Chapter 2: First-Order Differential Equations – Part 1

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First-Order Differential Equation

Throughout Chapter 2, we focus on solving the first-order ODE:

Problem

Find $y = \phi(x)$ satisfying

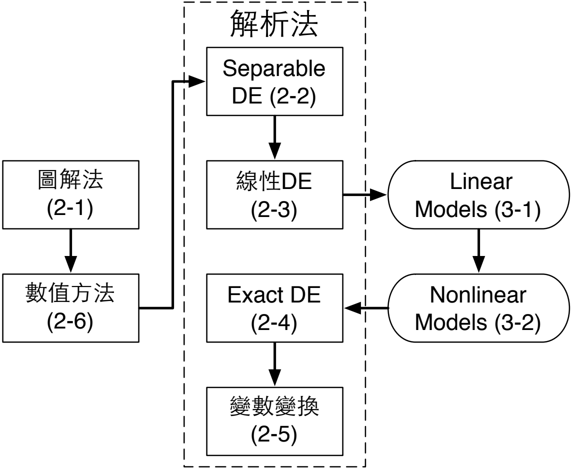
$$\frac{dy}{dx} = f(x, y), \text{ subject to } y(x_0) = y_0 \tag{1}$$

Methods of Solving First-Order ODE

1. Graphical Method (2-1)
2. Numerical Method (2-6, 9)
3. Analytic Method
   - Take antiderivative (*Calculus I, II*)
   - Separable Equations (2-2)
   - Solving Linear Equations (2-3)
   - Solving Exact Equations (2-4)
   - Solutions by Substitutions (2-5):
     homogeneous equations, Bernoulli's equation, $y' = Ax + By + C$.
4. Series Solution (6)
5. Transformation
   - Laplace Transform (7)
   - Fourier Series (11)
   - Fourier Transform (14)

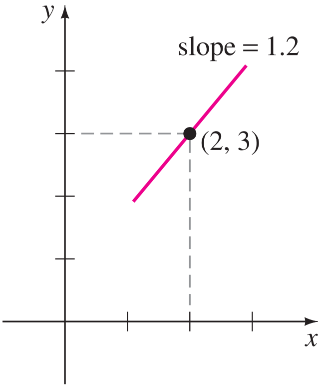
Organization of Lectures in Chapter 2 and 3

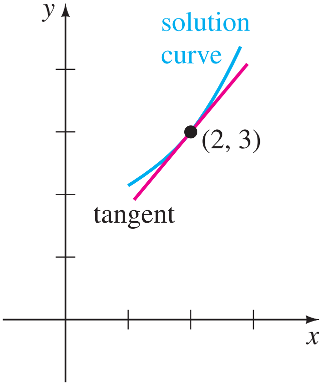
We will not follow the order in the textbook. Instead,



Example 1 (Zill&Wright p.36, Fig. 2.1.1.)

$$\frac{dy}{dx} = 0.2xy$$





Direction Fields

Key Observation

On the xy-plane, at a point (x_n, y_n), the first-order derivative

$$\left.\frac{dy}{dx}\right|_{x=x_n}$$

is the slope of the tangent line of the curve $y(x)$ at (x_n, y_n).

Hence, at every point on the xy-plane, one can *in principle* sketch an arrow indicating the direction of the tangent line.

From the initial point (x_0, y_0), one can connect all the arrows one by one and then sketch the solution curve. (土法煉鋼！)

Example 1 (Zill&Wright p.37, Fig. 2.1.3.)

