Chapter 2: First-Order Differential Equations – Part 2

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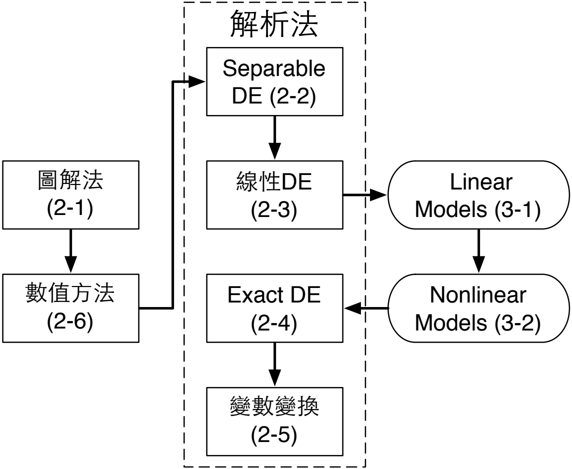
Organization of Lectures in Chapter 2 and 3

We will not follow the order in the textbook. Instead,



今天，我要出一道微分方程的考題，我可以從哪邊下手？

One proposal: reverse engineering – 先寫下解答，再反推回去方程式

1. Set up the solution curve: $G(x, y) = 0$ (can be an implicit solution) and an initial point (x_0, y_0).
2. Compute the **differential** of $G(x, y)$:

$$d(G(x, y)) = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$$

3. Since $G(x, y) = 0$, we have

$$0 = d(G(x, y)) = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$$

4. Let $\frac{\partial G}{\partial x} = M(x, y)$ and $\frac{\partial G}{\partial y} = N(x, y)$. Then, we have a DE:

$$M(x, y)\, dx + N(x, y)\, dy = 0 \implies \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

解題者觀點：看到一個一階常微分方程，若能將其化為

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

We can get the solution: $F(x, y) = c$, where $c = F(x_0, y_0)$.

Note: the function $F(x, y)$ you get may not be the same as the designer's choice $G(x, y)$.

Because the designer chose $G(x, y) = 0$ as his/her solution, while what you get is $F(x, y) = F(x_0, y_0)$.

Nevertheless, $G(x, y) = F(x, y) - F(x_0, y_0)$.

We shall develop a general method of solving this kind of DE based on the above observation.

Exact Differential and Exact Equation

Definition (Exact Equation)

A differential expression $M(x,y)\,dx + N(x,y)\,dy$ is an **Exact Differential** if it is the differential of some function $z = F(x,y)$, that is,

$$dz = M(x,y)\,dx + N(x,y)\,dy.$$

A first-order DE of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be an **Exact Equation** if the LHS is an exact differential.

Question: How to check if a differential expression is an exact differential?

Hint: $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$.

Criterion for an Exact Differential

Theorem

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives. Then,

$$M(x, y)\,dx + N(x, y)\,dy \text{ is an exact differential} \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof.

"$\Rightarrow$": Simply because $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$.

"$\Leftarrow$": We just need to construct a function $z = F(x, y)$ such that

$$dz = M(x, y)\,dx + N(x, y)\,dy.$$

In fact, this is the procedure of solving an exact DE. We will outline the procedure later.

Solving an Exact DE

Example

Solve $\left(e^{2y} - y\cos(xy)\right)\,dx + \left(2xe^{2y} - x\cos(xy) + 2y\right)\,dy = 0.$

A: Let $M(x, y) = e^{2y} - y\cos(xy)$ and $N(x, y) = 2xe^{2y} - x\cos(xy) + 2y$.

- Check if the DE is exact:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(e^{2y} - y\cos(xy)\right) = 2e^{2y} - \cos(xy) + xy\sin(xy)$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(2xe^{2y} - x\cos(xy) + 2y\right) = 2e^{2y} - \cos(xy) + xy\sin(xy)$$

- Since $M = \frac{\partial F}{\partial x}$ and we want to find F, why not integrate M with respect to x?

$$F(x, y) = \int \left\{e^{2y} - y\cos(xy)\right\}\,dx + g(y) = e^{2y}x - \sin(xy) + g(y).$$

Solving an Exact DE

Example

Solve $\left(e^{2y} - y\cos(xy)\right)\,dx + \left(2xe^{2y} - x\cos(xy) + 2y\right)\,dy = 0.$

A: Let $M(x, y) = e^{2y} - y\cos(xy)$ and $N(x, y) = 2xe^{2y} - x\cos(xy) + 2y.$ So far we found that $F(x, y) = e^{2y}x - \sin(xy) + g(y)$ where $g(y)$ is yet to be determined.

