Homework 4

Due: 1/3, 18:00

1. (**Laplace Transform of Periodic Extension**) [10]

A function $f(t) = e^{at}$ for $0 \le t < T$, and $f(t) = f(t-T)$ for $t \ge T$.
Evaluate $\mathscr{L}\{f(t)\}$.

Solution.

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{at} e^{-st} dt = \frac{1}{1-e^{-sT}} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^T = \boxed{\frac{1-e^{-(s-a)T}}{(s-a)\left(1-e^{-sT}\right)}}.$$

2. (**Inverse Laplace Transform**) [15]

Evaluate

$$\mathscr{L}^{-1}\left\{ \frac{1}{s^2} \tanh\left(\frac{s}{2}\right) \right\}.$$

Hint 1: $\tanh\left(\frac{s}{2}\right) = \frac{1-e^{-s}}{1+e^{-s}} = \frac{\left(1-e^{-s}\right)^2}{1-e^{-2s}}$.
Hint 2: $\mathscr{L}\{(f*f)(t)\} = \{F(s)\}^2$.

Solution.

$$\frac{1}{s^2}\tanh\left(\frac{s}{2}\right) = \frac{1}{s^2}\frac{1-e^{-s}}{1+e^{-s}} = \frac{1}{1-e^{-2s}} \frac{\left(1-e^{-s}\right)^2}{s^2} = \frac{1}{1-e^{-2s}} \left[F(s)\right]^2,$$

where $F(s) := \frac{1-e^{-s}}{s}$. Note that

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s} \implies f(t) := \mathscr{L}^{-1}\{F(s)\} = 1 - \mathcal{U}(t-1) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$[F(s)]^2 \xrightarrow{\mathscr{L}^{-1}} (f*f)(t) = \int_0^t f(\tau) f(t-\tau)\, d\tau = \begin{cases} \int_0^t d\tau = t, & 0 \le t < 1 \\ \int_{t-1}^1 d\tau = 2-t, & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\frac{1}{1-e^{-2s}}[F(s)]^2 \xrightarrow{\mathscr{L}^{-1}} \text{ 2-Periodic extension of } f(x)$$

$$= \boxed{\begin{cases} t, & 2n \le t < 2n+1 \\ 2-t, & 2n+1 \le t < 2n+2 \end{cases}, \quad n = 0, 1, 2, \ldots}$$

3. (**Fourier Series Expansion**) [15]

Expand $f(x) = xe^{-x}$, $0 < x < \pi$,

(a) in a Fourier cosine series. [5]

(b) in a Fourier sine series. [5]

(c) in a Fourier series. [5]

Solution.

(a) The Fourier cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} xe^{-x}\cos nx\,dx = \mathrm{Re}\left\{\frac{2}{\pi}\int_0^{\pi} xe^{-x}e^{inx}\,dx\right\}\\
&= \frac{2}{\pi}\mathrm{Re}\left\{\left[\frac{xe^{-x}e^{inx}}{-1+ni}\right]_0^{\pi} - \frac{1}{-1+ni}\int_0^{\pi} e^{-x}e^{inx}\,dx\right\}\\
&= \frac{2}{\pi}\mathrm{Re}\left\{\left[\frac{xe^{-x}e^{inx}}{-1+ni}\right]_0^{\pi} - \left[\frac{e^{-x}e^{inx}}{(-1+ni)^2}\right]_0^{\pi}\right\}\\
&= \frac{2}{\pi}\mathrm{Re}\left\{\frac{\pi e^{-\pi}\cos n\pi}{-1+ni} - \frac{e^{-\pi}\cos n\pi - 1}{1-n^2-2ni}\right\}\\
&= \boxed{\frac{2}{\pi}\left\{\frac{-\pi e^{-\pi}(-1)^n}{n^2+1} + \frac{(n^2-1)\left(e^{-\pi}(-1)^n - 1\right)}{(n^2+1)^2}\right\}, \quad n = 0, 1, 2, \ldots}
\end{aligned}$$

