Chapter 4: Higher-Order Differential Equations – Part 1

王奕翔

Department of Electrical Engineering
National Taiwan University

ihwang@ntu.edu.tw

October 17, 2013

Higher-Order Differential Equations

Most of this chapter deals with **linear** higher-order DE (except 4.10)

In our lecture, we skip 4.10 and focus on n-th order linear differential equations, where $n \geq 2$.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

Methods of Solving Linear Differential Equations

We shall gradually fill up this slide as the lecture proceeds.

Initial-Value Problem (IVP)

An n-th order initial-value problem associate with (1) takes the form:

Solve:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to:

$$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

Here (2) is a set of **initial conditions**.

Boundary-Value Problem (BVP)

Recall: in Chapter 1, we made 3 remarks on initial/boundary conditions

Remark (Initial vs. Boundary Conditions)

Initial Conditions: all conditions are at the **same** $x = x_0$.
Boundary Conditions: conditions can be at **different** x.

Remark (Number of Initial/Boundary Conditions)

"Usually" a n-th order ODE requires n initial/boundary conditions to specify an unique solution.

Remark (Order of the derivatives in the conditions

Initial/boundary conditions can be the value or the function of 0-th to $(n-1)$-th order derivatives, where n is the order of the ODE.

Boundary-Value Problem (BVP)

Example (Second-Order ODE)

Consider the following second-order ODE

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{3}$$

- IVP: solve (3) s.t. $y(x_0) = y_0,\ y'(x_0) = y_1.$
- BVP: solve (3) s.t. $y(a) = y_0,\ y(b) = y_1.$
- BVP: solve (3) s.t. $y'(a) = y_0,\ y(b) = y_1.$
- BVP: solve (3) s.t. $\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$

Existence and Uniqueness of the Solution to an IVP

Solve

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to

$$y(x_0) = y_0,\ \ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

Theorem

If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ are all continuous on an interval I, $a_n(x) \neq 0$ is not a zero function on I, and the initial point $x_0 \in I$, then the above IVP has a unique solution in I.

Existence and Uniqueness of the Solution to an IVP

Solve

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to

$$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

Throughout this lecture, we assume that on some common interval I,

- $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ are all continuous
- $a_n(x)$ is not a zero function, that is, $\exists x \in I$ such that $a_n(x) \neq 0$.

Existence and Uniqueness of the Solution to an BVP

Note: Unlike an IVP, even the n-th order ODE (1) satisfies the conditions in the previous theorem, a BVP corresponding to (1) may have many, one, or no solutions.

Example

Consider the 2nd-order ODE $\frac{d^2y}{dx^2} + y = 0$, whose general solution takes the form $y = c_1 \cos x + c_2 \sin x$. Find the solution(s) to an BVP subject to the following boundary conditions respectively

- $y(0) = 0, y(2\pi) = 0$ Plug it in $\Longrightarrow$ $c_1 = 0, c_1 = 0$ $\Longrightarrow$ c_2 is arbitrary $\Longrightarrow$ infinitely many solutions!
- $y(0) = 0, y(\pi/2) = 0$ Plug it in $\Longrightarrow$ $c_1 = 0, c_2 = 0$ $\Longrightarrow$ $c_1 = c_2 = 0$ $\Longrightarrow$ a unique solution!
- $y(0) = 0, y(2\pi) = 1$ Plug it in $\Longrightarrow$ $c_1 = 0, c_1 = 1$ $\Longrightarrow$ contradiction $\Longrightarrow$ no solutions!

Homogeneous Equation

Linear n-th order ODE takes the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

Homogeneous Equation: $g(x)$ in (1) is a zero function:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (4)$$

Nonhomogeneous Equation: $g(x)$ in (1) is **not** a zero function. Its *associated homogeneous equation* (4) is the one with the same coefficients except that $g(x)$ is a zero function

Later in the lecture we will see, when solving a nonhomogeneous equation, we must first solve its associated homogeneous equation (4).

Differential Operators

We introduce a **differential operator** D, which simply represent the operation of taking an ordinary differentiation:

Differential Operator

For a function $y = f(x)$, the differential operator D transforms the function $f(x)$ to its first-order derivative: $Dy := \frac{dy}{dx}$.

Higher-order derivatives can be represented compactly with D as well:

$$\frac{d^2 y}{dx^2} = D(Dy) =: D^2 y, \quad \frac{d^n y}{dx^n} =: D^n y$$

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y =: \left\{\sum_{i=0}^{n} a_i(x) D^i\right\} y$$

Differential Operators and Linear Differential Equations

Note: Polynomials of differential operators are differential operators.

