Solution to Homework 2

1. (**Substitution and Nonexact Differential Equation Made Exact**) [10]
Solve

$$\frac{dy}{dx} = 2 - 2e^{y} + 3e^{2x+y},\ y(0) = 0.$$

Solution.
Let $u := e^{2x}$, $v = e^{y}$, and hence

$$\begin{aligned} dy &= (2 - 2v + 3uv)\,dx,\ du = (2u)dx,\ dv = (v)dy \\ \implies &(2u)dv = v\,(2 - 2v + 3uv)\,du \\ \implies &v\,(2 - 2v + 3uv)\,du + (-2u)dv \end{aligned}$$

Let $M(u,v) := v(2 - 2v + 3uv)$, $N(u,v) := -2u$. Then,

$$\begin{aligned} &M_v = 2 - 2v + 3uv + v(3u - 2) = 2 - 4v + 6uv,\ N_u = -2 \\ \implies &\Delta = M_v - N_u = 4 - 4v + 6uv \neq 0, \quad \frac{\Delta}{M} = \frac{2}{v} \text{ only depends on } v \end{aligned}$$

Hence, we can solve the following to find a function $\mu(v)$ such that $\mu M du + \mu N dv$ is an exact differential:

$$\frac{d\mu}{dv} = -\frac{\Delta}{M}\mu = \frac{-2}{v}\mu \implies \ln|\mu| = -2\ln|v|.$$

Note that $v = e^{y} > 0$, and hence we can pick $\mu = \frac{1}{v^2}$, and get an exact equation

$$\left(\frac{2}{v} - 2 + 3u\right) du + \left(\frac{-2u}{v^2}\right) dv = 0.$$

To solve this exact equation, we find

$$F(u,v) = \int \left(\frac{2}{v} - 2 + 3u\right) du = \left(\frac{2}{v} - 2\right) u + \frac{3}{2}u^2 + g(v).$$

To determine $g(v)$, take the partial derivative w.r.t. v:

$$\frac{\partial F}{\partial v} = \left(\frac{-2}{v^2}\right) u + g'(v) = \frac{-2u}{v^2} \implies g'(v) = 0 \implies g(v) = \text{constant}.$$

Hence, we have $\left(\frac{2}{v} - 2\right) u + \frac{3}{2}u^2 = c$. Plug in the initial condition $u = e^0 = 1$, $v = e^0 = 1$, we get $c = \frac{3}{2}$, and

$$\begin{aligned} &\left(\frac{2}{v} - 2\right) u + \frac{3}{2}u^2 = \frac{3}{2} \implies v^{-1} = \frac{3}{4}u^{-1} - \frac{3}{4}u + 1 \implies e^{-y} = 1 + \frac{3}{4}\left(e^{-2x} - e^{2x}\right) \\ \implies &\boxed{y = -\ln\left(1 + \frac{3}{4}\left(e^{-2x} - e^{2x}\right)\right) = -\ln\left(1 - \frac{3}{2}\sinh 2x\right)}. \end{aligned}$$

Bonus. Solve $\frac{dy}{dx} = 2 - 2e^y + 3e^{x+y}$, $y(0) = 0$. [10]

Solution.

Let $u := e^x$, $v = e^y$, and hence

$$\begin{aligned} & dy = (2 - 2v + 3uv)\,dx,\ \ du = (u)dx,\ \ dv = (v)dy \\ \implies & (u)dv = v\,(2 - 2v + 3uv)\,du \\ \implies & \frac{dv}{du} = \frac{v\,(2-2v+3uv)}{u} = \frac{2}{u}v + \left(3 - \frac{2}{u}\right)v^2 \end{aligned}$$

Above is a Bernoulli's equation, and can be easily solved by substituting $w := v^{-1}$:

$$\begin{aligned} \frac{dv}{du} = \frac{2}{u}v + \left(3 - \frac{2}{u}\right)v^2 \implies & \frac{-1}{w^2}\frac{dw}{du} = \frac{2}{u}\frac{1}{w} + \left(3 - \frac{2}{u}\right)\frac{1}{w^2} \\ \implies & \frac{dw}{du} = -\frac{2}{u}w + \left(\frac{2}{u} - 3\right) \end{aligned}$$

