Chapter 2: First-Order Differential Equations – Part 1

王奕翔

Department of Electrical Engineering
National Taiwan University

ihwang@ntu.edu.tw

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First-Order Differential Equation

Throughout Chapter 2, we focus on solving the first-order ODE:

Problem

Find $y = \phi(x)$ satisfying

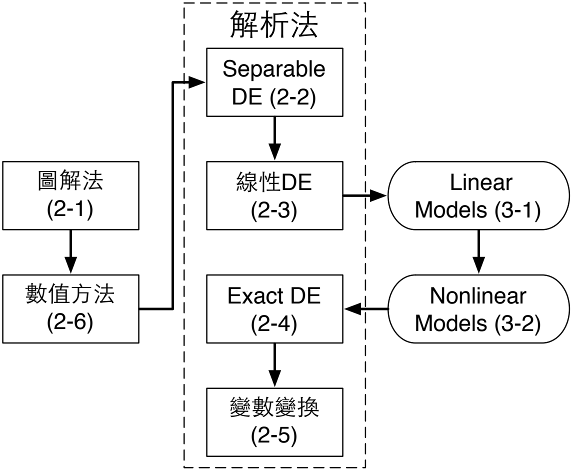
$$\frac{dy}{dx} = f(x, y), \text{ subject to } y(x_0) = y_0 \tag{1}$$

Methods of Solving First-Order ODE

1. Graphical Method (2-1)
2. Numerical Method (2-6, 9)
3. Analytic Method
   - Take antiderivative (*Calculus I, II*)
   - Separable Equations (2-2)
   - Solving Linear Equations (2-3)
   - Solving Exact Equations (2-4)
   - Solutions by Substitutions (2-5):
     homogeneous equations, Bernoulli's equation, $y' = Ax + By + C$.
4. Series Solution (6)
5. Transformation
   - Laplace Transform (7)
   - Fourier Series (11)
   - Fourier Transform (14)

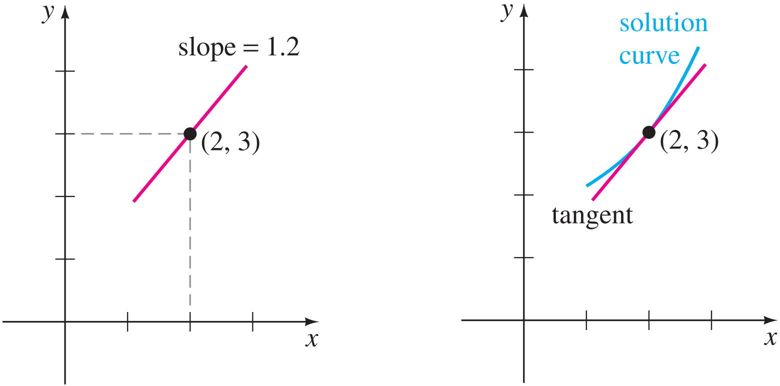
Organization of Lectures in Chapter 2 and 3

We will not follow the order in the textbook. Instead,



Example 1 (Zill&Wright p.36, Fig. 2.1.1.)

$$\frac{dy}{dx} = 0.2xy$$



Direction Fields

Key Observation

On the xy-plane, at a point (x_n, y_n), the first-order derivative

$$\left.\frac{dy}{dx}\right|_{x=x_n}$$

is the slope of the tangent line of the curve $y(x)$ at (x_n, y_n).

Hence, at every point on the xy-plane, one can *in principle* sketch an arrow indicating the direction of the tangent line.

From the initial point (x_0, y_0), one can connect all the arrows one by one and then sketch the solution curve. (土法煉鋼！)

Example 1 (Zill&Wright p.37, Fig. 2.1.3.)

$$\frac{dy}{dx} = 0.2xy$$

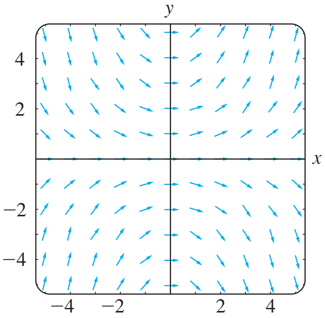


Figure : Direction Field

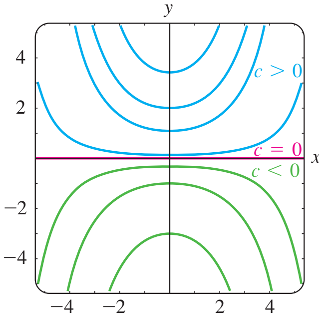
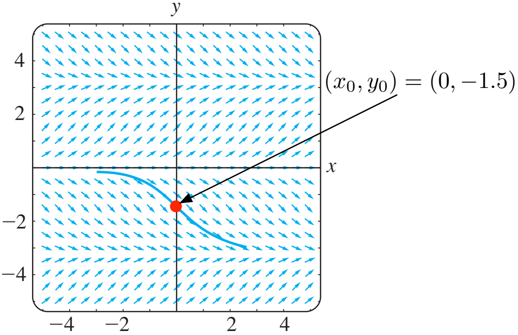


