Homework 3

Due: 12/6, 18:00

1. (**Solving a System of Linear Differential Equations**) [10]
Solve $x(t), y(t)$ in the following system of linear differential equations.

$$\begin{cases} t^2(x''+y'')+t(x'+y')+4x=t \\ t(x'+y')+y=\dfrac{1}{t^2} \end{cases}$$

Solution.
This is a system of Cauchy-Euler equations. For $t>0$, with the substation $t=e^u$, we get

$$t^2(x''+y'')+t(x'+y')+4x=t \implies \{D_u(D_u-1)+D_u\}\{x+y\}+4x=D_u^2\{x+y\}+4x=e^u$$

and

$$t(x'+y')+y=\frac{1}{t^2} \implies D_u\{x+y\}+y=e^{-2u}.$$

Hence,

$$\begin{cases} D_u^2\{x+y\}+4x=e^u & (1) \\ D_u\{x+y\}+y=e^{-2u} & (2) \end{cases}$$

Let us solve $x+y$ first:

$$(1)+4\times(2) \implies D_u^2\{x+y\}+4D_u\{x+y\}+4(x+y)=e^u+4e^{-2u}.$$

It is then straightforward to obtain

$$x+y=c_1e^{-2u}+c_2ue^{-2u}+2u^2e^{-2u}+\frac{1}{9}e^u.$$

Hence, from (2), we get $\boxed{\text{for } t>0}$,

$$\begin{aligned} y &= e^{-2u}-\left\{(-2)c_1e^{-2u}+(-2)c_2ue^{-2u}+c_2e^{-2u}+4ue^{-2u}-4u^2e^{-2u}+\frac{1}{9}e^u\right\} \\ &= (2c_1-c_2+1)e^{-2u}+(2c_2-4)ue^{-2u}+4u^2e^{-2u}-\frac{1}{9}e^u \\ &= \boxed{(2c_1-c_2+1)t^{-2}+(2c_2-4)(\ln t)t^{-2}+4(\ln t)^2t^{-2}-\frac{1}{9}t} \\ x &= (c_2-c_1-1)e^{-2u}+(4-c_2)ue^{-2u}-2u^2e^{-2u}+\frac{2}{9}e^u \end{aligned}$$

$$= \boxed{(c_2 - c_1 - 1)t^{-2} + (4 - c_2)(\ln t)t^{-2} - 2(\ln t)^2 t^{-2} + \frac{2}{9}t}$$

For $t < 0$, with the substation $t = -e^u$, we get

$$t^2(x''+y'')+t(x'+y')+4x = t \implies \{D_u(D_u - 1) + D_u\}\{x+y\}+4x = D_u^2\{x+y\}+4x = -e^u$$

and

$$t(x' + y') + y = \frac{1}{t^2} \implies D_u\{x+y\} + y = e^{-2u}.$$

Hence,

$$\begin{cases} D_u^2\{x+y\} + 4x = -e^u & (3) \\ D_u\{x+y\} + y = e^{-2u} & (4) \end{cases}$$

Let us solve $x + y$ first:

$$(3) + 4 \times (4) \implies D_u^2\{x+y\} + 4D_u\{x+y\} + 4(x+y) = e^u + 4e^{-2u}.$$

It is then straightforward to obtain

$$x + y = c_1 e^{-2u} + c_2 u e^{-2u} + 2u^2 e^{-2u} - \frac{1}{9}e^u.$$

Hence, from (3), we get $\boxed{\text{for } t < 0}$,

$$\begin{aligned} y &= e^{-2u} - \left\{(-2)c_1 e^{-2u} + (-2)c_2 u e^{-2u} + c_2 e^{-2u} + 4ue^{-2u} - 4u^2 e^{-2u} - \frac{1}{9}e^u\right\} \\ &= (2c_1 - c_2 + 1)e^{-2u} + (2c_2 - 4)ue^{-2u} + 4u^2 e^{-2u} + \frac{1}{9}e^u \\ &= \boxed{(2c_1 - c_2 + 1)t^{-2} + (2c_2 - 4)(\ln(-t))t^{-2} + 4(\ln(-t))^2 t^{-2} - \frac{1}{9}t} \\ x &= (c_2 - c_1 - 1)e^{-2u} + (4 - c_2)ue^{-2u} - 2u^2 e^{-2u} - \frac{2}{9}e^u \\ &= \boxed{(c_2 - c_1 - 1)t^{-2} + (4 - c_2)(\ln(-t))t^{-2} - 2(\ln(-t))^2 t^{-2} + \frac{2}{9}t} \end{aligned}$$

