Chapter 4: Higher-Order Differential Equations – Part 1

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October 3, 2013

Higher-Order Differential Equations

Most of this chapter deals with **linear** higher-order DE (except 4.10)

In our lecture, we skip 4.10 and focus on n-th order linear differential equations, where $n \geq 2$.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

Methods of Solving Linear Differential Equations

We shall gradually fill up this slide as the lecture proceeds.

Initial-Value Problem (IVP)

An n-th order initial-value problem associate with (1) takes the form:

Solve:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to:

$$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

Here (2) is a set of **initial conditions**.

Boundary-Value Problem (BVP)

Recall: in Chapter 1, we made 3 remarks on initial/boundary conditions

Remark (Initial vs. Boundary Conditions)

Initial Conditions: all conditions are at the **same** $x = x_0$.
Boundary Conditions: conditions can be at **different** x.

Remark (Number of Initial/Boundary Conditions)

"Usually" a n-th order ODE requires n initial/boundary conditions to specify an unique solution.

Remark (Order of the derivatives in the conditions

Initial/boundary conditions can be the value or the function of 0-th to $(n-1)$-th order derivatives, where n is the order of the ODE.

Boundary-Value Problem (BVP)

Example (Second-Order ODE)

Consider the following second-order ODE

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{3}$$

- IVP: solve (3) s.t. $y(x_0) = y_0,\ y'(x_0) = y_1$.
- BVP: solve (3) s.t. $y(a) = y_0,\ y(b) = y_1$.
- BVP: solve (3) s.t. $y'(a) = y_0,\ y(b) = y_1$.
- BVP: solve (3) s.t. $\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$

Existence and Uniqueness of the Solution to an IVP

Solve

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to

$$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

Theorem

If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ are all continuous on an interval I, $a_n(x) \neq 0$ is not a zero function on I, and the initial point $x_0 \in I$, then the above IVP has a unique solution in I.

Existence and Uniqueness of the Solution to an IVP

Solve

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to

$$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

Throughout this lecture, we assume that on some common interval I,

- $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ are all continuous
- $a_n(x)$ is not a zero function, that is, $\exists x \in I$ such that $a_n(x) \neq 0$.

Existence and Uniqueness of the Solution to an BVP

Note: Unlike an IVP, even the n-th order ODE (1) satisfies the conditions in the previous theorem, a BVP corresponding to (1) may have many, one, or no solutions.

Example

Consider the 2nd-order ODE $\frac{d^2y}{dx^2} + y = 0$, whose general solution takes the form $y = c_1 \cos x + c_2 \sin x$. Find the solution(s) to an BVP subject to the following boundary conditions respectively

- $y(0) = 0, y(2\pi) = 0$ Plug it in $\Longrightarrow c_1 = 0, c_1 = 0$ $\Longrightarrow$ c_2 is arbitrary $\Longrightarrow$ infinitely many solutions!
- $y(0) = 0, y(\pi/2) = 0$ Plug it in $\Longrightarrow c_1 = 0, c_1 + c_2 = 0$ $\Longrightarrow$ $c_1 = c_2 = 0$ $\Longrightarrow$ a unique solution!
- $y(0) = 0, y(2\pi) = 1$ Plug it in $\Longrightarrow c_1 = 0, c_1 = 1$ $\Longrightarrow$ contradiction $\Longrightarrow$ no solutions!

Homogeneous Equation

Linear n-th order ODE takes the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

Homogeneous Equation: $g(x)$ in (1) is a zero function:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (4)$$

Nonhomogeneous Equation: $g(x)$ in (1) is **not** a zero function. Its *associated homogeneous equation* (4) is the one with the same coefficients except that $g(x)$ is a zero function

Later in the lecture we will see, when solving a nonhomogeneous equation, we must first solve its associated homogeneous equation (4).

Differential Operators

We introduce a **differential operator** D, which simply represent the operation of taking an ordinary differentiation:

Differential Operator

For a function $y = f(x)$, the differential operator D transforms the function $f(x)$ to its first-order derivative: $Dy := \dfrac{dy}{dx}$.

Higher-order derivatives can be represented compactly with D as well:

$$\frac{d^2y}{dx^2} = D(Dy) =: D^2y, \quad \frac{d^ny}{dx^n} =: D^ny$$

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y =: \left\{\sum_{i=0}^{n} a_i(x)D^i\right\}y$$

Differential Operators and Linear Differential Equations

Note: Polynomials of differential operators are differential operators.

Let $L := \sum_{i=0}^{n} a_i(x) D^i$ be an n-th order differential operator.

Then we can compactly represent the linear differential equation (1) and the homogeneous linear DE (4) as

$$L(y) = g(x), \quad L(y) = 0$$

respectively.

Linearity and Superposition Principle

$L := \sum_{i=0}^{n} a_i(x) D^i$ is a **linear operator**: for two functions $f_1(x), f_2(x)$,

$$L(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 L(f_1) + \lambda_2 L(f_2).$$

For any homogeneous linear equation (4), that is, $L(y) = 0$, we obtain the following superposition principle.

Theorem (Superposition Principle: Homogeneous Equations)

Let $f_1, f_2, \dots, f_k$ be solutions to the homogeneous n-th order linear equation $L(y) = 0$ on an interval I, that is,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (4)$$

then the linear combination $f = \sum_{i=1}^{k} \lambda_i f_i$ is also a solution to (4).

Linear Dependence and Independence of Functions

In Linear Algebra, we learned that one can view the collection of all *functions* defined on a common interval as a **vector space**, where linear dependence and independence can be defined respectively.