- To find $g(y)$, we use the fact that $N = \frac{\partial F}{\partial y}$:

$$2xe^{2y} - x\cos(xy) + 2y = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(e^{2y}x - \sin(xy) + g(y)\right)$$
$$= 2xe^{2y} - x\cos(xy) + g'(y) \implies \frac{dg}{dy} = 2y \implies g(y) = y^2$$

Hence, $F(x, y) = xe^{2y} - \sin(xy) + y^2$, and the implicit solution is

$$xe^{2y} - \sin(xy) + y^2 = c.$$

Solving an Exact DE $M(x, y)\,dx + N(x, y)\,dy = 0$

Goal: Find $z = F(x, y)$ such that $dz = M(x, y)\,dx + N(x, y)\,dy = 0$.

General Procedure of Solving an DE

1. Transform DE into the differential form: $M(x, y)\,dx + N(x, y)\,dy = 0$.
2. Verify if it is exact: $\dfrac{\partial M}{\partial y} \overset{?}{=} \dfrac{\partial N}{\partial x}$
3. Integrate M with respect to x (or N with respect to y):
   $F(x, y) = \int M dx + g(y)$ (or $F(x, y) = \int N dy + h(x)$)
4. Take partial derivative with respect to y (or x):
   $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M dx\right) + g'(y) = N(x, y) \qquad \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\int N dy\right) + h'(x) = M(x, y)$$
   $$\implies g(y) = \int \left(N - \frac{\partial}{\partial y}\int M dx\right) dy \qquad \implies h(x) = \int \left(M - \frac{\partial}{\partial x}\int N dy\right) dx$$

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y},\ y(1) = -1$

A:

$$\frac{dy}{dx} = \frac{20}{xy} - \frac{2y}{x} - \frac{3x}{y} = \frac{20 - 2y^2 - 3x^2}{xy}$$

$$\Longrightarrow \overbrace{(3x^2 + 2y^2 - 20)}^{M(x,y)}\, dx + \overbrace{(xy)}^{N(x,y)}\, dy = 0$$

Check if this equation is exact: $\frac{\partial M}{\partial y} = 4y \neq \frac{\partial N}{\partial x} = y$.

Can we make it exact, by multiplying both M and N with some $\mu(x, y)$?

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y} \implies \overbrace{(3x^2 + 2y^2 - 20)}^{M(x,y)}\,dx + \overbrace{(xy)}^{N(x,y)}\,dy = 0$

Goal: find $\mu(x, y)$ such that $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$. Let $\mu_x := \frac{\partial \mu}{\partial x}$, $\mu_y := \frac{\partial \mu}{\partial y}$.

$$\frac{\partial(\mu M)}{\partial y} = \mu_y M + M_y \mu = (3x^2 + 2y^2 - 20)\mu_y + 4y\mu$$

$$\frac{\partial(\mu N)}{\partial x} = \mu_x N + N_x \mu = (xy)\mu_x + y\mu$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \iff (3x^2 + 2y^2 - 20)\mu_y + 4y\mu = (xy)\mu_x + y\mu$$

This is a **PDE**?! How to solve it?

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y} \implies \overbrace{(3x^2 + 2y^2 - 20)}^{M(x,y)}\,dx + \overbrace{(xy)}^{N(x,y)}\,dy = 0$

Focus on finding a function $\mu(x, y)$ such that

$$(3x^2 + 2y^2 - 20)\mu_y + 4y\mu = (xy)\mu_x + y\mu$$

Let's make some restriction: how about finding μ that only depends on x?

$$4y\mu = (xy)\mu_x + y\mu \implies xy\frac{d\mu}{dx} = 3y\mu \implies \frac{d\mu}{dx} = \frac{3\mu}{x} \implies \mu = x^3 \quad \text{(works!)}$$

How about finding μ that only depends on y?

$$(3x^2 + 2y^2 - 20)\mu_y + 4y\mu = y\mu \implies \frac{d\mu}{dy} = -\frac{3y}{3x^2 + 2y^2 - 20}\mu \quad \text{(still hard!)}$$

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y} \implies \overbrace{x^3(3x^2 + 2y^2 - 20)}^{\widetilde{M}(x,y)}\, dx + \overbrace{(x^4 y)}^{\widetilde{N}(x,y)}\, dy = 0$

Finally we multiply both $M(x, y)$ and $N(x, y)$ with $\mu(x) = x^3$ (see above).