Let $L := \sum_{i=0}^{n} a_i(x) D^i$ be an n-th order differential operator.

Then we can compactly represent the linear differential equation (1) and the homogeneous linear DE (4) as

$$L(y) = g(x), \quad L(y) = 0$$

respectively.

Linearity and Superposition Principle

$L := \sum_{i=0}^{n} a_i(x) D^i$ is a **linear operator**: for two functions $f_1(x), f_2(x)$,

$$L(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 L(f_1) + \lambda_2 L(f_2).$$

For any homogeneous linear equation (4), that is, $L(y) = 0$, we obtain the following superposition principle.

Theorem (Superposition Principle: Homogeneous Equations)

Let $f_1, f_2, \dots, f_k$ be solutions to the homogeneous n-th order linear equation $L(y) = 0$ on an interval I, that is,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (4)$$

then the linear combination $f = \sum_{i=1}^{k} \lambda_i f_i$ is also a solution to (4).

Linear Dependence and Independence of Functions

In Linear Algebra, we learned that one can view the collection of all *functions* defined on a common interval as a **vector space**, where linear dependence and independence can be defined respectively.

Definition (Linear Dependence and Independence)

A set of functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ are linearly dependent on an interval I if $\exists\ c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0,\ \forall\ x \in I,$$

that is, the linear combination is a zero function. If the set of functions is not linearly dependent, it is linearly independent.

Example:

- $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, $I = (-\pi, \pi)$: Linearly dependent
- $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^3$, $I = \mathbb{R}$: Linearly independent.

Linear Independence of Solutions to (4)

Consider the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (4)$$

Given n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$, we would like to test if they are independent or not.

Of course we can always go back to the definition but it is clumsy...

Recall: In Linear Algebra, to test if n vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly independent, we can compute the determinant of the matrix

$$\mathbf{V} := \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

If $\det \mathbf{V} = 0$, they are linearly dependent; if $\det \mathbf{V} \neq 0$, they are linearly independent.

Criterion of Linearly Independent Solutions

Consider the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (4)$$

To test the linear independence of n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ to (4), we can use the following theorem.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ *be* n *solutions to the homogeneous linear* n*-th order DE* (4) *on an interval* I*. They are linearly independent on* I

$$\iff W(f_1, f_2, \ldots, f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ {f_1}' & {f_2}' & \cdots & {f_n}' \\ \vdots & \vdots & & \vdots \\ {f_1}^{(n-1)} & {f_2}^{(n-1)} & \cdots & {f_n}^{(n-1)} \end{vmatrix} \neq 0.$$

Fundamental Set of Solutions

We are interested in describing the *solution space*, that is, the subspace spanned by the solutions to the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0. \tag{4}$$

How?

Recall: In Linear Algebra, we describe a subspace by its *basis*: any vector in the subspace can be represented by a linear combination of the elements in the basis, and these elements are linearly independent.

Similar things can be done here.

Definition (Fundamental Set of Solutions)

Any set $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ of n linearly independent solutions to the homogeneous linear n-th order DE (4) on an interval I is called a fundamental set of solutions.

General Solutions to Homogeneous Linear DE

General solution to an n-th order ODE:
An n-parameter family of solutions that can contains *all* solutions.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a fundamental set of solutions to the homogeneous linear n-th order DE (4) *on an interval I. Then the **general solution** to* (4) *is*

$$y = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x),$$

where $\{c_i \mid i = 1, 2, \ldots, n\}$ are arbitrary constants.

Examples

Example

Consider the DE

$$\frac{d^2 y}{dx^2} = y.$$

Check that both $y = e^x$ and $y = e^{-x}$ are solutions to the equation. Derive the general solution to the DE.

A: The linear DE is homogeneous.
We see that $\frac{d^2}{dx^2} e^x = \frac{d}{dx} e^x = e^x$, and $\frac{d^2}{dx^2} e^{-x} = \frac{d}{dx} - e^{-x} = e^{-x}$. Hence they are both solutions to the homogeneous linear second-order DE. Since

$$\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0,$$

the two solutions are linearly independent. Hence, the general solution can be written as $\boxed{y = c_1 e^x + c_2 e^{-x},\ \ c_1, c_2 \in \mathbb{R}}$.

General Solutions to Nonhomogeneous Linear DE

Nonhomogeneous linear n-th order ODE takes the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

or equivalently, $L(y) = g(x), \ L := \sum_{i=0}^{n} a_i(x)D^i$

where $g(x)$ is not a zero function.

How to find its general solution?

