solutions of the homogeneous linear DE $L\{y\} = 0$.

Then a particular solution $y_p = \sum_{i=0}^{n} u_i(x) f_i(x)$ can be found by the following formula regarding $\{{u_1}', {u_2}', \ldots, {u_n}'\}$:

$$
{u_i}' = \frac{W_i}{W}, \quad W = \begin{vmatrix} f_1 & \cdots & f_n \\ {f_1}' & \cdots & {f_n}' \\ \vdots & & \vdots \\ {f_1}^{(n-1)} & \cdots & {f_n}^{(n-1)} \end{vmatrix}
$$

W_i is W with the i-th column replaced by $\begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{g(x)}{a_n(x)} \end{bmatrix}^T$.

A Key Fact

Fact

For $k = 1, 2, \ldots, n$,

$$D^k\{u_i f_i\} = u_i D^k\{f_i\} + \sum_{j=0}^{k-1} D^j\left\{{u_i}' {f_i}^{(k-1-j)}\right\},$$

which implies

$$L\{u_i f_i\} = \sum_{j=1}^{n-1} L_j\left\{{u_i}' {f_i}^{(j-1)}\right\} + a_n(x){u_i}'{f_i}^{(n-1)},$$

where $L_j := \sum_{l=j}^{n} a_l(x) D^{l-j}$.

Proof: Exercise.

Examples

Example

Derive the general solution of $4y'' + 36y = \csc 3x$.

Example

Derive the general solution of $y'' - y = \frac{1}{x}$.

Example

Derive the general solution of $x^2 y'' - 3xy' + 3y = 2x^4 e^x$.

Short Recap

- Method of Undetermined Coefficients
- Annihilator Operator
- Variation of Parameters

Self-Practice Exercises

4-4: 1, 7, 13, 27, 29, 35

4-5: 15, 21, 25, 49, 65, 69, 71

4-6: 1, 3, 5, 9, 17, 21, 25, 27

4-7: 21, 29, 35