Figure : Family of Solution Curves

Example 2 (Zill&Wright p.37-38, Fig. 2.1.4.)

$$\frac{dy}{dx} = \sin y, \;\; y(0) = -1.5$$



Euler's Method

The graphical method of "connecting arrows" on the directional field can be mathematically thought of as follows:

Initial Point:	(x_0, y_0)
x Increment:	$x_1 = x_0 + h$
y Increment:	$y_1 = y_0 + h\left(\left.\frac{dy}{dx}\right\rvert_{x=x_0}\right) = y_0 + hf(x_0, y_0)$
Second Point:	(x_1, y_1)
$\vdots$	$\vdots$

Euler's Method

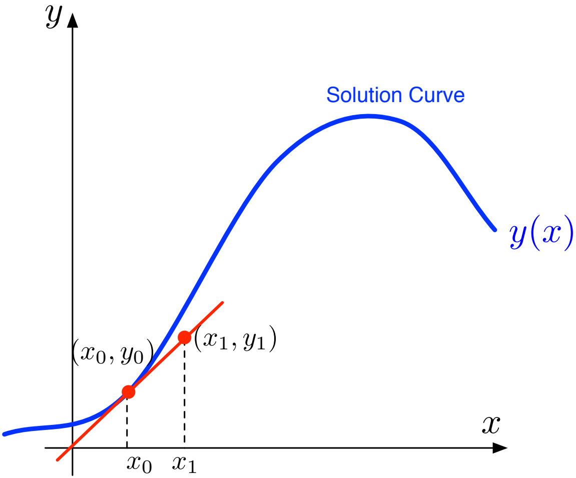
Recursive Formula

Let $h > 0$ be the recursive step size,

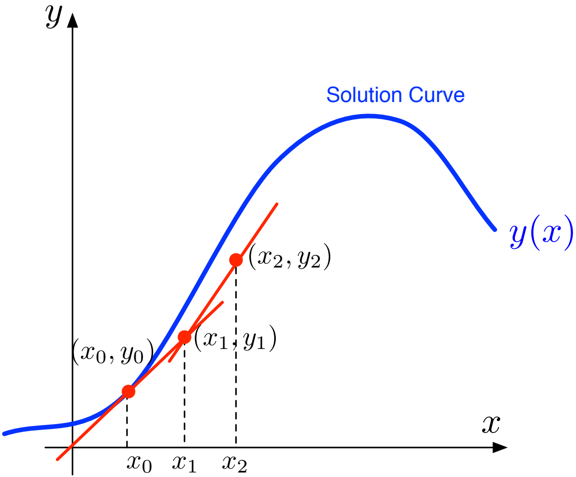
$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + hf(x_n, y_n), \qquad \forall\ n \geq 0$$

$$x_{n-1} = x_n - h, \qquad y_{n-1} = y_n - hf(x_n, y_n), \qquad \forall\ n \leq 0$$

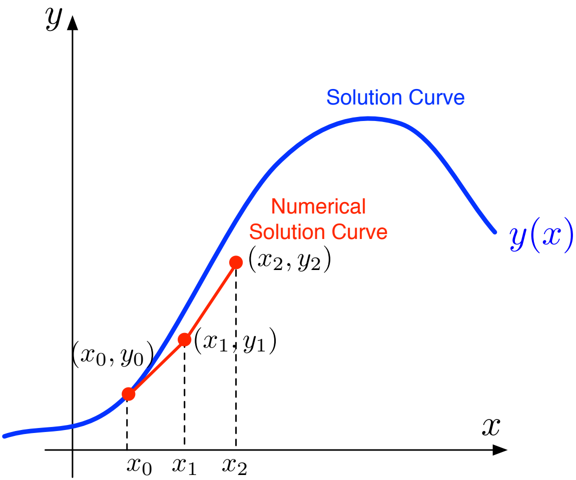
Illustration



Illustration



Illustration



Remarks

- The approximate numerical solution converges to the actual solution as $h \to 0$.
- Euler's method is just one simple numerical method for solving differential equations. Chapter 9 of the textbook introduces more advanced methods.