2. (**Solving a Nonlinear Differential Equation**) [10]

Solve $y(t)$ in the following initial value problem:

$$y'' = -\frac{gR^2}{y^2},\ y(0) = R,\ y'(0) = 2\sqrt{gR}.$$

Solution.

Let $y' = u$, and we get

$$u\frac{du}{dy} = -gR^2\frac{1}{y^2} \implies u\,du = -gR^2\frac{dy}{y^2} \implies u^2 = 2gR^2\frac{1}{y} + C.$$

Since $u(0) = 2\sqrt{gR}$, $y(0) = R$, we get $C = 2gR$. Hence,

$$\frac{dy}{dt} = \sqrt{2gR}\sqrt{1+\frac{R}{y}} \implies \sqrt{2gR}\,dt = \frac{dy}{\sqrt{1+\frac{R}{y}}} \implies \sqrt{2gR}t = \int\frac{dy}{\sqrt{1+\frac{R}{y}}}$$

To compute $\int\frac{dy}{\sqrt{1+\frac{R}{y}}}$, set $u := \sqrt{1+\frac{R}{y}}$, and

$$y = \frac{R}{u^2-1} = \frac{R}{2}\left\{\frac{1}{u-1} - \frac{1}{u+1}\right\} \implies dy = \frac{R}{2}\left\{\frac{1}{(u+1)^2} - \frac{1}{(u-1)^2}\right\}du.$$

Hence,

$$\begin{aligned}
\int\frac{dy}{\sqrt{1+\frac{R}{y}}} &= \frac{R}{2}\int\frac{1}{u}\left\{\frac{1}{(u+1)^2} - \frac{1}{(u-1)^2}\right\}du \\
&= \frac{R}{2}\int\frac{1}{u}\frac{-4u}{(u+1)^2(u-1)^2}\,du = 2R\int\frac{-1}{(u+1)^2(u-1)^2}\,du \\
&= \frac{R}{2}\int\left\{\frac{-1}{(u+1)^2} + \frac{-1}{(u-1)^2} - \frac{1}{u+1} + \frac{1}{u-1}\right\}du \\
&= \frac{R}{2}\left\{\frac{1}{u+1} + \frac{1}{u-1} - \ln|u+1| + \ln|u-1|\right\} \\
&= y\sqrt{1+\frac{R}{y}} + \frac{R}{2}\ln\left\{2\frac{y}{R} + 1 - 2\sqrt{\left(\frac{y}{R}\right)^2 + \frac{y}{R}}\right\}
\end{aligned}$$

Hence,

$$\boxed{\sqrt{2gR}t = y\sqrt{1+\frac{R}{y}} + \frac{R}{2}\ln\left\{2\frac{y}{R} + 1 - 2\sqrt{\left(\frac{y}{R}\right)^2 + \frac{y}{R}}\right\} - \left(\sqrt{2}R + \frac{R}{2}\ln\left(3-2\sqrt{2}\right)\right)}.$$

3. (**Power Series Solution about an Ordinary Point**) [10]

Solve the DE below using power series centered at $x = 0$.

$$(x^2 + x - 2)y'' - 2(2x + 1)y' + 6y = 0.$$

Solution.