Idea:

- Find the general solution y_c to the *homogeneous* equation $L(y) = 0$.
- Find **a** solution y_p to the *nonhomogeneous* equation $L(y) = g(x)$.
- The general solution $y = y_c + y_p$.

General Solutions to Nonhomogeneous Linear DE

Nonhomogeneous :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

Homogeneous :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (4)$$

Theorem

Let y_p be any *particular solution to the nonhomogeneous linear n-th order DE* (1) *on an interval I, and y_c be the* general solution *to the associated homogeneous linear n-th order DE* (4) *on I, then the general solution to* (1) *is*

$$y = y_c + y_p.$$

Proof of the Theorem

Proof: Let $y = f(x)$ be any solution to the nonhomogeneous linear n-th order DE (1), that is, $L(y) = g(x)$.

Now, since both y_p and f are solutions to $L(y) = g(x)$, we have

$$0 = L(f) - L(y_p) = L(f - y_p).$$

Hence, $(f - y_p)$ is a solution to the homogeneous linear n-th order DE (4).

Therefore, any solution to (1) can be represented by the sum of a solution to (4) and the particular solution y_p.

Examples

Example

Consider the DE

$$\frac{d^2y}{dx^2} = y + 9.$$

Derive the general solution to the DE.

A: The linear DE is nonhomogeneous. The associated homogeneous equation $\frac{d^2y}{dx^2} = y$ has the following general solution:

$$y = c_1 e^x + c_2 e^{-x}, \ c_1, c_2 \in \mathbb{R}.$$

There is an obvious particular solution $y = -9$.

Hence, the general solution can be written as

$$\boxed{y = c_1 e^x + c_2 e^{-x} - 9, \ c_1, c_2 \in \mathbb{R}}$$

Superposition Principle for Nonhomogeneous Equations

For nonhomogeneous linear differential equations, we have the following superposition principle.

Theorem (Superposition Principle: Nonhomogeneous Equations)

Let $f_i(x)$ be a particular solution to the nonhomogeneous n-th order linear equation $L(y) = g_i(x)$ on an interval I, for $i = 1, 2, \ldots, k$. Then the linear combination $f = \sum_{i=1}^{k} \lambda_i f_i$ is a particular solution to the nonhomogeneous n-th order linear equation

$$L(y) = \sum_{i=1}^{k} \lambda_i g_i(x).$$

Finding a New Solution

Recall: the fundamental set of solutions of the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (4)$$

contains n linearly independent solutions.

Now suppose we already have k $(1 \le k < n)$ linearly independent solutions $\{f_1, f_2, \ldots, f_k\}$. How do we find another one f_{k+1} so that the $(k+1)$ solutions $\{f_1, f_2, \ldots, f_{k+1}\}$ remain linearly independent?

Second Order Equation

We begin with the simplest case: $n = 2$ and $k = 1$. Consider the following homogeneous linear second order DE

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Suppose we already have a solution $y = f_1(x)$. How do we find another solution $y = f_2(x)$, such that f_1 and f_2 are linearly independent?

Idea: Let $\boxed{f_2(x) = u(x)f_1(x)}$, and make use of the fact that

$$a_2(x)\frac{d^2}{dx^2}f_1 + a_1(x)\frac{d}{dx}f_1 + a_0(x)f_1 = 0$$

to reduce the second order DE into a first order DE of u !

Example

Example

$f_1(x) = x^2$ is a solution of the second order DE $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$. Find the general solution of the above DE for $x > 0$.

A: We need to find a fundamental set of solutions, which contains two linearly independent solutions. Now we have only one. To find a second one, let us set substitute $y = f_1 u = x^2 u$:

$$\frac{dy}{dx} = 2xu + x^2 u', \quad \frac{d^2y}{dx^2} = (2u + 2xu') + (2xu' + x^2 u'') = 2u + 4xu' + x^2 u''$$

$$\Longrightarrow \; x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2 u + 4x^3 u' + x^4 u'' - 6x^2 u - 3x^3 u' + 4x^2 u$$

$$= x^3 u' + x^4 u'' = 0$$

$$\Longrightarrow \; v + xv' = 0 \quad (\text{Set } v := u')$$

$$\Longrightarrow \; \text{one such } v = \frac{1}{x} \Longrightarrow \text{one such } u = \ln x.$$

Example

Example

$f_1(x) = x^2$ is a solution of the second order DE $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$. Find the general solution of the above DE for $x > 0$.

We find a second solution $y = f_2(x) = x^2 \ln x$ on $x \in (0, \infty)$, and the general solution is

$$\boxed{y = c_1x^2 + c_2x^2 \ln x}.$$

Question:
How about the more complicated case, when $n > 2$ and $k > 1$?