(b) The Fourier sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} xe^{-x}\sin nx\,dx = \mathrm{Im}\left\{\frac{2}{\pi}\int_0^{\pi} xe^{-x}e^{inx}\,dx\right\}\\
&= \frac{2}{\pi}\mathrm{Im}\left\{\left[\frac{xe^{-x}e^{inx}}{-1+ni}\right]_0^{\pi} - \frac{1}{-1+ni}\int_0^{\pi} e^{-x}e^{inx}\,dx\right\}\\
&= \frac{2}{\pi}\mathrm{Im}\left\{\left[\frac{xe^{-x}e^{inx}}{-1+ni}\right]_0^{\pi} - \left[\frac{e^{-x}e^{inx}}{(-1+ni)^2}\right]_0^{\pi}\right\}\\
&= \frac{2}{\pi}\mathrm{Im}\left\{\frac{\pi e^{-\pi}\cos n\pi}{-1+ni} - \frac{e^{-\pi}\cos n\pi - 1}{1-n^2-2ni}\right\}\\
&= \boxed{-\frac{2}{\pi}\left\{\frac{n\pi e^{-\pi}(-1)^n}{n^2+1} + \frac{2n\left(e^{-\pi}(-1)^n - 1\right)}{(n^2+1)^2}\right\}, \quad n = 1, 2, \ldots}
\end{aligned}$$

(c) The Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2nx}$, where

$$c_n = \frac{1}{\pi}\int_0^{\pi} xe^{-x}e^{-i2nx}\,dx = \frac{1}{\pi}\left\{\left[\frac{xe^{-x}e^{-i2nx}}{-1-2ni}\right]_0^{\pi} - \frac{1}{-1-2ni}\int_0^{\pi} e^{-x}e^{-i2nx}\,dx\right\}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left\{\left[\frac{xe^{-x}e^{-i2nx}}{-1-2ni}\right]_0^{\pi} - \left[\frac{e^{-x}e^{-i2nx}}{(-1-2ni)^2}\right]_0^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{\frac{\pi e^{-\pi}}{-1-2ni} - \frac{e^{-\pi}-1}{1-4n^2-4ni}\right\} \\
&= \boxed{\frac{1}{\pi}\left\{\frac{\pi e^{-\pi}(-1+2ni)}{4n^2+1} - \frac{(1-4n^2+4ni)\left(e^{-\pi}-1\right)}{(4n^2+1)^2}\right\}, \quad n \in \mathbb{Z}}
\end{aligned}
$$

4. (**Wave Equation**) [10]

Solve the following boundary-value problem:

$$\begin{array}{|rl|}\hline \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad 0 < x < \frac{\pi}{2}, \quad t > 0 \\ \text{subject to}: & u(0,t) = 0, \quad u_x\left(\frac{\pi}{2}, t\right) = 0, \quad t > 0 \\ & u(x,0) = 1, \quad u_t(x,0) = 0, \quad 0 < x < \frac{\pi}{2} \\ \hline \end{array}$$

Solution.

Let $u(x,t) = X(x)T(t)$, $X \neq 0$, $T \neq 0$. Then,

$$a^2 u_{xx} = u_{tt} \implies a^2 X''T = XT'' \implies \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$
$$\implies \begin{cases} X'' + \lambda X &= 0 \\ T'' + a^2 \lambda T &= 0 \end{cases}$$

First we solve $X'' + \lambda X$ subject to $X(0) = X'(\frac{\pi}{2}) = 0$:

- $\lambda = 0$: $X = c_1 + c_2 x$.

$$X(0) = 0 \implies c_1 = 0, \quad X'\left(\frac{\pi}{2}\right) = 0 \implies c_2 = 0 \quad \text{(Contradiction)}$$

- $\lambda = -\alpha^2 < 0$: $X = c_1 \cosh \alpha x + c_2 \sinh \alpha x$.

$$X(0) = 0 \implies c_1 = 0, \quad X'\left(\frac{\pi}{2}\right) = 0 \implies \alpha c_2 \cosh \frac{\alpha\pi}{2} = 0 \implies c_2 = 0$$
$$\text{(Contradiction)}$$

- $\lambda = \alpha^2 < 0$: $X = c_1 \cos \alpha x + c_2 \sin \alpha x$.

$$X(0) = 0 \implies c_1 = 0, \quad X'\left(\frac{\pi}{2}\right) = 0 \implies \alpha c_2 \cos \frac{\alpha\pi}{2} = 0.$$

  Since $X \neq 0$, we have $\alpha = 1, 3, 5, \ldots$, and $X = c_2 \sin \alpha x$. For the chosen λ, $T = c_3 \cos a\alpha t + c_4 \sin a\alpha t$. Note that $T'(0) = 0 \implies c_4 = 0$, and hence $T = c_3 \cos a\alpha t$.

