Chapter 1: Introduction

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Differential Equations

Definition (Differential Equations)

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is called a differential equation (DE).

Example

$\frac{dy}{dx} = xy$, where $y = f(x)$ is a function of x.
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, where $u = \phi(x, y)$ is a function of x and y.

Exercise

Identify the independent and dependent variables in the above two DE's.

Classification of DE

- By type
- By order
- By linearity

Type of Differential Equations

Ordinary Differential Equation

- Involving ordinary derivatives
- Only 1 independent variable
- Examples:

$$\frac{dy}{dx} = xy, \ \ \frac{dy}{dx}\frac{dz}{dx} = 3xy^2/z$$

Partial Differential Equation

- Involving partial derivatives
- $2+$ independent variables
- Example:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Order of Differential Equations

Order: the highest derivative of the equation.

Example (ODE)

$$\frac{d^2 y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

$\Longrightarrow$ order is 2.

Example (PDE)

$$x\frac{\partial^2 u}{\partial x \partial y} = uy^2$$

$\Longrightarrow$ order is 2.

Linearity of Differential Equations

Every ODE of a function $y = f(x)$ with order n can be written in the following *general form*:

$$F\left(x, y, y', \dots, y^{(n)}\right) = 0.$$

Definition (Linear ODE)

The ODE is linear $\iff$ F is linear in $\{y, y', \dots, y^{(n)}\}$.

Hence a linear ODE can be written more explicitly as follows:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \qquad (1)$$

- The coefficients a_i's are only function of x, not y.

Linearity of Differential Equations

Discussion

If both $y = f_1(x)$ and $y = f_2(x)$ satisfy (1), does a linear combination of f_1 and f_2 satisfy (1)?

A: Not necessarily.

Example (Nonlinear ODE)

$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right) - 4y = e^x$ is linear

$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right) - 4y = e^y$ is nonlinear

$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 - 4y = e^x$ is nonlinear

Explicit vs. Implicit Solutions

Definition

- Explicit solution: solutions can be expressed *explicitly* as $y = \phi(x)$.
- Implicit solution: solutions in the form of a relation $G(x, y) = 0$.

Example

Consider the following ODE $\frac{dy}{dx} = -\frac{x}{y}$.

- Both $y = \phi_1(x) = \sqrt{1 - x^2}$ and $y = \phi_2(x) = -\sqrt{1 - x^2}$ are explicit solutions
- The relation $x^2 + y^2 - 1 = 0$ is an implicit solution.
  Because $x^2 + y^2 - 1 = 0 \implies 2xdx + 2ydy = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$.

Trivial Solutions

Example

Consider the ODE

$$\frac{dy}{dx} = x\sqrt{y}.$$

Verify that both $y = \frac{x^4}{16}$ and $y = 0$ are solutions.

A:

$$y = \frac{x^4}{16} \implies \frac{dy}{dx} = \frac{x^3}{4};\ x\sqrt{y} = x\sqrt{x^4/16} = x \cdot x^2/4 = \frac{x^3}{4}$$

Hence $y = \frac{x^4}{16}$ is a solution.
Also, trivially $y = 0$ is a solution. We call $y = 0$ a *trivial solution*.

Initial-Value Problems

- A differential equation usually has more than one solution.
- For example, consider $\frac{dy}{dx} = 1$. We can derive a family of solutions:

$$\{y = x + c, \; c \in \mathbb{R}\}.$$

  Because all parallel lines have the same slope.
- In real applications, we need some *conditions* to specify a unique solution.
- The **initial value** is one of them(not necessarily at $x = 0$):

$$y(0) = 2 \implies c = 0 + 2 = 2 \implies \text{unique solution: } y = x + 2$$
$$y(2) = -1 \implies c = -1 - 2 = -3 \implies \text{unique solution: } y = x - 3.$$

Number of Initial/Boundary Conditions vs. Order

Fact (Number of Initial/Boundary Conditions)

Usually a n-th order ODE requires n initial/boundary conditions to specify an unique solution.

Example

For the ODE $y'' = 2$, the family of solutions take the form $y = x^2 + bx + c$.

- Initial condition: $y(1) = -1, y'(1) = 3 \implies b = 1, c = -3$.
- Boundary condition: $y(0) = 3, y'(1) = 3 \implies b = 1, c = -3$.

Some Remarks on Initial/Boundary Conditions

Remark (Initial vs. Boundary Conditions)

Initial Conditions: all conditions are at the same $x = x_0$.
Boundary Conditions: conditions can be at different x.

Remark (Order of the derivatives in the conditions

Initial/boundary conditions can be the value or the function of 0-th to $(n-1)$-th order derivatives, where n is the order of the ODE.

Applications of Differential Equations in Economics

Thomas Malthus (馬爾薩斯) 人口論 (1798): The growth rate of population is proportional to total population.

Population Growth Model (T. Malthus, 1978)

Let $P(t)$ denote the total population at time t. Then,

$$\frac{dP}{dt} = kP, \text{ for some } k > 0. \tag{2}$$

Applications of Differential Equations in Physics

Radioactive Decay 放射性物質衰變

Let $A(t)$ denote the amount of some radioactive substance remaining at time t. Then,

$$\frac{dA}{dt} = kA, \text{ for some } k < 0. \tag{3}$$

道理類似人口論

Applications of Differential Equations in Physics

Newton's Law of Cooling/Warming 冷卻/升溫定律

Let $T(t)$ denote the temperature of an object at time t, and T_m denote the fixed temperature of the surroundings. Then,

$$\frac{dT}{dt} = k(T - T_m), \text{ for some } k < 0. \tag{4}$$

溫差越大, 散熱/吸熱越快

Remark

- Equations (2) – (4) all take the same form
- As long as we know how to solve one, we can solve all others.
- This is the power of mathematical modeling.

Methods of Studying Differential Equations

1. Analytic Approach 解析方法
2. Qualitative Approach 定性分析
3. Numerical Methods 數值方法

Although the main focus of this course lies in 1, the other two are very useful in research and solving real engineering problems.

Because models in real life are mostly analytically unmanageable.

Short Recap

- 什麼是微分方程？
- Dependent vs. Independent Variables
- ODE vs. PDE
- Linear vs. Nonlinear DE
- Order of DE
- Initial-Value Problems
- Initial vs. Boundary Conditions
- Number of Conditions vs. Order
- Modeling with DE

Self-Practice Exercises

1-1: 1, 5, 9, 15, 27, 29, 33

1-2: 3, 11, 21, 31

1-3: 3, 15, 17, 21