In this section we focus on solving (that is, giving general solutions to) **Homogeneous Linear Equations with Constant Coefficients**

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0, \tag{5}$$

which is a homogeneous linear DE with constant real coefficients.

In the textbook, it tells us (without much reasoning) what the form of the general solution should look like, and then we analyze the particular structure of a give equation to derive the exact form.

In this lecture, we try to provide more reasoning, so that you get a clearer big picture.

Second Order Equation

We begin with some examples of second order equations.

Example

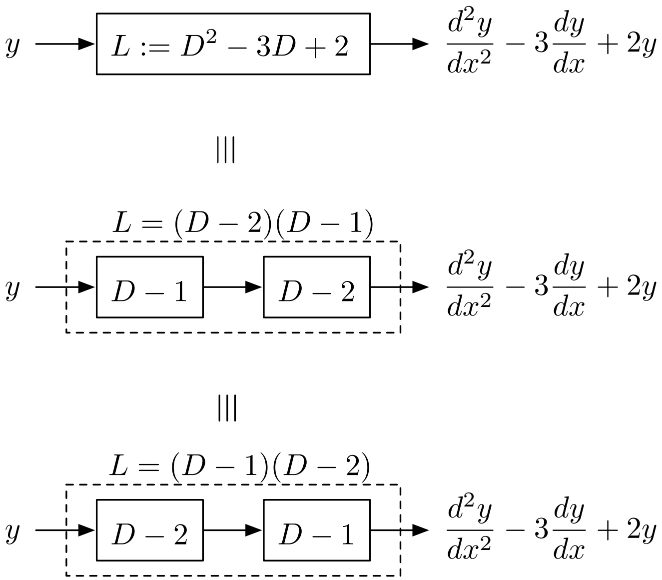
Find the general solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0.$

A: Let us use the differential operator to rewrite this DE as follows:

$$(D^2 - 3D + 2)y = 0.$$

Note that $L := D^2 - 3D + 2 = (D-1)(D-2)$.

We can view the second-order differential operator L as a concatenation of two first-order differential operators: $(D-1)$ and $(D-2)$!


y
$L := D^2 - 3D + 2$
$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$
$|||$
$L = (D-2)(D-1)$
y
$D - 1$
$D - 2$
$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$
$|||$
$L = (D-1)(D-2)$
y
$D - 2$
$D - 1$
$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$

Second Order Equation

Example

Find the general solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0.$

A: We have found the equivalent forms of the above equation

$$Ly = 0 \quad \equiv \quad (D-2)\left\{(D-1)y\right\} = 0 \quad \equiv \quad (D-1)\left\{(D-2)y\right\} = 0$$

where $L := D^2 - 3D + 2 = (D-1)(D-2)$.

Observation:

- If f_1 is a solution to $(D-1)y = 0$, it is also a solution to $Ly = 0$. A solution: $f_1 = e^x$.
- If f_2 is a solution to $(D-2)y = 0$, it is also a solution to $Ly = 0$. A solution: $f_2 = e^{2x}$.

Second Order Equation

Example

Find the general solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$.

A: So far we have found two solutions to $(D^2 - 3D + 2)y = 0$:

$$f_1 = e^x, \quad \text{corresponds to } (D-1)y = 0$$
$$f_2 = e^{2x}, \quad \text{corresponds to } (D-2)y = 0.$$

f_1 and f_2 are linearly independent (**Exercise**: check!) and hence $\{f_1, f_2\}$ is a fundamental set of solutions.

$\implies$ The general solution:

$$y = c_1 f_1 + c_2 f_2 = \boxed{c_1 e^x + c_2 e^{2x}}, \quad c_1, c_2 \in \mathbb{R}.$$

How we solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

1. Use a polynomial of D,

$$L := p(D) = D^2 - 3D + 2,$$

   to rewrite the DE into the form $Ly = 0$.

2. Factor $p(D) = (D-1)(D-2)$.
3. Observe that a solution to either $(D-1)y = 0$ or $(D-2)y = 0$ will be a solution to $Ly = 0$.
4. Find two solutions $f_1 = e^x$ and $f_2 = e^{2x}$, corresponding to $(D-1)y = 0$ and $(D-2)y = 0$ respectively.
5. Check that f_1 and f_2 are linearly independent, and hence they form a fundamental set of solutions.
6. Finally we get the general solution $\boxed{y = c_1 e^x + c_2 e^{2x}}$.