We then solve it by the procedures discussed before:

$$\widetilde{N} = \frac{\partial F}{\partial y} \implies F(x, y) = \int \widetilde{N} dy = \frac{1}{2} x^4 y^2 + h(x)$$

$$\widetilde{M} = \frac{\partial F}{\partial x} \implies x^3(3x^2 + 2y^2 - 20) = \frac{\partial}{\partial x}\left(\frac{1}{2} x^4 y^2\right) + \frac{dh}{dx} = 2x^3 y^2 + \frac{dh}{dx}$$

$$\implies \frac{dh}{dx} = 3x^5 - 20x^3 \implies h(x) = \frac{1}{2} x^6 - 5x^4$$

$$\implies F(x, y) = \frac{1}{2} x^4 y^2 + \frac{1}{2} x^6 - 5x^4$$

Nonexact DE Made Exact

Example

Solve $\frac{dy}{dx} = \frac{20}{xy} - \frac{2y}{x} - \frac{3x}{y}, \ y(1) = -1$

We arrive at an implicit solution: $F(x, y) = \frac{1}{2}x^4y^2 + \frac{1}{2}x^6 - 5x^4 = c$.

Plug in the initial condition, we get $\frac{1}{2}x^4y^2 + \frac{1}{2}x^6 - 5x^4 = c = -4$.

To get an explicit solution, we see that

$$\begin{aligned} \frac{1}{2}x^4y^2 + \frac{1}{2}x^6 - 5x^4 = -4 &\implies y^2 = 10 - x^2 - 8x^{-4} \\ &\implies y = \pm\sqrt{10 - x^2 - 8x^{-4}} \\ &\implies y = -\sqrt{10 - x^2 - 8x^{-4}} \end{aligned}$$

Exercise. Find an interval of definition for the above solution.

Nonexact DE $M(x, y)\,dx + N(x, y)\,dy = 0$ Made Exact

Nonexact DE: $M_y - N_x := \Delta(x, y) \neq 0$

Key Idea 1: Introduce a function $\mu(x, y)$ (*integrating factor*) to ensure

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \iff \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

However in general this is a PDE which may be hard to solve.

Key Idea 2: Restrict $\mu(x, y)$ to be $\mu(x)$ or $\mu(y)$.

$$\begin{aligned}
\text{Plan A: } \mu(x, y) = \mu(x) &\implies \mu_y = 0 \implies \mu M_y = \mu_x N + \mu N_x \\
&\implies \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = \frac{\Delta}{N}\mu \\
\text{Plan B: } \mu(x, y) = \mu(y) &\implies \mu_x = 0 \implies \mu_y M + \mu M_y = \mu N_x \\
&\implies \frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu = -\frac{\Delta}{M}\mu
\end{aligned}$$

Nonexact DE $M(x,y)\,dx + N(x,y)\,dy = 0$ Made Exact

Nonexact DE: $M_y - N_x := \Delta(x,y) \neq 0$

$$\text{Plan A: } \mu(x,y) = \mu(x) \implies \frac{d\mu}{dx} = \frac{\Delta}{N}\mu$$

$$\text{Plan B: } \mu(x,y) = \mu(y) \implies \frac{d\mu}{dy} = -\frac{\Delta}{M}\mu$$

Key Idea 3: Which plan should we choose? Choose it based on $\Delta(x,y)$:

- If $\dfrac{\Delta}{N}$ only depends on x, then $\dfrac{d\mu}{dx} = \dfrac{\Delta}{N}\mu$ is separable. Plan A!
- If $\dfrac{\Delta}{M}$ only depends on y, then $\dfrac{d\mu}{dy} = -\dfrac{\Delta}{N}\mu$ is separable. Plan B!

今天，我要出一道微分方程的考題，我可以從哪邊下手？

One proposal: reverse engineering –
先寫下解答，再反推回去方程式

Another proposal: substitution of variables –
先寫下簡單的方程式，再把其中的 x 與 y 代換成 x, y 的函數

1. Write down a simple DE: $\dfrac{du}{dx} = f(u, x)$.
2. Replace u by $G(x, y)$:
$$\frac{d(G(x,y))}{dx} = f(G(x,y), x) \implies \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = f(G(x,y), x)$$
3. We get a new DE: $\dfrac{dy}{dx} = \dfrac{f(G(x,y), x) - G_x(x,y)}{G_y(x,y)}$.

解題者觀點：將上述方程式化為 u 和 x 的方程式 –

$$\frac{dy}{dx} = \frac{f(G(x,y),x) - G_x(x,y)}{G_y(x,y)} \implies \frac{du}{dx} = f(u,x)$$

Key: setting $u := G(x,y)$. 但，要找到合適的G，非常困難！

We can only "guess" based on **inspection** and **experience**.

