Chapter 2: First-Order Differential Equations – Part 2

王奕翔

Department of Electrical Engineering
National Taiwan University

ihwang@ntu.edu.tw

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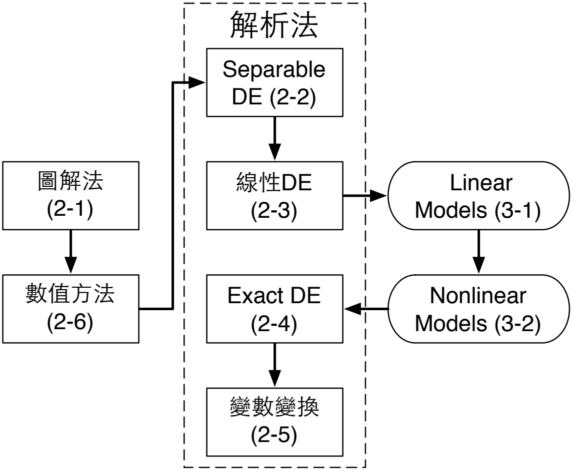
Organization of Lectures in Chapter 2 and 3

We will not follow the order in the textbook. Instead,



今天，我要出一道微分方程的考題，我可以從哪邊下手？

One proposal: reverse engineering – 先寫下解答，再反推回去方程式

1. Set up the solution curve: $G(x, y) = 0$ (can be an implicit solution) and an initial point (x_0, y_0).
2. Compute the **differential** of $G(x, y)$:

$$d(G(x,y)) = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$$

3. Since $G(x, y) = 0$, we have

$$0 = d(G(x,y)) = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$$

4. Let $\frac{\partial G}{\partial x} = M(x, y)$ and $\frac{\partial G}{\partial y} = N(x, y)$. Then, we have a DE:

$$M(x,y)\,dx + N(x,y)\,dy = 0 \implies \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

解題者觀點：看到一個一階常微分方程，若能將其化為

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

We can get the solution: $F(x, y) = c$, where $c = F(x_0, y_0)$.

Note: the function $F(x, y)$ you get may not be the same as the designer's choice $G(x, y)$.

Because the designer chose $G(x, y) = 0$ as his/her solution, while what you get is $F(x, y) = F(x_0, y_0)$.

Nevertheless, $G(x, y) = F(x, y) - F(x_0, y_0)$.

We shall develop a general method of solving this kind of DE based on the above observation.

Exact Differential and Exact Equation

Definition (Exact Equation)

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **Exact Differential** if it is the differential of some function $z = F(x, y)$, that is,

$$dz = M(x, y)\,dx + N(x, y)\,dy.$$

A first-order DE of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be an **Exact Equation** if the LHS is an exact differential.

Question: How to check if a differential expression is an exact differential?

Hint: $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$.

Criterion for an Exact Differential

Theorem

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives. Then,

$$M(x,y)\,dx + N(x,y)\,dy \text{ is an exact differential } \Longleftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof.

"$\Rightarrow$": Simply because $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$.

"$\Leftarrow$": We just need to construct a function $z = F(x,y)$ such that

$$dz = M(x,y)\,dx + N(x,y)\,dy.$$

In fact, this is the procedure of solving an exact DE. We will outline the procedure later.

Solving an Exact DE

Example

Solve $\left(e^{2y} - y\cos(xy)\right)\,dx + \left(2xe^{2y} - x\cos(xy) + 2y\right)\,dy = 0.$

A: Let $M(x, y) = e^{2y} - y\cos(xy)$ and $N(x, y) = 2xe^{2y} - x\cos(xy) + 2y$.

- Check if the DE is exact:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(e^{2y} - y\cos(xy)\right) = 2e^{2y} - \cos(xy) + xy\cos(xy)$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(2xe^{2y} - x\cos(xy) + 2y\right) = 2e^{2y} - \cos(xy) + xy\cos(xy)$$

- Since $M = \frac{\partial F}{\partial x}$ and we want to find F, why not integrate M with respect to x?

$$F(x, y) = \int \left\{e^{2y} - y\cos(xy)\right\}\,dx + g(y) = e^{2y}x - \sin(xy) + g(y).$$

Solving an Exact DE

Example

Solve $\left(e^{2y} - y\cos(xy)\right)\,dx + \left(2xe^{2y} - x\cos(xy) + 2y\right)\,dy = 0.$

A: Let $M(x, y) = e^{2y} - y\cos(xy)$ and $N(x, y) = 2xe^{2y} - x\cos(xy) + 2y$. So far we found that $F(x, y) = e^{2y}x - \sin(xy) + g(y)$ where $g(y)$ is yet to be determined.