$p(D) = a_2 D^2 + a_1 D + a_0$ Has Two Distinct Real Roots

For a homogeneous linear second order DE with constant coefficients $Ly = 0$, where (WLOG we assume $a_2 = 1$)

$$L := p(D) = a_2 D^2 + a_1 D + a_0 = D^2 + a_1 D + a_0 :$$

Fact

If $p(D)$ has two distinct real roots m_1 and m_2, then we can use the above mentioned method to get a general solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

What if $p(D)$ has

- Two repeated real roots, or
- Two conjugate complex roots?

$p(D)$ Has Two Conjugate Complex Roots $\alpha \pm i\beta$

Suppose $p(D)$ has two conjugate complex roots

$$m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta,\ \alpha, \beta \in \mathbb{R}.$$

If we slightly extend our discussion to complex-valued DE, it is not hard to see that the previous method works again and we get a general (complex-valued) solution

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x},\ C_1, C_2 \in \mathbb{C}.$$

Still we need to get back to the real domain ...

So, let's do some further manipulation by using the fact that

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

$p(D)$ Has Two Conjugate Complex Roots $\alpha \pm i\beta$

The general solution to $Ly = 0$ where $L = p(D)$ is

$$\begin{aligned} y &= C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x} \\ &= C_1 e^{\alpha x} (\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x} (\cos\beta x - i\sin\beta x) \\ &= (C_1 + C_2)\, e^{\alpha x}\cos\beta x + i\,(C_1 - C_2)\, e^{\alpha x}\sin\beta x \end{aligned}$$

To get a real-valued solution, there are two choices:

- Pick $C_1 + C_2 = 1$, $C_1 - C_2 = 0$: we get $y = f_1(x) = e^{\alpha x}\cos\beta x$.
- Pick $C_1 + C_2 = 0$, $C_1 - C_2 = -i$: we get $y = f_2(x) = e^{\alpha x}\sin\beta x$.

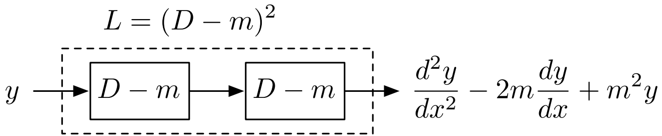
Since f_1 and f_2 are linearly independent, the general real-valued solution to $Ly = 0$ where $L = p(D)$ is

$$\boxed{y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x}, \quad c_1, c_2 \in \mathbb{R}.$$

$p(D)$ Has Two Repeated Real Roots m

Suppose $p(D)$ has two repeat real roots m, which means that

$$p(D) = (D - m)^2.$$

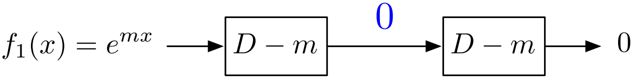


From the previous discussion, we see that $y = f_1(x) = e^{mx}$ is a solution to $(D - m)y = 0$ and hence it is also a solution to $(D - m)^2 y = 0$.

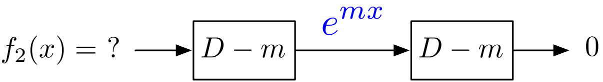
Question:
How to find another solution $y = f_2(x)$ so that f_1 and f_2 are linearly independent?

$p(D)$ Has Two Repeated Real Roots m

$f_1(x) = e^{mx}$ is a solution to $(D-m)^2 y = 0$ because:



Why not find some $f_2(x)$ such that after the first $\boxed{D-m}$ block, the outcome is $f_1(x) = e^{mx}$?



We only need to solve a first order linear DE!

$$\underbrace{e^{-mx}}_{\text{integrating factor}} \quad f_2(x) = \int \overbrace{\underbrace{e^{-mx}}_{\text{integrating factor}} \quad e^{mx}}^{1} dx \implies f_2(x) = xe^{mx}.$$

$p(D)$ Has Two Repeated Real Roots m

We have found two solutions to $(D-m)^2 y = 0$:

$$f_1(x) = e^{mx}, \quad f_2(x) = xe^{mx},$$

and they are linearly independent (check!).

Hence the general solution to $p(D)y = 0$ is

$$y = c_1 e^{mx} + c_2 x e^{mx} = \boxed{(c_1 + c_2 x)e^{mx}}.$$

Summary: Second Order Equation $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 = 0$

Define the following (quadratic) polynomial

$$p(D) := a_2 D^2 + a_1 D + a_0.$$

Roots of $p(D)$	**General Solution**
Distinct real roots $m_1, m_2 \in \mathbb{R}$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Conjugate complex roots $\alpha \pm i\beta,\ \alpha, \beta \in \mathbb{R}$	$y = c_1 e^{\alpha x} \sin \beta x + c_2 e^{\alpha x} \cos \beta x$
Repeated real roots $m \in \mathbb{R}$	$y = (c_1 + c_2 x) e^{mx}$

n-th Order Equation $a_n \frac{d^n y}{dx^n} + \cdots + a_1 \frac{dy}{dx} + a_0 = 0$

Define

$$p(D) := a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 = \sum_{i=0}^{n} a_i D^i$$

and rewrite the n-th order equation as $\boxed{p(D)y = 0}$.