$$\frac{dy}{dx} = 0.2xy$$

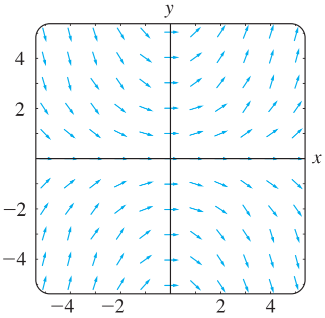


Figure : Direction Field

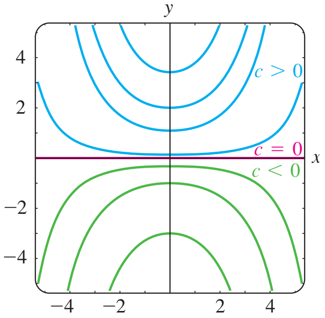
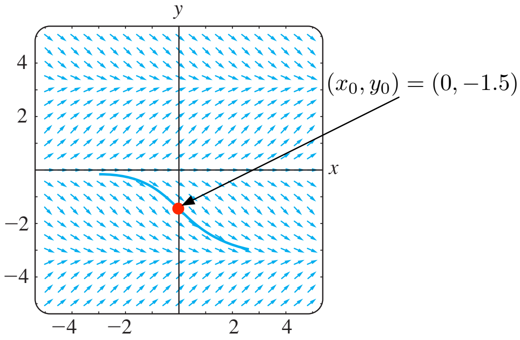


Figure : Family of Solution Curves

Example 2 (Zill&Wright p.37-38, Fig. 2.1.4.)

$$\frac{dy}{dx} = \sin y, \;\; y(0) = -1.5$$



Euler's Method

The graphical method of "connecting arrows" on the directional field can be mathematically thought of as follows:

Initial Point:	(x_0, y_0)
x Increment:	$x_1 = x_0 + h$
y Increment:	$y_1 = y_0 + h\left(\left.\frac{dy}{dx}\right\|_{x=x_0}\right) = y_0 + hf(x_0, y_0)$
Second Point:	(x_1, y_1)
$\vdots$	$\vdots$

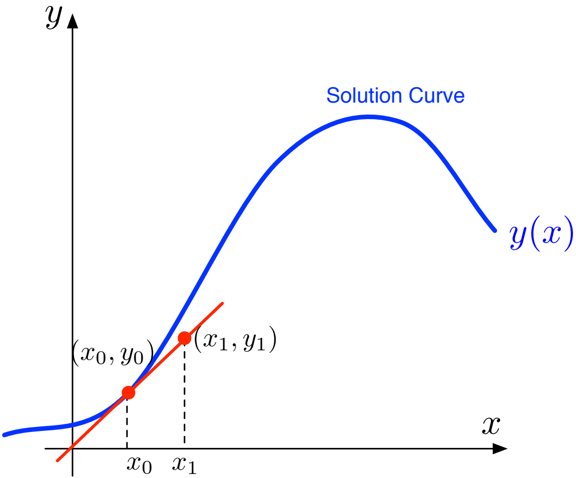
Euler's Method

Recursive Formula

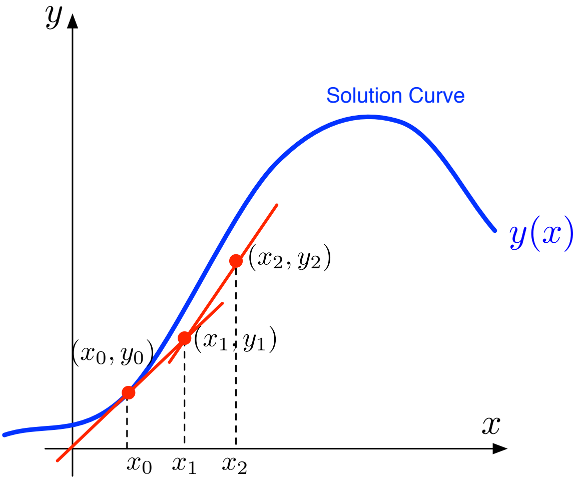
Let $h > 0$ be the recursive step size,

$$
\begin{aligned}
x_{n+1} &= x_n + h, \qquad & y_{n+1} &= y_n + hf(x_n, y_n), \qquad & \forall\ n \geq 0 \\
x_{n-1} &= x_n - h, \qquad & y_{n-1} &= y_n - hf(x_n, y_n), \qquad & \forall\ n \leq 0
\end{aligned}
$$