- To find $g(y)$, we use the fact that $N = \frac{\partial F}{\partial y}$:

$$2xe^{2y} - x\cos(xy) + 2y = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(e^{2y}x - \sin(xy) + g(y)\right)$$

$$= 2xe^{2y} - x\cos(xy) + g'(y) \implies \frac{dg}{dy} = 2y \implies g(y) = y^2$$

Hence, $F(x, y) = xe^{2y} - \sin(xy) + y^2$, and the implicit solution is

$$xe^{2y} - \sin(xy) + y^2 = c.$$

Solving an Exact DE $M(x, y)\,dx + N(x, y)\,dy = 0$

Goal: Find $z = F(x, y)$ such that $dz = M(x, y)\,dx + N(x, y)\,dy = 0$.

General Procedure of Solving an DE

1. Transform DE into the differential form: $M(x, y)\,dx + N(x, y)\,dy = 0$.
2. Verify if it is exact: $\dfrac{\partial M}{\partial y} \stackrel{?}{=} \dfrac{\partial N}{\partial x}$
3. Integrate M with respect to x (or N with respect to y):
   $F(x, y) = \int M dx + g(y)$ (or $F(x, y) = \int N dy + h(x)$)
4. Take partial derivative with respect to y (or x):
   $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M dx\right) + g'(y) = N(x, y) \qquad \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\int N dy\right) + h'(x) = M(x, y)$$
   $$\Longrightarrow\ g(y) = \int \left(N - \frac{\partial}{\partial y}\int M dx\right) dy \qquad \Longrightarrow\ h(x) = \int \left(M - \frac{\partial}{\partial x}\int N dy\right) dx$$

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y}, \ y(1) = -1$

A:

$$\frac{dy}{dx} = \frac{20}{xy} - \frac{2y}{x} - \frac{3x}{y} = \frac{20 - 2y^2 - 3x^2}{xy}$$

$$\implies \overbrace{(3x^2 + 2y^2 - 20)}^{M(x,y)} dx + \overbrace{(xy)}^{N(x,y)} dy = 0$$

Check if this equation is exact: $\frac{\partial M}{\partial y} = 4y \neq \frac{\partial N}{\partial x} = y$.

Can we make it exact, by multiplying both M and N with some $\mu(x, y)$?

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y} \implies \overbrace{(3x^2 + 2y^2 - 20)}^{M(x,y)}\, dx + \overbrace{(xy)}^{N(x,y)}\, dy = 0$

Goal: find $\mu(x, y)$ such that $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$. Let $\mu_x := \frac{\partial \mu}{\partial x}$, $\mu_y := \frac{\partial \mu}{\partial y}$.

$$\frac{\partial(\mu M)}{\partial y} = \mu_y M + M_y \mu = (3x^2 + 2y^2 - 20)\mu_y + 4y\mu$$

$$\frac{\partial(\mu N)}{\partial x} = \mu_x N + N_x \mu = (xy)\mu_x + y\mu$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \iff (3x^2 + 2y^2 - 20)\mu_y + 4y\mu = (xy)\mu_x + y\mu$$

This is a **PDE**?! How to solve it?

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y} \implies \overbrace{(3x^2 + 2y^2 - 20)}^{M(x,y)}\, dx + \overbrace{(xy)}^{N(x,y)}\, dy = 0$

Focus on finding a function $\mu(x, y)$ such that

$$(3x^2 + 2y^2 - 20)\mu_y + 4y\mu = (xy)\mu_x + y\mu$$

Let's make some restriction: how about finding μ that only depends on x?

$$4y\mu = (xy)\mu_x + y\mu \implies xy\frac{d\mu}{dx} = 3y\mu \implies \frac{d\mu}{dx} = \frac{3\mu}{x} \implies \mu = x^3 \quad \text{(works!)}$$

How about finding μ that only depends on y?

$$(3x^2 + 2y^2 - 20)\mu_y + 4y\mu = y\mu \implies \frac{d\mu}{dy} = -\frac{3y}{3x^2 + 2y^2 - 20}\mu \quad \text{(still hard!)}$$

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y} \implies \overbrace{x^3(3x^2 + 2y^2 - 20)}^{\widetilde{M}(x,y)}\,dx + \overbrace{(x^4 y)}^{\widetilde{N}(x,y)}\,dy = 0$

Finally we multiply both $M(x, y)$ and $N(x, y)$ with $\mu(x) = x^3$ (see above).