Plug in $y = \sum_{n=0}^{\infty} c_n x^n$ we get

$$\begin{aligned}
0 &= (x^2 + x - 2)y'' - 2(2x + 1)y' + 6y \\
&= \left(x^2 y'' - 4xy' + 6y\right) + (xy'' - 2y') - 2y'' \\
&= \sum_{n=0}^{\infty} \{n(n-1) - 4n + 6\} c_n x^n + \sum_{n=0}^{\infty} \{(n+1)n - 2(n+1)\} c_{n+1} x^n \\
&\quad + \sum_{n=0}^{\infty} -2(n+2)(n+1) c_{n+2} x^n \\
&= \sum_{n=0}^{\infty} \{(n-2)(n-3)c_n + (n+1)(n-2)c_{n+1} - 2(n+2)(n+1)c_{n+2}\} x^n
\end{aligned}$$

Hence, for $n = 0, 1, 2, \ldots$,

$$(n-2)(n-3)c_n + (n+1)(n-2)c_{n+1} - 2(n+2)(n+1)c_{n+2} = 0$$

$n = 0$	$6c_0 - 2c_1 - 4c_2 = 0$
$n = 1$	$2c_1 - 2c_2 - 12c_3 = 0$
$n = 2$	$-24c_4 = 0$
$n = 3$	$4c_4 - 40c_5 = 0$
$n = 4$	$2c_4 + 10c_5 - 60c_6 = 0$
$\vdots$	$\vdots$

It is straightforward to see that $0 = c_4 = c_5 = c_6 = \ldots$. Final solution is

$$\boxed{c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad \begin{cases} 3c_0 = c_1 + 2c_2 \\ 6c_3 = c_1 - c_2 \end{cases}}$$

4. (**Method of Frobenius**) [10]

Use the method of Frobenius, find two linearly independent solutions of the following DE about the singular point $x = 0$.

$$xy'' + y = 0.$$

Hint. Recall that if $r_1 > r_2$ are the two roots of the indicial equation and $r_1 - r_2 \in \mathbb{Z}$,

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1},\ c_0 \neq 0, \quad y_2(x) = \underbrace{C}_{\text{can be } 0} y_1(x) \ln x + \sum_{n=0}^{\infty} d_n x^{n+r_2},\ d_0 \neq 0.$$

Solution.

In the class we have found that the two roots of the indicial equation is $r = 1, 0$. For $r = 1$, we have found a series solution

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1}, \quad c_{n+1} = \frac{-c_n}{(n+1)(n+2)},\ n = 0, 1, 2, \ldots.$$

For $r = 0$, we set

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} d_n x^n,\ d_0 \neq 0.$$

Therefore,

$$y_2{}'' = C y_1{}'' \ln x + C \sum_{n=0}^{\infty} (2n+1) c_n x^{n-1} + \sum_{n=0}^{\infty} n(n-1) d_n x^{n-2}.$$

Hence,

$$0 = xy_2{}'' + y_2 = C\left(xy_1{}'' + y_1\right) \ln x + C \sum_{n=0}^{\infty} (2n+1) c_n x^n + \sum_{n=0}^{\infty} n(n-1) d_n x^{n-1} + \sum_{n=0}^{\infty} d_n x^n$$

$$= \sum_{n=0}^{\infty} \left\{ C c_n (2n+1) + (n+1) n d_{n+1} + d_n \right\} x^n$$

Therefore,

$$y_2(x) = C \ln x \sum_{n=0}^{\infty} c_n x^{n+1} + \sum_{n=0}^{\infty} d_n x^n$$

$$\text{where} \quad c_{n+1} = \frac{-c_n}{(n+1)(n+2)},\ C c_n (2n+1) + (n+1) n d_{n+1} + d_n,\ n = 0, 1, 2, \ldots$$

5. (**Laplace Transform and its Inverse Transform**) [20]

Evaluate the following:

(a) $\mathscr{L}\left\{(t+\cos t)\sinh 2t\right\}$ [5]

(b) $\mathscr{L}\left\{(t^3-3t^2+3t-1)\mathcal{U}(t-2)\right\}$ [5]

(c) $\mathscr{L}^{-1}\left\{\dfrac{3e^{-s}}{(s^3-1)}\right\}$ [5]

(d) $\mathscr{L}^{-1}\left\{\dfrac{6s^2-14}{(s-3)^2(s^2+2s+5)}\right\}$ [5]

Solution.