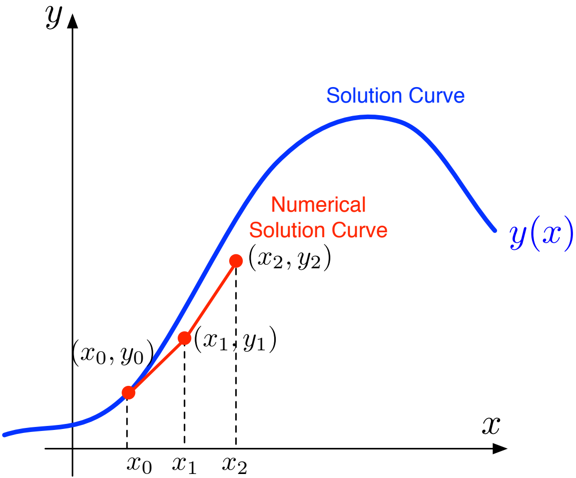
Illustration



Illustration



Illustration



Remarks

- The approximate numerical solution converges to the actual solution as $h \to 0$.
- Euler's method is just one simple numerical method for solving differential equations. Chapter 9 of the textbook introduces more advanced methods.

Solving (1) Analytically

Recall the first-order ODE (1) we would like to solve

Problem

Find $y = \phi(x)$ satisfying

$$\frac{dy}{dx} = f(x, y), \text{ subject to } y(x_0) = y_0 \tag{1}$$

We start by inspecting the equation and see if we can identify some special structure of it.

When $f(x, y)$ depends only on x

If $f(x, y) = g(x)$, then by what we learn in Calculus I & II,

$$\frac{dy}{dx} = g(x) \implies y(x) = \int_{x_0}^{x} g(t)\, dt + y_0$$

Method: Direct Integration

In the first-order ODE (1), if $f(x, y) = g(x)$ only depends on x, it can be solved by directly integrating the function $g(x)$.

When $f(x, y)$ depends only on x

Example

Solve

$$\frac{dy}{dx} = \frac{1}{x} + e^x, \text{ subject to } y(-1) = 0.$$

A: From calculus we know that the

$$\int \frac{1}{x} dx = \ln|x|, \int e^x dx = e^x$$

Plugging in the initial condition, we have

$$y(x) = \ln|x| + e^x - \frac{1}{e}, \ x < 0.$$

When $f(x, y)$ depends only on y

If $f(x, y) = h(y)$, then

$$\frac{dy}{dx} = h(y) \implies \frac{dy}{h(y)} = dx \overset{\text{integrate both sides}}{\implies} \int_{y_0}^{y} \frac{dy}{h(y)} = x - x_0$$

Assume that the antiderivative (不定積分、反導函數) of $1/h(y)$ is $H(y)$. That is,

$$\int \frac{1}{h(y)} dy = H(y).$$

Then, we have

$$H(y) - H(y_0) = x - x_0 \implies y(x) = H^{-1}(x - x_0 + H(y_0))$$

When $f(x, y)$ depends only on y

Example

Solve $\frac{dy}{dx} = (y-1)^2$.

A: Use the same principle, we have

$$\begin{aligned} \frac{dy}{dx} = (y-1)^2 &\implies \frac{dy}{(y-1)^2} = dx, \quad y \neq 1 \\ &\implies \frac{1}{1-y} = x + c, \quad \text{for some constant } c \\ &\implies y = 1 - \frac{1}{x+c}, \quad \text{for some constant } c, \quad \text{or } y = 1 \end{aligned}$$

Note: How about the constant function $y = 1$?
$\implies$ $y = 1$ is called a **singular solution**.

Table of Integrals

Function	**Antiderivative**
u^n	$\frac{u^{n+1}}{n+1} + C, \ n \neq -1$
u^{-1}	$\ln\lvert u\rvert + C$
a^u	$\frac{a^u}{\ln a} + C$
$\sin u$	$-\cos u + C$
$\cos u$	$\sin u + C$
$\tan u$	$-\ln\lvert\cos u\rvert + C$
$\cot u$	$\ln\lvert\sin u\rvert + C$
$\frac{1}{a^2+u^2}$	$\frac{1}{a}\tan^{-1}\frac{u}{a} + C$
$\frac{1}{\sqrt{a^2-u^2}}$	$\sin^{-1}\frac{u}{a} + C$
$\vdots$	$\vdots$

Separable Equations

Definition (Separable Equations)

If in (1) the function $f(x, y)$ on the right hand side takes the form $f(x, y) = g(x)h(y)$,, we call the first-order ODE **separable**, or to have **separable variables**.