$p(D)$: a polynomial of order n with real-valued coefficients.

- $p(D)$ has n roots in the complex domain (counting the **multiplicity**)
- Complex roots of $p(D)$ must appear in conjugate pairs.

Example: $p(D) = (D-1)^3(D-2)^1(D^2-2D+2)^2$ is a polynomial of order 8, and has the following roots

1	multiplicity 3
2	multiplicity 1
$1 \pm i$	multiplicity 2 for each.

Finding the General Solution of $p(D)y = 0$

High-level Idea: let $p(D)$ have n_1 distinct real roots $\{m_i \mid i \in [1:n_1]\}$, and n_2 distinct pairs of conjugate complex roots $\{\alpha_j \pm i\beta_j \mid j \in [1:n_2]\}$.

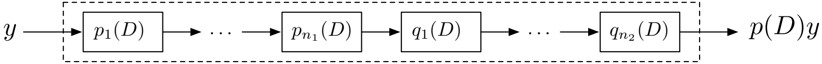
1. Factorize $p(D) = \sum_{i=0}^{n} a_i D^i$ as

$$p(D) = a_n \left(\prod_{i=1}^{n_1} \overbrace{(D - m_i)^{k_i}}^{p_i(D)} \right) \left(\prod_{j=1}^{n_2} \overbrace{\left(D^2 - 2\alpha_j D + \alpha_j^2 + \beta_j^2\right)^{l_j}}^{q_j(D)} \right)$$

$$= a_n \prod_{i=1}^{n_1} p_i(D) \prod_{j=1}^{n_2} q_j(D), \text{ where } n = \sum_{i=1}^{n_1} k_i + 2\sum_{j=1}^{n_2} l_j.$$

2. For each $i \in [1:n_1]$, find k_i linearly independent solutions of $p_i(D)y = 0$.
3. For each $j \in [1:n_2]$, find $2l_j$ linearly independent solutions of $q_j(D)y = 0$.
4. Combine them all to get n linearly independent solutions of $p(D)y = 0$.

$$p(D) := \sum_{i=0}^{n} a_i D^i,\ a_n = 1$$



$$p_i(D) := (D - m_i)^{k_i},\ i \in [1:n_1]; \quad q_j(D) := \left(D^2 - 2\alpha_j D + \alpha_j^2 + \beta_j^2\right)^{l_j},\ j \in [1:n_2].$$

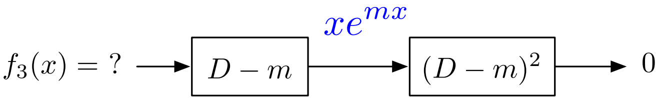
$p(D)$ have n_1 distinct real roots $\{m_i \mid i \in [1:n_1]\}$, and n_2 distinct pairs of conjugate complex roots $\{\alpha_j \pm i\beta_j \mid j \in [1:n_2]\}$.

Note: The solutions of different blocks in the above diagram will be linearly independent.

Solve $(D-m)^k\, y = 0$

$k=2$: two linearly independent solutions $f_1(x) = e^{mx}$ and $f_2(x) = xe^{mx}$.

$k=3$: Look at the diagram below:

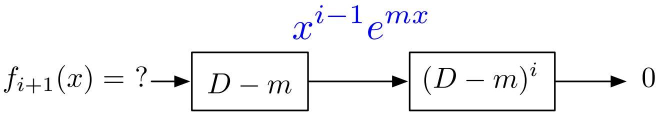


We only need to solve a first order linear DE!

$$\underbrace{e^{-mx}}_{\text{integrating factor}} f_3(x) = \int \overbrace{\underbrace{e^{-mx}}_{\text{integrating factor}} \; xe^{mx}}^{x}\, dx \implies f_3(x) = x^2 e^{mx}/2.$$

We can drop the factor of 2 and pick $f_3(x) = x^2 e^{mx}$.