To solve the above linear first-order DE, we first find an integrating factor by solving the following:

$$\frac{d\mu}{du} = \frac{2}{u}\mu \implies \frac{d\mu}{\mu} = \frac{2du}{u} \implies \ln|\mu| = 2\ln|u|.$$

We pick $\mu = u^2$, and we get

$$w = \frac{1}{u^2}\int u^2\left(\frac{2}{u} - 3\right)du = \frac{1}{u^2}\left(u^2 - u^3 + c\right).$$

Plug in the initial condition $w = v^{-1} = e^0 = 1$, $u = e^0 = 1$, and we get $c = 1$. Therefore, $w = 1 - u + u^{-2}$, and hence

$$e^{-y} = 1 - e^x + e^{-2x} \implies \boxed{y = -\ln\left(1 - e^x + e^{-2x}\right)}.$$

2. (**Method of Substitution**) [20]

Solve

(a) [10]

$$\frac{dy}{dx} = \frac{2}{x} + \left(3 - \frac{1}{x}\right) y + xy^2.$$

(b) [10]

$$\frac{dy}{dx} = 2e^{x^2} + (2x+3)y + e^{-x^2}y^2,\ y(0) = 1.$$

Hint: Choose appropriate $f(x)$ and use the substitution $u = f(x)y$ to convert the equation to the form $u' = P(u)$, where $P(u)$ is a polynomial of u.

Solution.

(a) We manipulate the original equation as follows:

$$\begin{aligned}\frac{dy}{dx} = \frac{2}{x} + \left(3 - \frac{1}{x}\right) y + xy^2 &\implies x\frac{dy}{dx} = 2 + (3x-1)\,y + x^2y^2 \\ &\implies y + x\frac{dy}{dx} = 2 + 3xy + x^2y^2 \\ &\implies \frac{d(xy)}{dx} = 2 + 3xy + x^2y^2\end{aligned}$$

Hence, use the substitution $u = xy$, we get

$$\begin{aligned}&\frac{du}{dx} = 2 + 3u + u^2 = (u+1)(u+2) \implies du\left(\frac{1}{u+1} - \frac{1}{u+2}\right) = dx \\ &\implies \ln|u+1| - \ln|u+2| = x + c \implies \frac{u+1}{u+2} = 1 - \frac{1}{u+2} = Ce^x,\ C \neq 0 \\ &\implies u = xy = \frac{1}{1-Ce^x} - 2 \implies \boxed{y = \frac{1}{x - Cxe^x} - \frac{2}{x},\ C \neq 0}.\end{aligned}$$

(b) We manipulate the original equation as follows:

$$\begin{aligned}\frac{dy}{dx} = 2e^{x^2} + (2x+3)y + e^{-x^2}y^2 &\implies e^{-x^2}\frac{dy}{dx} = 2 + 2xe^{-x^2}y + 3e^{-x^2}y + e^{-2x^2}y^2 \\ &\implies e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = 2 + 3e^{-x^2}y + e^{-2x^2}y^2 \\ &\implies \frac{d\left(e^{-x^2}y\right)}{dx} = 2 + 3e^{-x^2}y + e^{-2x^2}y^2\end{aligned}$$

Hence, use the substitution $u = e^{-x^2}y$, we get

$$\frac{du}{dx} = 2 + 3u + u^2 = (u+1)(u+2) \implies du\left(\frac{1}{u+1} - \frac{1}{u+2}\right) = dx$$

$$\implies \ln|u+1| - \ln|u+2| = x + c$$

Plug in the initial condition $x = 0$, $y = 1$, $u = 1$, we get $c = \ln(2/3)$. Hence,

$$\frac{u+1}{u+2} = 1 - \frac{1}{u+2} = \frac{2}{3}e^x \implies u = e^{-x^2}y = \frac{1}{1-\frac{2}{3}e^x} - 2$$

$$\implies \boxed{y = \frac{3e^{x^2}}{3-2e^x} - 2e^{x^2},\ x \in (-\infty, \ln(3/2))}.$$

3. (**General Solution of Homogenous Linear Differential Equations**) [10]
Find the general solutions of the following:

(a) [5]

$$y^{(4)} - 6y''' + 15y'' - 18y' + 10y = 0.$$

(b) [5]

$$(x-1)^2 y'' + (x-1)y' + 4y = 0.$$

Solution.