Hence, we have $\lambda_n = (2n-1)^2$, $n \in \mathbb{N}$, $X_n(x) = \sin(2n-1)x$, $T_n(t) = \cos(2n-1)at$, and

$$u_n(x,t) = A_n \sin(2n-1)x \cos(2n-1)at, \quad u(x,t) = \sum_{n=1}^{\infty} A_n \sin(2n-1)x \cos(2n-1)at.$$

Finally, we plug in the last condition:

$$u(x,0) = 1 \implies 1 = \sum_{n=1}^{\infty} A_n \sin(2n-1)x.$$

Note that from Example 2 on Page 434 of the textbook, the Fourier sine series of the function 1 on $(0, \pi)$ is

$$\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-(-1)^k}{k}\sin kx = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right\}.$$

Therefore, on $(0, \frac{\pi}{2})$,

$$1 = \sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)}\sin(2n-1)x = \sum_{n=1}^{\infty}A_n\sin(2n-1)x \implies A_n = \frac{4}{\pi(2n-1)},$$

and

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)}\sin(2n-1)x\cos(2n-1)at}.$$

You can also obtain the coefficients A_n by the orthogonal series expansion, because $\{\sin(2n-1)x \mid n \in \mathbb{N}\}$ is an orthogonal set on $(0, \frac{\pi}{2})$.

5. (**Laplace's Equation**) [10]

Solve the following boundary-value problem:

$$\boxed{\begin{array}{rl} \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \text{subject to}: & u_x(0,y)=u(0,y), \quad u_x(a,y)=G(y), \quad 0<y<b \\ & u(x,0)=f(x), \quad u(x,b)=g(x), \quad 0<x<a \end{array}}$$

Solution.

From the homogeneous boundary conditions, we get

$$\boxed{u(x,y)=\sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi}{b}x+B_n\sinh\frac{n\pi}{b}x\right)\sin\frac{n\pi}{b}y}.$$

Plug in the boundary condition at $x=0$:

$$u_x(0,y)=u(0,y) \implies \sum_{n=1}^{\infty}\frac{n\pi}{b}B_n\sin\frac{n\pi}{b}y=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{b}y \implies A_n=\frac{n\pi}{b}B_n.$$

Plug in the boundary condition at $x=a$:

$$\begin{aligned} & u_x(a,y)=G(y) \\ \implies & G(y)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi}{b}y=\sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi}{b}a+B_n\sinh\frac{n\pi}{b}a\right)\sin\frac{n\pi}{b}y \\ \implies & A_n\cosh\frac{n\pi}{b}a+B_n\sinh\frac{n\pi}{b}a=a_n=\frac{2}{b}\int_0^b G(y)\sin\frac{n\pi}{b}y\,dy. \end{aligned}$$

Therefore,

$$\boxed{A_n=\frac{n\pi}{b}B_n, \quad B_n=\frac{\frac{2}{b}\int_0^b G(y)\sin\frac{n\pi}{b}y\,dy}{\frac{n\pi}{b}\cosh\frac{n\pi}{b}a+\sinh\frac{n\pi}{b}a}}.$$

6. (**Wave Equation on an Infinite String**) [10]

Solve the following boundary-value problem:

$$\boxed{\begin{array}{rl} \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad -\infty < x < \infty, \quad t > 0 \\ \text{subject to}: & u(\pm\infty, t) = u_x(\pm\infty, t) = 0, \quad t > 0 \\ & u(x,0) = e^{-|x|}, \quad u_t(x,0) = 0, \quad -\infty < x < \infty \end{array}}$$

Solution.

Take the Fourier transform w.r.t. x on both sides of the PDE and let $u(x,t) \xrightarrow{\mathscr{F}} U(\alpha,t)$:

$$-a^2\alpha^2 U(\alpha,t) = \frac{d^2U}{dt^2} \implies U(\alpha,t) = C_1(\alpha)\cos a\alpha t + C_2(\alpha)\sin a\alpha t.$$

With the Fourier transform,

$$\begin{aligned} u(x,0) = e^{-|x|} \implies U(\alpha,0) &= \int_{-\infty}^{\infty} e^{-|x|}e^{-i\alpha x}\,dx = \int_0^{\infty} e^{-x}e^{-i\alpha x}\,dx + \int_0^{\infty} e^{-x}e^{i\alpha x}\,dx \\ &= 2\int_0^{\infty} e^{-x}\cos\alpha x\,dx = \frac{2}{1+\alpha^2} \\ u_t(x,0) = 0 \implies \frac{dU}{dt}(\alpha,0) &= 0. \end{aligned}$$