(a)

$$\begin{aligned}\mathscr{L}\left\{(t+\cos t)\sinh 2t\right\} &= \frac{1}{2}\mathscr{L}\left\{(t+\cos t)e^{2t}\right\} - \frac{1}{2}\mathscr{L}\left\{(t+\cos t)e^{-2t}\right\}\\ &= \frac{1}{2}\mathscr{L}\left\{(t+\cos t)\right\}|_{s\to s-2} - \frac{1}{2}\mathscr{L}\left\{(t+\cos t)\right\}|_{s\to s+2}\\ &= \boxed{\frac{1}{2}\left\{\frac{1}{(s-2)^2} - \frac{1}{(s+2)^2} + \frac{s-2}{(s-2)^2+1} - \frac{s+2}{(s+2)^2+1}\right\}}\end{aligned}$$

(b)

$$\begin{aligned}\mathscr{L}\left\{(t^3-3t^2+3t-1)\mathcal{U}(t-2)\right\} &= \mathscr{L}\left\{(t-1)^3\mathcal{U}(t-2)\right\} = \mathscr{L}\left\{(t+1-2)^3\mathcal{U}(t-2)\right\}\\ &= e^{-2s}\mathscr{L}\left\{(t+1)^3\right\} = \boxed{e^{-2s}\left\{\frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s}\right\}}\end{aligned}$$

(c)

$$\frac{3}{(s^3-1)} = \frac{1}{s-1} - \frac{s+2}{s^2+s+1} = \frac{1}{s-1} - \frac{\left(s+\frac{1}{2}\right)+\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Hence,

$$\mathscr{L}^{-1}\left\{\frac{3}{(s^3-1)}\right\} = e^t - e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right),$$

and

$$\mathscr{L}^{-1}\left\{\frac{3e^{-s}}{(s^3-1)}\right\} = \boxed{\left\{e^{t-1} - e^{-\frac{t-1}{2}}\cos\left(\frac{\sqrt{3}}{2}(t-1)\right) - \sqrt{3}e^{-\frac{t-1}{2}}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\right\}\mathcal{U}(t-1)}.$$

(d)

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{6s^2-14}{(s-3)^2(s^2+2s+5)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s-3}+\frac{2}{(s-3)^2}+\frac{-(s+1)}{(s+1)^2+4}\right\}\\ &= \boxed{e^{3t}+2te^{3t}-e^{-t}\cos 2t}.\end{aligned}$$

6. (**Solving IVP with Laplace Transform**) [10]

Solve the following initial value problem: $y(\pi) = 1,\ y'(\pi) = -1$,

$$y'' + 4y' + 4y = g(t) = \begin{cases} \cos 2t & t < 2\pi \\ e^{-(t-2\pi)} \cos 2t & t \geq 2\pi \end{cases}$$

Solution.

Let $\tau := t - \pi$, and convert the original IVP into the following one:

$$\begin{aligned} \text{Solve}: &\quad y'' + 4y' + 4y = \cos 2\tau (1 - \mathcal{U}(\tau - \pi)) + e^{-(\tau-\pi)} \cos 2\tau\, \mathcal{U}(\tau - \pi) \\ \text{subject to}: &\quad y(0) = 1,\ y'(0) = -1 \end{aligned}$$

Using the Laplace transform, we get

$$\begin{aligned} \left(1 - e^{-\pi s}\right) \frac{s}{s^2+4} + e^{-\pi s} \frac{s+1}{(s+1)^2+4} &= \left(s^2 Y(s) - s + 1\right) + 4\left(sY(s) - 1\right) + 4Y(s) \\ &= (s+2)^2 Y(s) - s - 3 \end{aligned}$$