Example (Are the following equations separable?)

- $\frac{dy}{dx} = x + y$ No.
- $\frac{dy}{dx} = e^{x+y}$ Yes.
- $\frac{dy}{dx} = x + y + xy + 1$ Yes, $\because x + y + xy + 1 = (x + 1)(y + 1)$.
- $\frac{dy}{dx} = x + y + xy$ No.

Separable Equations

General Procedure of Solving a Separable DE

1. 分別移項: $\frac{dy}{h(y)} = \frac{dx}{g(x)}$. 若分母會為零, check singular solutions!
2. 兩邊積分: $\int \frac{dy}{h(y)} = \int \frac{dx}{g(x)} \implies H(y) = G(x) + c.$
3. 代入條件: $c = H(y_0) - G(x_0)$.
4. 取反函數: $y = H^{-1}(G(x) + H(y_0) - G(x_0))$.
   Don't forget to check singular solutions!

Example

Example

Solve $\frac{dy}{dx} = k\frac{x}{y}$ subject to (i) $k = -1, y(-1) = -1$; (ii) $k = 1, y(0) = 1$.

A: $\frac{dy}{dx} = k\frac{x}{y} \implies \int y\ dy = \int kx\ dx \implies y^2 = kx^2 + c.$

Note that we require $y \neq 0$ so that the derivate $\frac{dy}{dx}$ is well-defined.

(i) Plug in the initial condition, we have: $c = 1 + 1 = 2$.
Hence $y = -\sqrt{2 - x^2}$, for $x \in \left(-\sqrt{2}, \sqrt{2}\right)$.

(ii) Plug in the initial condition, we have: $c = 0 + 1 = 1$.
Hence $y = \sqrt{x^2 + 1}$, for $x \in \mathbb{R}$.

Example

Example

Solve $\dfrac{dy}{dx} = x\sqrt{y}$ subject to $y(0) = 0$.

A: $\dfrac{dy}{dx} = x\sqrt{y} \overset{y\neq 0}{\Longrightarrow} \displaystyle\int y^{-1/2}\, dy = \int x\, dx \implies 2\sqrt{y} = \frac{1}{2}x^2 + c.$
Plug in the initial condition, we have $c = 0 \implies y = x^4/16$.

Check the singular solution $y = 0$
$\implies y = 0$ is also a solution (trivial, singular).

Both $y = x^4/16$ and $y = 0$ are solutions to the initial-value problem.

More Examples

Example

Solve $\dfrac{dy}{dx} = e^{x+y}$, subject to $y(0) = 0$.

Example

Solve $\dfrac{dy}{dx} = x + y + xy + 1$, subject to $y(0) = -1$.

Example

Solve $\dfrac{dy}{dx} = y^2 + 1$, subject to $y(0) = 0$.

Linear Equations

Linear First-Order ODE: such an ODE takes the following general form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The Standard Form of a Linear First-Order ODE

Find $y = \phi(x)$ satisfying

$$\frac{dy}{dx} + P(x)y = f(x), \text{ subject to } y(x_0) = y_0 \tag{2}$$

Useful Observations

Consider the derivative of the product of $y(x)$ and some function $\mu(x)$:

$$\begin{aligned}\frac{d(\mu y)}{dx} &= \mu\frac{dy}{dx} + y\frac{d\mu}{dx} \\ &= \mu\left\{-P(x)y + f(x)\right\} + y\frac{d\mu}{dx} \quad \text{(Plug in (2))} \\ &= \mu(x)f(x) + \left\{\frac{d\mu}{dx} - P(x)\mu\right\}y\end{aligned}$$

Observation: If we can force the term $\left\{\frac{d\mu}{dx} - P(x)\mu\right\}$ to zero, then we can solve $\mu(x)y(x)$ by directly integrating $\mu(x)f(x)$!

$\implies$ We need to solve an auxiliary (輔助的) DE first: $\dfrac{d\mu}{dx} = P(x)\mu$.