In this lecture we cover 3 classes of DE where we know how to pick G:

- $\frac{dy}{dx} = f(Ax + By + C)$ and some other special equations
- Homogeneous Equations
- Bernoulli's Equation

Solve $\frac{dy}{dx} = f(Ax + By + C)$

Obviously, we shall set $u := Ax + By + C$. We have:

$$u = Ax + By + C \implies \frac{du}{dx} = A + B\frac{dy}{dx} = A + Bf(u).$$

The new DE is easy to solve by separation of variables, since

$$\frac{du}{dx} = A + Bf(u)$$

is separable.

Example

Example

Solve $\frac{dy}{dx} = (-2x+y)^2 - 7,\ y(0) = 0.$

A: Set $u = -2x + y \implies \frac{du}{dx} = -2 + \frac{dy}{dx} = u^2 - 9 = (u-3)(u+3).$

We solve u as follows:

$$\frac{du}{(u-3)(u+3)} = dx,\ u \neq \pm 3 \implies \int \frac{1}{6}\left(\frac{1}{u-3} - \frac{1}{u+3}\right) du = x + c$$
$$\implies \frac{1}{6}\ln|u-3| - \frac{1}{6}\ln|u+3| = x + c.$$

Plug in the initial condition $y(0) = 0 \implies u(0) = 0$, we get $c = 0$ and

$$\frac{3-u}{3+u} = e^{6x} \implies u = 3\frac{1-e^{6x}}{1+e^{6x}} \implies \boxed{y = 2x + 3\frac{1-e^{6x}}{1+e^{6x}}.}$$

Homogeneous Functions

Definition (Homogeneous Function)

A function $f(x, y)$ is **homogeneous** of degree α if for all x, y,

$$f(tx, ty) = t^{\alpha} f(x, y) \text{ for some } \alpha.$$

Example (Determine if a function is homogeneous and its degree α)

$f(x, y) = x^3 + y^3 + xy^2$	$f(tx, ty) = t^3 f(x, y)$	Yes, $\alpha = 3$.
$f(x, y) = x^3 + y^3 + x^2$	$f(tx, ty) = t^3(x^3 + y^3) + t^2 x^2$	No.
$f(x, y) = \sqrt{x^5 + x^2 y^3}$	$f(tx, ty) = t^{2.5} f(x, y)$	Yes, $\alpha = 2.5$.
$f(x, y) = e^{x+y}$	$f(tx, ty) = e^t f(x, y)$	No.
$f(x, y) = (x + \sqrt{xy})e^{\frac{2y}{x}}$	$f(tx, ty) = tf(x, y)$	Yes, $\alpha = 1$.

Homogeneous Functions

Definition (Homogeneous Function)

A function $f(x, y)$ is **homogeneous** of degree α if for all x, y,

$$f(tx, ty) = t^{\alpha} f(x, y) \text{ for some } \alpha.$$

Lemma

*If a function $f(x, y)$ is **homogeneous** of degree α, then*

$$f(x, y) = x^{\alpha} f(1, y/x) = y^{\alpha} f(x/y, 1).$$

Proof. The first equality is proved by setting $t = 1/x$ and hence $f(1, y/x) = (1/x)^{\alpha} f(x, y) \implies f(x, y) = x^{\alpha} f(1, y/x)$. The second equality is proved similarly by setting $t = 1/y$.

Homogeneous Equations

Definition (Homogeneous Equation)

Consider a DE in the differential form: $M(x, y)\,dx + N(x, y)\,dy = 0$.
If both M and N are homogeneous of the same degree α, we called this DE **homogeneous**.

From the previous Lemma, we get

$$M(x,y) = x^\alpha M(1, y/x) = y^\alpha M(x/y, 1) \qquad N(x,y) = x^\alpha N(1, y/x) = y^\alpha N(x/y, 1)$$

Hence, $M(x, y)\,dx + N(x, y)\,dy = 0$ implies

$$M(1, y/x)\,dx + N(1, y/x)\,dy = M(x/y, 1)\,dx + N(x/y, 1)\,dy = 0.$$

A natural substitution: Set $u := y/x$ or $v := x/y$.