We then solve it by the procedures discussed before:

$$\widetilde{N} = \frac{\partial F}{\partial y} \implies F(x, y) = \int \widetilde{N} dy = \frac{1}{2}x^4y^2 + h(x)$$

$$\widetilde{M} = \frac{\partial F}{\partial x} \implies x^3(3x^2 + 2y^2 - 20) = \frac{\partial}{\partial x}\left(\frac{1}{2}x^4y^2\right) + \frac{dh}{dx} = 2x^3y^2 + \frac{dh}{dx}$$

$$\implies \frac{dh}{dx} = 3x^5 - 20x^3 \implies h(x) = \frac{1}{2}x^6 - 5x^4$$

$$\implies F(x, y) = \frac{1}{2}x^4y^2 + \frac{1}{2}x^6 - 5x^4$$

Nonexact DE Made Exact

Example

Solve $\dfrac{dy}{dx} = \dfrac{20}{xy} - \dfrac{2y}{x} - \dfrac{3x}{y}, \; y(1) = -1$

We arrive at an implicit solution: $F(x, y) = \frac{1}{2}x^4y^2 + \frac{1}{2}x^6 - 5x^4 = c$.

Plug in the initial condition, we get $\frac{1}{2}x^4y^2 + \frac{1}{2}x^6 - 5x^4 = c = -4$.

To get an explicit solution, we see that

$$\begin{aligned} \frac{1}{2}x^4y^2 + \frac{1}{2}x^6 - 5x^4 = -4 &\implies y^2 = 10 - x^2 - 8x^{-4} \\ &\implies y = \pm\sqrt{10 - x^2 - 8x^{-4}} \\ &\implies y = -\sqrt{10 - x^2 - 8x^{-4}} \end{aligned}$$

Exercise. Find an interval of definition for the above solution.

Nonexact DE $M(x, y)\,dx + N(x, y)\,dy = 0$ Made Exact

Nonexact DE: $M_y - N_x := \Delta(x, y) \neq 0$

Key Idea 1: Introduce a function $\mu(x, y)$ (*integrating factor*) to ensure

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \iff \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

However in general this is a PDE which may be hard to solve.

Key Idea 2: Restrict $\mu(x, y)$ to be $\mu(x)$ or $\mu(y)$.

$$\begin{aligned}
\text{Plan A: } \mu(x, y) = \mu(x) &\implies \mu_y = 0 \implies \mu M_y = \mu_x N + \mu N_x \\
&\implies \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = \frac{\Delta}{N}\mu \\
\text{Plan B: } \mu(x, y) = \mu(y) &\implies \mu_x = 0 \implies \mu_y M + \mu M_y = \mu N_x \\
&\implies \frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu = -\frac{\Delta}{M}\mu
\end{aligned}$$

Nonexact DE $M(x,y)\,dx + N(x,y)\,dy = 0$ Made Exact

Nonexact DE: $M_y - N_x := \Delta(x,y) \neq 0$

$$\text{Plan A: } \mu(x,y) = \mu(x) \implies \frac{d\mu}{dx} = \frac{\Delta}{N}\mu$$

$$\text{Plan B: } \mu(x,y) = \mu(y) \implies \frac{d\mu}{dy} = -\frac{\Delta}{M}\mu$$

Key Idea 3: Which plan should we choose? Choose it based on $\Delta(x,y)$:

- If $\dfrac{\Delta}{N}$ only depends on x, then $\dfrac{d\mu}{dx} = \dfrac{\Delta}{N}\mu$ is separable. Plan A!
- If $\dfrac{\Delta}{M}$ only depends on y, then $\dfrac{d\mu}{dy} = -\dfrac{\Delta}{N}\mu$ is separable. Plan B!

Short Recap

- Exact Differential
- Exact Equation
- Nonexact Equation made Exact: Integrating Factor

Self-Practice Exercises

2-4: 1, 7, 9, 11, 13, 15, 17, 27, 33, 35, 39