Example

Example (A Linear First-Order ODE, part 1)

Solve $\dfrac{dy}{dx} = x + y,\ \ y(0) = 2.$

Deriving the Auxiliary DE:

$$\begin{aligned} \frac{d(\mu y)}{dx} &= \mu \frac{dy}{dx} + y \frac{d\mu}{dx} \\ &= \mu \{x + y\} + y \frac{d\mu}{dx} \quad \left(\text{Plug in } \frac{dy}{dx} = x + y\right) \\ &= \mu x + \left\{ \frac{d\mu}{dx} + \mu \right\} y \end{aligned} \tag{3}$$

$\Longrightarrow$ Auxiliary DE: find some $\mu(x)$ such that $\dfrac{d\mu}{dx} + \mu = 0.$

Solving an Auxiliary DE to Find an *Integrating Factor*

Auxiliary DE

Find an $\mu(x)$ satisfying

$$\frac{d\mu}{dx} = P(x)\mu$$

Note that we only need to find one such μ (called an **integrating factor**)

A: This is easy to solve by Separation of Variables:

$$\frac{d\mu}{\mu} = P(x)\,dx \implies \ln|\mu| = \int P(x)\,dx + c$$

We shall pick $c = 0$ and restrict μ to be positive to get one solution:

$$\mu(x) = e^{\int P(x)\,dx}.$$

Example (continued)

Example (A Linear First-Order ODE, part 2)

Solve $\dfrac{dy}{dx} = x + y, \; y(0) = 2.$

Solving the Auxiliary DE: Find some $\mu(x)$ such that $\dfrac{d\mu}{dx} + \mu = 0.$
This is easy to solve by Separation of Variables:

$$\frac{d\mu}{\mu} = -dx \implies \ln|\mu| = -x + c \overset{c=0,\mu>0}{\implies} \mu(x) = e^{-x}.$$

Solving the Original DE: Plugging $\mu(x) = e^{-x}$ into (3), we have

$$\frac{d\left(e^{-x}y\right)}{dx} = xe^{-x}$$
$$\implies e^{-x}y = -xe^{-x} - e^{-x} + 3 \quad \text{Plug in } y(0) = 2 \text{ to find the constant } 3$$
$$\implies y = -x - 1 + 3e^{x}, \; x \in \mathbb{R}$$

Singular Points

Consider a general linear first-order ODE:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

When rewriting the original linear equation into its standard form, that is, when we what to figure out how to represent $\frac{dy}{dx}$ in terms of linear combinations of y and functions of x, we need to divide everything by the coefficient $a_0(x)$:

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}.$$

Here we implicitly impose the condition that $a_1(x) \neq 0$.
The points at which $a_1(x) = 0$ are called **singular points**.

Solving the Linear First-Order ODE

General Procedure of Solving a Linear First-Order ODE

1. 寫成標準式: Rewrite the give ODE into the form $\frac{dy}{dx} + P(x)y = f(x)$.
   若分母$= 0$, exclude the singular points from the interval of solutions.
2. 導出輔助式: Introduce an integrating factor $\mu(x)$ and derive the auxiliary equation of μ to find μ such that $\frac{d(\mu y)}{dx} = \mu(x)f(x)$.
3. 解輔助式: Find one μ satisfying the auxiliary DE $\frac{d\mu}{dx} = P(x)\mu$.
4. 解原式: Plug in the integrating factor $\mu(x)$ we found and solve $\mu(x)y$ by directly integrating $\mu(x)f(x)$.
   Check if the singular points can be included into the interval of solutions.

Example

Example (A Linear First-Order ODE, part 1)

Solve $(x^2 - 9)\dfrac{dy}{dx} + xy = 0,\ y(0) = 2.$

Deriving the Auxilary DE:

$$\begin{aligned}\frac{d(\mu y)}{dx} &= \mu\frac{dy}{dx} + y\frac{d\mu}{dx} \\ &= \mu\left\{\frac{-xy}{x^2-9}\right\} + y\frac{d\mu}{dx} \quad \left(\text{Plug in } \frac{dy}{dx} = \frac{-xy}{x^2-9},\ x \neq \pm 3\right) \\ &= \left\{\frac{d\mu}{dx} - \mu\frac{x}{x^2-9}\right\} y \qquad (4)\end{aligned}$$