Hence,

$$\begin{aligned} Y(s) &= \frac{s+3}{(s+2)^2} + \left(1 - e^{-\pi s}\right) \frac{s}{(s^2+4)(s+2)^2} + e^{-\pi s} \frac{s+1}{\left[(s+1)^2+4\right](s+2)^2} \\ &= \frac{1}{s+2} + \frac{1}{(s+2)^2} + \frac{1 - e^{-\pi s}}{4} \left\{ \frac{1}{s^2+4} - \frac{1}{(s+2)^2} \right\} \\ &\quad + e^{-\pi s} \left\{ \frac{\frac{-3}{25}(s+1) + \frac{8}{25}}{(s+1)^2+4} + \frac{\frac{3}{25}(s+2) - \frac{2}{25}}{(s+2)^2} \right\} \end{aligned}$$

Taking the inverse transform, we get

$$\begin{aligned} y(\tau) &= e^{-2\tau} + \tau e^{-2\tau} + \frac{1}{8} \left\{ \sin 2\tau - \sin 2(\tau - \pi) \mathcal{U}(\tau - \pi) \right\} \\ &\quad - \frac{1}{4} \left\{ \tau e^{-2\tau} - (\tau - \pi) e^{-2(\tau-\pi)} \mathcal{U}(\tau - \pi) \right\} \\ &\quad + \left\{ -\frac{3}{25} e^{-(\tau-\pi)} \cos 2(\tau - \pi) + \frac{4}{25} e^{-(\tau-\pi)} \sin 2(\tau - \pi) \right\} \mathcal{U}(\tau - \pi) \\ &\quad + \left\{ \frac{3}{25} e^{-2(\tau-\pi)} - \frac{2}{25} (\tau - \pi) e^{-2(\tau-\pi)} \right\} \mathcal{U}(\tau - \pi) \end{aligned}$$

and

$$\begin{aligned} y(t) &= e^{-2(t-\pi)} + \frac{3}{4}(t - \pi) e^{-2(t-\pi)} + \frac{1}{8} \sin 2t (1 - \mathcal{U}(t - 2\pi)) \\ &\quad + \left\{ -\frac{3}{25} e^{-(t-2\pi)} \cos 2t + \frac{4}{25} e^{-(t-2\pi)} \sin 2t \right\} \mathcal{U}(t - 2\pi) \\ &\quad + \left\{ \frac{3}{25} e^{-2(t-2\pi)} - \frac{17}{100} (t - 2\pi) e^{-2(t-2\pi)} \right\} \mathcal{U}(t - 2\pi) \end{aligned}$$

7. (**What Laplace Transform does not Take Care of**) [10]
Solve the following initial value problem: $y(0) = 1,\ y'(0) = 3,$

$$y'' - y = g(t) = \begin{cases} 0 & t < 0 \\ t^3 & t \geq 0 \end{cases}$$

Hint. Be careful at the solution you found: it has to satisfy $y'' - y = 0$ for $t < 0$.

Solution.

For $t \geq 0$, using the Laplace transform, we get

$$\begin{aligned} \frac{6}{s^4} &= \left(s^2 Y(s) - s - 3\right) - Y(s) = (s^2 - 1)Y(s) - (s+3) \\ \implies Y(s) &= \frac{s+3}{s^2-1} + \frac{6}{s^4(s^2-1)} = \frac{2}{s-1} - \frac{1}{s+1} + \frac{3}{s-1} - \frac{3}{s+1} - \frac{6}{s^2} - \frac{6}{s^4} \\ \implies y(t) &= 5e^t - 4e^{-t} - 6t - t^3 \end{aligned}$$

For $t < 0$, let $\tau = -t$, and we get

$$y'' - y = 0,\ \tau > 0, \quad y(0) = 1,\ y'(0) = 3.$$

Using the Laplace transform, we get

$$\begin{aligned} 0 &= \left(s^2 Y(s) - s - 3\right) - Y(s) = (s^2 - 1)Y(s) - (s+3) \\ \implies Y(s) &= \frac{s+3}{s^2-1} = \frac{2}{s-1} - \frac{1}{s+1} \\ \implies y(t) &= 2e^t - e^{-t}. \end{aligned}$$

Hence,

$$\boxed{y(t) = \begin{cases} 5e^t - 4e^{-t} - 6t - t^3, & t \geq 0 \\ 2e^t - e^{-t}, & t < 0 \end{cases}}$$