Definition (Linear Dependence and Independence)

A set of functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ are linearly dependent on an interval I if $\exists$ $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \ \forall \ x \in I,$$

that is, the linear combination is a zero function. If the set of functions is not linearly dependent, it is linearly independent.

Example:

- $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, $I = (-\pi, \pi)$: Linearly dependent
- $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^3$, $I = \mathbb{R}$: Linearly independent.

Linear Independence of Solutions to (4)

Consider the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (4)$$

Given n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$, we would like to test if they are independent or not.

Of course we can always go back to the definition but it is clumsy...

Recall: In Linear Algebra, to test if n vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly independent, we can compute the determinant of the matrix

$$\mathbf{V} := \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

If $\det \mathbf{V} = 0$, they are linearly dependent; if $\det \mathbf{V} \neq 0$, they are linearly independent.

Criterion of Linearly Independent Solutions

Consider the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (4)$$

To test the linear independence of n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ to (4), we can use the following theorem.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be n solutions to the homogeneous linear n-th order DE (4) on an interval I. They are linearly independent on I

$$\iff W(f_1, f_2, \ldots, f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \neq 0.$$

Fundamental Set of Solutions

We are interested in describing the *solution space*, that is, the subspace spanned by the solutions to the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0. \tag{4}$$

How?

Recall: In Linear Algebra, we describe a subspace by its *basis*: any vector in the subspace can be represented by a linear combination of the elements in the basis, and these elements are linearly independent.

Similar things can be done here.

Definition (Fundamental Set of Solutions)

Any set $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ of n linearly independent solutions to the homogeneous linear n-th order DE (4) on an interval I is called a fundamental set of solutions.

General Solutions to Homogeneous Linear DE

General solution to an n-th order ODE:
An n-parameter family of solutions that can contains *all* solutions.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ *be a fundamental set of solutions to the homogeneous linear* n*-th order DE* (4) *on an interval* I*. Then the* ***general solution*** *to* (4) *is*

$$y = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x),$$

where $\{c_i \mid i = 1, 2, \ldots, n\}$ *are arbitrary constants.*

Examples

Example

Consider the DE

$$\frac{d^2y}{dx^2} = y.$$

Check that both $y = e^x$ and $y = e^{-x}$ are solutions to the equation. Derive the general solution to the DE.

A: The linear DE is homogeneous.

We see that $\frac{d^2}{dx^2} e^x = \frac{d}{dx} e^x = e^x$, and $\frac{d^2}{dx^2} e^{-x} = \frac{d}{dx} - e^{-x} = e^{-x}$. Hence they are both solutions to the homogeneous linear second-order DE. Since

$$\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0,$$

the two solutions are linearly independent. Hence, the general solution can be written as $\boxed{y = c_1 e^x + c_2 e^{-x}, \ \ c_1, c_2 \in \mathbb{R}}$.

General Solutions to Nonhomogeneous Linear DE

Nonhomogeneous linear n-th order ODE takes the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

or equivalently, $L(y) = g(x), \; L := \sum_{i=0}^{n} a_i(x)D^i$

where $g(x)$ is not a zero function.

How to find its general solution?

Idea:

- Find the general solution y_c to the *homogeneous* equation $L(y) = 0$.
- Find **a** solution y_p to the *nonhomogeneous* equation $L(y) = g(x)$.
- The general solution $y = y_c + y_p$.

General Solutions to Nonhomogeneous Linear DE

Nonhomogeneous :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

Homogeneous :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (4)$$

Theorem

Let y_p *be* any *particular solution to the nonhomogeneous linear* n*-th order DE* (1) *on an interval* I*, and* y_c *be the* general solution *to the associated homogeneous linear* n*-th order DE* (4) *on* I*, then the general solution to* (1) *is*

$$y = y_c + y_p.$$

Proof of the Theorem

Proof: Let $y = f(x)$ be any solution to the nonhomogeneous linear n-th order DE (1), that is, $L(y) = g(x)$.

Now, since both y_p and f are solutions to $L(y) = g(x)$, we have

$$0 = L(f) - L(y_p) = L(f - y_p).$$

Hence, $(f - y_p)$ is a solution to the homogeneous linear n-th order DE (4).

Therefore, any solution to (1) can be represented by the sum of a solution to (4) and the particular solution y_p.

Examples

Example

Consider the DE

$$\frac{d^2y}{dx^2} = y + 9.$$

Derive the general solution to the DE.

A: The linear DE is nonhomogeneous. The associated homogeneous equation $\frac{d^2y}{dx^2} = y$ has the following general solution:

$$y = c_1 e^x + c_2 e^{-x},\ \ c_1, c_2 \in \mathbb{R}.$$

There is an obvious particular solution $y = -9$.

Hence, the general solution can be written as

$$\boxed{y = c_1 e^x + c_2 e^{-x} - 9,\ \ c_1, c_2 \in \mathbb{R}}$$

Superposition Principle for Nonhomogeneous Equations

For nonhomogeneous linear differential equations, we have the following superposition principle.

Theorem (Superposition Principle: Nonhomogeneous Equations)

Let $f_i(x)$ be a particular solution to the nonhomogeneous n-th order linear equation $L(y) = g_i(x)$ on an interval I, for $i = 1, 2, \ldots, k$. Then the linear combination $f = \sum_{i=1}^{k} \lambda_i f_i$ is a particular solution to the nonhomogeneous n-th order linear equation

$$L(y) = \sum_{i=1}^{k} \lambda_i g_i(x).$$

Short Recap

- Initial-Value Problems (IVP) vs. Boundary-Value Problems (BVP)
- Homogeneous vs Nonhomogeneous Linear ODE
- Fundamental set of solutions and General Solutions

Self-Practice Exercises

4-1: 1, 9, 13, 17, 21, 25, 35