Solve $(D-m)^k\, y = 0$



We can repeat this procedure and find k linearly independent solutions:

$$\boxed{f_1(x) = e^{mx},\ f_2(x) = xe^{mx},\ f_3(x) = x^2 e^{mx}, \ldots,\ f_k(x) = x^{k-1} e^{mx}}.$$

Solve $\left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^l \; y = 0$

$$D^2 - 2\alpha D + \alpha^2 + \beta^2 = (D - m)(D - \overline{m}), \text{ where } m = \alpha + i\beta \in \mathbb{C}.$$

$$\therefore \left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^l = (D - m)^l (D - \overline{m})^l$$

We can repeat the previous discussion and get $2l$ linearly independent solutions (in $\mathbb{C}$):

$$F_1(x) = e^{mx},\; F_2(x) = xe^{mx}, \ldots,\; F_l(x) = x^{l-1} e^{mx}$$
$$\overline{F}_1(x) = e^{\overline{m}x},\; \overline{F}_2(x) = xe^{\overline{m}x}, \ldots,\; \overline{F}_l(x) = x^{l-1} e^{\overline{m}x}$$

For each $j \in [1 : l]$, use F_j and $\overline{F}_j$ to generate two real-valued solutions:

$$f_{2j-1}(x) = \frac{1}{2} F_j(x) + \frac{1}{2} \overline{F}_j(x) = \mathrm{Re}\left\{F_j(x)\right\} = x^{j-1} e^{\alpha x} \cos \beta x$$
$$f_{2j}(x) = \frac{-i}{2} F_j(x) + \frac{i}{2} \overline{F}_j(x) = \mathrm{Im}\left\{F_j(x)\right\} = x^{j-1} e^{\alpha x} \sin \beta x.$$

Solve $\left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^l \; y = 0$

Here are $2l$ linearly independent real-valued solutions:

$$\boxed{\left\{ x^{j-1} e^{\alpha x} \cos \beta x,\; x^{j-1} e^{\alpha x} \sin \beta x \;\middle|\; j = 1, 2, \ldots, l \right\}}$$

Examples

Example

Solve the IVP $4y'' + 4y' + 17y = 0,\ y(0) = -1,\ y'(0) = 2.$

A: Write down the associated polynomial

$$p(D) = 4D^2 + 4D + 17 = (2D+1)^2 + 16.$$

$p(D)$ has two roots: $-\frac{1}{2} \pm 2i$. Hence according to the previous discussion, the general solution is $y = c_1 f_1(x) + c_2 f_2(x)$, where

$$f_1(x) = e^{-\frac{1}{2}x} \cos 2x, \quad f_2(x) = e^{-\frac{1}{2}x} \sin 2x.$$

Plug in the initial conditions, we get

$$\begin{cases} c_1 f_1(0) + c_2 f_2(0) = -1 \\ c_1 {f_1}'(0) + c_2 {f_2}'(0) = 2 \end{cases} \implies c_1 = -1, c_2 = \frac{3}{4}$$

Examples

Example

Solve the IVP $y''' + y'' - 2y = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 4$.

A: Write down the associated polynomial

$$p(D) = D^3 + D^2 - 2 = (D-1)(D^2 + 2D + 2).$$

$p(D)$ has three roots: $1, -1 \pm i$. Hence according to the previous discussion, the general solution is $y = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x)$, where

$$f_1(x) = e^x, \quad f_2(x) = e^{-x}\cos x, \quad f_3(x) = e^{-x}\sin x.$$

Plug in the initial conditions, we get

$$\begin{cases} c_1 f_1(0) + c_2 f_2(0) + c_3 f_3(0) = 1 \\ c_1 {f_1}'(0) + c_2 {f_2}'(0) + c_3 {f_3}'(0) = 2 \\ c_1 {f_1}''(0) + c_2 {f_2}''(0) + c_3 {f_3}''(0) = 4 \end{cases} \implies c_1 = 2, c_2 = -1, c_3 = -1.$$

Homogeneous Linear DE with Variable Coefficients

In general, a homogeneous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (4)$$

may have *variable* coefficients, that is, $a_i(x)$ is not a constant function, for $i = 0, 1, \ldots, n$.

If the coefficients are not constants, it is usually hard to find closed-form solutions. Instead, we shall see in Chapter 6 that the best we expect is to find a solution in the form of *infinite series*.

There is one exception: when

$$a_i(x) = a_i x^i, \ \forall \ i = 0, 1, \ldots, n.$$

Cauchy-Euler Equation

Leonhard Euler

Augustin-Louis Cauchy

Cauchy-Euler Equation

Definition (Cauchy-Euler Equation)

A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

is called a Cauchy-Euler equation.

We first focus on finding the general solution of a **homogeneous** Cauchy-Euler equation (see below) in this lecture. In the next lecture we discuss how to find a particular solution of a nonhomogeneous Cauchy-Euler equation.

$$\boxed{a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0}$$

Change of Variable

Goal: Find the general solution of a homogeneous Cauchy-Euler equation on the interval $(0, \infty)$.