Solving (1) Analytically

Recall the first-order ODE (1) we would like to solve

Problem

Find $y = \phi(x)$ satisfying

$$\frac{dy}{dx} = f(x, y), \text{ subject to } y(x_0) = y_0 \tag{1}$$

We start by inspecting the equation and see if we can identify some special structure of it.

When $f(x, y)$ depends only on x

If $f(x, y) = g(x)$, then by what we learn in Calculus I & II,

$$\frac{dy}{dx} = g(x) \implies y(x) = \int_{x_0}^{x} g(t)\,dt + y_0$$

Method: Direct Integration

In the first-order ODE (1), if $f(x, y) = g(x)$ only depends on x, it can be solved by directly integrating the function $g(x)$.

When $f(x, y)$ depends only on x

Example

Solve

$$\frac{dy}{dx} = \frac{1}{x} + e^x, \text{ subject to } y(-1) = 0.$$

A: From calculus we know that the

$$\int \frac{1}{x} dx = \ln |x|, \int e^x dx = e^x$$

Plugging in the initial condition, we have

$$y(x) = \ln |x| + e^x - \frac{1}{e}, \ x < 0.$$

When $f(x, y)$ depends only on y

If $f(x, y) = h(y)$, then

$$\frac{dy}{dx} = h(y) \implies \frac{dy}{h(y)} = dx \overset{\text{integrate both sides}}{\implies} \int_{y_0}^{y} \frac{dy}{h(y)} = x - x_0$$

Assume that the antiderivative (不定積分、反導函數) of $1/h(y)$ is $H(y)$. That is,

$$\int \frac{1}{h(y)}\, dy = H(y).$$

Then, we have

$$H(y) - H(y_0) = x - x_0 \implies y(x) = H^{-1}(x - x_0 + H(y_0))$$

When $f(x, y)$ depends only on y

Example

Solve $\dfrac{dy}{dx} = (y-1)^2$.

A: Use the same principle, we have

$$\frac{dy}{dx} = (y-1)^2 \implies \frac{dy}{(y-1)^2} = dx, \; y \neq 1$$
$$\implies \frac{1}{1-y} = x + c, \text{ for some constant } c$$
$$\implies y = 1 - \frac{1}{x+c}, \text{ for some constant } c, \text{ or } y = 1$$

Note: How about the constant function $y = 1$?
$\implies$ $y = 1$ is called a **singular solution**.

Table of Integrals

Function	Antiderivative
u^n	$\dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
u^{-1}	$\ln\lvert u\rvert + C$
a^u	$\dfrac{a^u}{\ln a} + C$
$\sin u$	$-\cos u + C$
$\cos u$	$\sin u + C$
$\tan u$	$-\ln\lvert\cos u\rvert + C$
$\cot u$	$\ln\lvert\sin u\rvert + C$
$\dfrac{1}{a^2+u^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{u}{a} + C$
$\dfrac{1}{\sqrt{a^2-u^2}}$	$\sin^{-1}\dfrac{u}{a} + C$
$\vdots$	$\vdots$

Separable Equations

Definition (Separable Equations)

If in (1) the function $f(x, y)$ on the right hand side takes the form $f(x, y) = g(x)h(y)$,, we call the first-order ODE **separable**, or to have **separable variables**.

Example (Are the following equations separable?)

- $\frac{dy}{dx} = x + y$ No.
- $\frac{dy}{dx} = e^{x+y}$ Yes.
- $\frac{dy}{dx} = x + y + xy + 1$ Yes, $\because x + y + xy + 1 = (x + 1)(y + 1)$.
- $\frac{dy}{dx} = x + y + xy$ No.

Separable Equations

General Procedure of Solving a Separable DE

1. 分別移項: $\dfrac{dy}{h(y)} = \dfrac{dx}{g(x)}$. 若分母會為零, check singular solutions!
2. 兩邊積分: $\displaystyle\int \frac{dy}{h(y)} = \int \frac{dx}{g(x)} \implies H(y) = G(x) + c.$
3. 代入條件: $c = H(y_0) - G(x_0)$.
4. 取反函數: $y = H^{-1}(G(x) + H(y_0) - G(x_0))$.
   Don't forget to check singular solutions!