Hence, $C_1(\alpha) = \frac{2}{1+\alpha^2}$, $C_2(\alpha) = 0$, and

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\alpha^2}\cos a\alpha t\, e^{i\alpha x}\,d\alpha = \boxed{\frac{2}{\pi}\int_0^{\infty} \frac{1}{1+\alpha^2}\cos a\alpha t\,\cos\alpha x\,d\alpha}.$$

7. (**Laplace's Equation on a Semi-Infinite Plate**) [15]

Solve the following boundary-value problem:

$$\boxed{\begin{aligned}\text{Solve } u(x,y):&\quad u_{xx}+u_{yy}=0,\quad 0<x<\pi,\quad y>0\\ \text{subject to}:&\quad u(0,y)=ye^{-y},\quad u(\pi,y)=0,\quad y>0\\ &\quad u(x,0)=xe^{-x},\quad u(x,\infty)=u_y(x,\infty)=0,\quad 0<x<\pi\end{aligned}}$$

Solution.

We shall break it into two sub-problems:

Sub-Problem 1:

$$\begin{aligned}\text{Solve } u(x,y):&\quad u_{xx}+u_{yy}=0,\quad 0<x<\pi,\quad y>0\\ \text{subject to}:&\quad u(0,y)=ye^{-y},\quad u(\pi,y)=0,\quad y>0\\ &\quad u(x,0)=0,\quad u(x,\infty)=u_y(x,\infty)=0,\quad 0<x<\pi\end{aligned}$$

For this problem, the homogeneous boundary conditions are given at $y=0,\infty$, and at $y=0$ the condition is placed at u. Hence we use Fourier sine transform to solve this problem.

Take the Fourier sine transform w.r.t. y on both sides of the PDE and let $u(x,y)\xrightarrow{\mathscr{F}}U(x,\alpha)$:

$$\frac{d^2U}{dx^2}-\alpha^2U(x,\alpha)+\alpha u(x,0)=0\implies U(x,\alpha)=C_1(\alpha)\cosh\alpha x+C_2(\alpha)\sinh\alpha x.$$

With the Fourier sine transform,

$$\begin{aligned}u(0,y)=ye^{-y}&\implies U(0,\alpha)=\int_0^\infty ye^{-y}\sin\alpha y\,dy\\ &\qquad\qquad\quad=\mathscr{L}\{y\sin\alpha y\}_{s=1}=\frac{2\alpha}{(1+\alpha^2)^2}\\ u(\pi,y)=0&\implies U(\pi,\alpha)=0.\end{aligned}$$

Hence, $C_1(\alpha)=\frac{2\alpha}{(1+\alpha^2)^2}$, $C_2(\alpha)=-\frac{\cosh\alpha\pi}{\sinh\alpha\pi}\frac{2\alpha}{(1+\alpha^2)^2}$, and

$$\boxed{u_1(x,y)=\frac{2}{\pi}\int_0^\infty\frac{2\alpha}{(1+\alpha^2)^2}\left(\cosh\alpha x-\frac{\cosh\alpha\pi}{\sinh\alpha\pi}\sinh\alpha x\right)\sin\alpha y\,d\alpha}.$$

Sub-Problem 2:

$$\begin{aligned}\text{Solve } u(x,y):&\quad u_{xx}+u_{yy}=0,\quad 0<x<\pi,\quad y>0\\ \text{subject to}:&\quad u(0,y)=0,\quad u(\pi,y)=0,\quad y>0\\ &\quad u(x,0)=xe^{-x},\quad u(x,\infty)=u_y(x,\infty)=0,\quad 0<x<\pi\end{aligned}$$

For this problem, the homogeneous boundary conditions are given at $x=0,\pi$. Hence we use separation of variables and Fourier series to solve this problem.

Following the steps in Lecture Note 14, Page 45–47, we get

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin nx \, e^{-ny},$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} xe^{-x} \sin nx \, dx = -\frac{2}{\pi}\left\{ \frac{n\pi e^{-\pi}(-1)^n}{n^2+1} + \frac{2n\left(e^{-\pi}(-1)^n - 1\right)}{(n^2+1)^2} \right\}.$$

Hence,

$$\boxed{u_2(x,y) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{ \frac{n\pi e^{-\pi}(-1)^n}{n^2+1} + \frac{2n\left(e^{-\pi}(-1)^n - 1\right)}{(n^2+1)^2} \right\} \sin nx \, e^{-ny}}.$$

Finally, the solution to the original problem is $\boxed{u(x,y) = u_1(x,y) + u_2(x,y)}$.