$\Longrightarrow$ Auxiliary DE: find some $\mu(x)$ such that $\dfrac{d\mu}{dx} - \mu\dfrac{x}{x^2-9} = 0.$

Example (continued)

Example (A Linear First-Order ODE, part 2)

Solve $(x^2-9)\dfrac{dy}{dx}+xy=0,\ \ y(0)=2.$

Solving the Auxiliary DE: Find some $\mu(x)$ such that $\dfrac{d\mu}{dx}-\mu\dfrac{x}{x^2-9}=0.$

$$\frac{d\mu}{\mu}=\frac{x}{x^2-9}\,dx=\left\{\frac{\frac{1}{2}}{x-3}+\frac{\frac{1}{2}}{x+3}\right\}dx$$

$$\implies \ln|\mu|=\frac{1}{2}\ln|x-3|+\frac{1}{2}\ln|x+3|$$

$$\implies \mu(x)=\sqrt{9-x^2},\ \ -3<x<3 \quad (\text{Because initial point is } x=0!)$$

Example (continued)

Example (A Linear First-Order ODE, part 3)

Solve $(x^2 - 9)\dfrac{dy}{dx} + xy = 0,\ \ y(0) = 2.$

Solving the Original DE: Plugging $\mu(x) = \sqrt{9 - x^2}$ into (4), we have

$$\frac{d\left(\sqrt{9-x^2}\,y\right)}{dx} = 0$$

$$\implies \sqrt{9-x^2}\,y = 6 \quad \text{Plug in } y(0) = 2 \text{ to find the constant } 6$$

$$\implies y = \frac{6}{\sqrt{9-x^2}}, \quad -3 < x < 3$$

A Remark on Singular Points

In the above example, the solution is undefined at the singular points $x = \pm 3$. However, it is possible that the final solution can be define at the excluded singular points. See Example 3 in Section 2-3 on Page 57.

This becomes important if the initial point is a singular point.

Example

Solve $x\dfrac{dy}{dx} = 4y + x^6 e^x,\ \ y(0) = 0.$

Using the same method, we have $y = x^4(xe^x - e^x + c)$. But the interval of solution cannot contain $x = 0$. However, one can check that the function and its derivative are both continuous at $x = 0$. Hence we can include it and find that $y = x^4(xe^x - e^x + c)$ is a solution to the initial-value problem for any $c \in \mathbb{R}$.

A Trick

When a first-order ODE (1) is not linear in terms of $y' = \frac{dy}{dx}$ and y but linear in terms of x and $\frac{dx}{dy} = \frac{1}{y'}$, we can first solve x as a function of y and then take the inverse function to find $y(x)$.

Example

Solve $\frac{dy}{dx} = \frac{1}{x+y}$.

A: We already know that the solution to $\frac{dx}{dy} = x + y$ is

$$x = -y - 1 + ce^y,$$

which is an implicit solution.

Discontinuous Coefficients

What if coefficients are discontinuous?

Example

Solve $\dfrac{dy}{dx} + y = f(x),\ \ y(1) = 1 - e^{-1},\ f(x) = \begin{cases} 1, & x \leq 1 \\ 0, & x > 1 \end{cases}$

If they are piecewise continuous and only discontinuous at finitely many points, we can solve the equations on each interval and "stitch" them together.

See Example 6 in Section 2-3 on Page 59.

Solutions Defined by Integrals

What if we cannot find a closed-form antiderivative?
For example, $\int e^{-t^2}\,dt$.

We can express the solution in terms of integrals.

Some classes of these integrals are defined as **special functions**, and many properties are derived. For example, error functions, Bessel functions, Gamma functions, etc.

See Example 7 in Section 2-3 on Page 60.

Short Recap

- First-Order ODE
- Graphical Methods: solution curves without a solution
- A Numerical Method: Euler's method
- Separable Equations: solve by separation of variables
- Linear Equations: solve an auxiliary DE to find an integrating factor
- Watch out: singular solutions and interval of the solution

Self-Practice Exercises

2-2: 1, 9, 13, 19, 25, 27, 31, 39, 41, 49

2-3: 3, 9, 13, 17, 25, 29, 35, 39