Solving a Homogeneous Equation

To solve a homogeneous equation $M(x,y)\,dx + N(x,y)\,dy = 0$, first we set $u := y/x$ and we get

$$
\begin{aligned}
M(x,y)\,dx + N(x,y)\,dy = 0 &\implies M(1, y/x)\,dx + N(1, y/x)\,dy = 0 \\
&\implies M(1,u)\,dx + N(1,u)\,dy = 0 \\
\text{移項} &\implies \frac{dy}{dx} = \frac{-M(1,u)}{N(1,u)} \\
\tfrac{dy}{dx} = \tfrac{d(ux)}{dx} = x\tfrac{du}{dx} + u &\implies x\frac{du}{dx} + u = \frac{-M(1,u)}{N(1,u)} \\
\text{移項} &\implies \boxed{\frac{du}{dx} = -\frac{1}{x}\left\{u + \frac{M(1,u)}{N(1,u)}\right\}}
\end{aligned}
$$

This new equation is separable and hence easy to solve.

Note: we can also begin with setting $v := x/y$, depending on which will lead to a simpler from.

Example

Example

Solve $(x^2+y^2)\,dx+(x^2-xy)\,dy=0,\ y(1)=0$

A: Note that this equation is not exact, $\Delta = M_y - N_x = y - 2x$, and hence both $\frac{\Delta}{N}$ and $\frac{\Delta}{M}$ will depend on x and y. 2-4 technique won't work!

Instead, we see that this equation is homogeneous. Hence we set $u := y/x$, i.e., $y = ux$, and get

$$
\begin{aligned}
& x^2(1+u^2)\,dx + x^2(1-u)\,d(ux) = 0 \\
\overset{d(ux)=udx+xdu}{\implies}\ & (1+u^2)\,dx + (1-u)(udx+xdu) = 0 \\
\implies\ & (1+u)\,dx + (1-u)xdu = 0 \implies \frac{dx}{x} + \frac{1-u}{1+u}du \\
\overset{u(1)=y(1)/1=0}{\implies}\ & \ln|x| - u + 2\ln|1+u| = c = 0 \\
\implies\ & \frac{x(1+y/x)^2}{e^{y/x}} = 1 \implies \boxed{x^2+y^2 = xe^{\frac{y}{x}}}
\end{aligned}
$$

Bernoulli's Equation

Definition (Bernoulli's Equation)

The DE $\dfrac{dy}{dx} + P(x)y = f(x)y^r$ where $r \in \mathbb{R}$ is any real number.

For $r = 0, 1$, the equation is linear.

For $r \neq 0, 1$, we shall use the substitution $u := y^{1-r}$ to make it linear:

$$u = y^{1-r} \implies y = u^{\frac{1}{1-r}} \implies \begin{cases} \frac{dy}{dx} = \frac{1}{1-r} u^{\frac{r}{1-r}} \frac{du}{dx} \\ P(x)y = P(x)u^{\frac{1}{1-r}} \\ f(x)y^r = f(x)u^{\frac{r}{1-r}} \end{cases}$$

$$\frac{dy}{dx} + P(x)y = f(x)y^r \implies \frac{1}{1-r} u^{\frac{r}{1-r}} \frac{du}{dx} + P(x)u^{\frac{1}{1-r}} = f(x)u^{\frac{r}{1-r}}$$

$$\implies \boxed{\frac{du}{dx} + (1-r)P(x)u = (1-r)f(x)} : \text{ Linear!}$$

Example

Example

Solve $x\dfrac{dy}{dx} + y = x^2y^2$, $y(1) = 1$

A: Rewrite the equation into $\dfrac{dy}{dx} + \dfrac{y}{x} = xy^2 \implies$ Bernoulli, $r = 2$.
Hence, we set $u = y^{1-r} = 1/y$. ($y \neq 0$)

$$\frac{dy}{dx} = \frac{d(u^{-1})}{dx} = -\frac{1}{u^2}\frac{du}{dx} \implies \frac{1}{u^2}\frac{du}{dx} + \frac{1}{ux} = \frac{x}{u^2} \implies \boxed{\frac{du}{dx} + x^{-1}u = x}$$

Solve u (exercise!) and we get $u = 2x - x^2$,

$$\implies \boxed{y = \frac{1}{2x - x^2},\ \ 0 < x < 2.}$$

Alternative Substitution

Example

Solve $x\dfrac{dy}{dx} + y = x^2y^2,\ y(1) = 1$

There is actually a much simpler approach, if you find a better substitution!

Can you find it? (exercise!)

Short Recap

- Exact differential and exact equation
- Nonexact equation made exact: integrating factor
- Substitution of variables – simplify your equation
- $\frac{dy}{dx} = f(Ax + By + C)$
- Homogeneous equations
- Bernoulli's equation

Self-Practice Exercises

2-4: 1, 7, 9, 11, 13, 15, 17, 27, 33, 35, 39

2-5: 1, 7, 9, 13, 17, 19, 21, 25, 27, 35