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0$$

Idea: Convert a Cauchy-Euler equation into a linear equation with constant coefficients, by substituting $\boxed{x = e^t} \implies \boxed{\frac{dx}{dt} = e^t = x}$.

Observation

With the substitution $\boxed{x = e^t} \implies \boxed{\frac{dx}{dt} = e^t = x}$,

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = x\frac{dy}{dx} \implies x\frac{dy}{dx} = \frac{dy}{dt} := D_t y$$

$$D_t^2 y = \frac{dx}{dt}\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x\left(\frac{dy}{dx} + x\frac{d^2 y}{dx^2}\right) = D_t y + x^2\frac{d^2 y}{dx^2}$$

$$\implies x^2\frac{d^2 y}{dx^2} = D_t(D_t - 1)y$$

$$D_t^2(D_t - 1)y = \frac{dx}{dt}\frac{d}{dx}\left(x^2\frac{d^2 y}{dx^2}\right) = x\left(2x\frac{d^2 y}{dx^2} + x^2\frac{d^3 y}{dx^3}\right)$$

$$= 2D_t(D_t - 1)y + x^3\frac{d^3 y}{dx^3}$$

$$\implies x^3\frac{d^3 y}{dx^3} = D_t(D_t - 1)(D_t - 2)y$$

Conversion

Fact

With $x = e^t$, $x^k D_x^k = D_t(D_t - 1)\cdots(D_t - k + 1)$, for all integer $k \geq 1$.

Proof: For $k = 1$, the fact is true. Suppose the fact holds for $k = h \geq 1$. Then we have $x^h D_x^h\ y = D_t(D_t - 1)\cdots(D_t - h + 1)\ y$.

Take the derivative with respect to x on both sides, we get

$$\left(hx^{h-1}D_x^h + x^h D_x^{h+1}\right) y = \frac{dt}{dx} D_t^2(D_t - 1)\cdots(D_t - h + 1)y$$
$$\Longrightarrow \left(hx^h D_x^h + x^{h+1} D_x^{h+1}\right) y = D_t^2(D_t - 1)\cdots(D_t - h + 1)y$$
$$\Longrightarrow \left(hD_t(D_t - 1)\cdots(D_t - h + 1) + x^{h+1} D_x^{h+1}\right) y = D_t^2(D_t - 1)\cdots(D_t - h + 1)y$$
$$\Longrightarrow x^{h+1} D_x^{h+1} = D_t(D_t - 1)\cdots(D_t - h + 1)(D_t - h)$$

Hence we prove the fact by induction.

Convert into a Equation with Constant Coefficients

Based on the above fact, with the substitution $\boxed{x = e^t}$, we convert a Cauchy-Euler Equation

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0$$

into a linear differential equation (with respect to t) with constant coefficients

$$L_t \; y = 0, \quad \boxed{L_t := \sum_{i=0}^{n} a_i D_t (D_t - 1) \cdots (D_t - i + 1)}$$

Mapping of Solutions: with $x = e^t$,

t **Domain**	$t^k e^{mt}$	$t^k e^{\alpha t} \cos \beta t$	$t^k e^{\alpha t} \sin \beta t$
x **Domain**	$(\ln x)^k x^m$	$(\ln x)^k x^\alpha \cos(\beta \ln x)$	$(\ln x)^k x^\alpha \sin(\beta \ln x)$

Solutions for $x < 0$

So far we give the general solution of the Cauchy-Euler equation for $x \in (0, \infty)$. How about the solution for $x < 0$?

Idea: Change of variable – substitute $x = -u$, and solve the new Cauchy-Euler Equation for $u > 0$.

Conversion:

$$D_x^k = (-1)^k D_u^k.$$

Short Recap

- Initial-Value Problems (IVP) vs. Boundary-Value Problems (BVP)
- Homogeneous vs Nonhomogeneous Linear ODE
- Fundamental set of solutions and General Solutions
- Linearity and Superposition Principle
- General Solution of Homogeneous Linear Equation with Constant Coefficients: Usage of Polynomial of Differential Operator D
- General Solution of Homogeneous Cauchy-Euler Equation: Substitution $x = e^t$ and $x^k D_x^k = \prod_{i=1}^{k} (D_t - i + 1)$

Self-Practice Exercises

4-1: 1, 9, 13, 17, 21, 25, 35

4-2: 1, 3, 13, 17, 19

4-3: 3, 5, 17, 21, 25, 31, 37, 51, 57

4-7: 1, 5, 15, 25, 31, 41