(a) The corresponding polynomial is

$$D^4 - 6D^3 + 15D^2 - 18D + 10 = \left(D^2 - 2D + 2\right)\left(D^2 - 4D + 5\right),$$

and it has four complex roots: $1 \pm i,\ 2 \pm i$. Hence, the general solution is

$$\boxed{y = c_1 e^x \cos x + c_2 e^x \sin x + c_3 e^{2x} \cos x + c_4 e^{2x} \sin x,\ \ c_1, c_2, c_3, c_4 \in \mathbb{R}}.$$

(b) First let $x > 1$. With the substation $x - 1 = e^t$, we convert the original DE into

$$\left(D_t(D_t - 1) + D_t + 4\right)\{y\} = \left(D_t^2 + 4\right)\{y\} = 0.$$

The polynomial $D_t^2 + 4$ has two roots $\pm 2i$. Hence, the general solution is

$$y = c_1 \cos 2t + c_2 \sin 2t = \boxed{c_1 \cos\left(2\ln(x-1)\right) + c_2 \sin\left(2\ln(x-1)\right),\ \ c_1, c_2 \in \mathbb{R},\ \ x > 1}.$$

If we let $x < 1$, then use the substation $x - 1 = -e^t$, we convert the original DE into

$$\left(D_t(D_t - 1) + D_t + 4\right)\{y\} = \left(D_t^2 + 4\right)\{y\} = 0,$$

the same as above. Hence, the general solution is

$$y = c_1 \cos 2t + c_2 \sin 2t = \boxed{c_1 \cos\left(2\ln(1-x)\right) + c_2 \sin\left(2\ln(1-x)\right),\ \ c_1, c_2 \in \mathbb{R},\ \ x < 1}.$$

4. (**An IVP of Homogeneous Linear DE with Constant Coefficients**) [15]
Consider the following IVP:

$$\begin{array}{ll} \text{Solve} & y^{(4)} + 4y = 0 \\ \text{subject to} & y(x_0) = 1,\ y'(x_0) = r,\ y''(x_0) = r^2,\ y'''(x_0) = r^3 \end{array}$$

(a) Find the 4 *complex* roots for the polynomial $D^4 + 4$: m_1, m_2, m_3, m_4, where $m_2 = m_1^*$, $m_4 = m_3^*$. [5]

(b) From the lecture we know that $\{e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, e^{m_4 x}\}$ is a fundamental set of solutions in the complex domain $\mathbb{C}$. Hence the general solution in the complex domain can be represented as

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x},\ C_i \in \mathbb{C},\ i = 1, 2, 3, 4. \tag{1}$$

Please give the necessary and sufficient condition for y being a real-valued function, in terms of the relationships among $\{C_1, C_2, C_3, C_4\}$. [5]

(c) Use the form in (1) to find out the unique solution of the IVP. [5]

Hint: Use Cramer's Rule to solve $\{C_1, C_2, C_3, C_4\}$, and use the following fact:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i).$$

Solution.

(a) $D^4 + 4 = D^4 + 4D^2 + 4 - 4D^2 = \left(D^2 + 2\right)^2 - (2D)^2 = \left(D^2 + 2D + 2\right)\left(D^2 - 2D + 2\right)$, and hence the four roots are:

$$\boxed{m_1 = -1 + i,\ m_2 = m_1^* = -1 - i,\ m_3 = 1 + i,\ m_4 = m_3^* = 1 - i}.$$

(b) Note that since $x \in \mathbb{R}$, $e^{m_2 x} = e^{m_1^* x} = \left(e^{m_1 x}\right)^*$ and $e^{m_4 x} = e^{m_3^* x} = \left(e^{m_3 x}\right)^*$. Therefore,

$$\begin{aligned} y &= C_1 e^{m_1 x} + C_2 \left(e^{m_1 x}\right)^* + C_3 e^{m_3 x} + C_4 \left(e^{m_3 x}\right)^*, \\ y^* &= C_2^* e^{m_1 x} + C_1^* \left(e^{m_1 x}\right)^* + C_4^* e^{m_3 x} + C_3^* \left(e^{m_3 x}\right)^*. \end{aligned}$$

With the above observation, we have $y \in \mathbb{R} \iff y - y^* = 0$

$$\begin{aligned} &\iff \left(C_1 - C_2^*\right) e^{m_1 x} + \left(C_2 - C_1^*\right)\left(e^{m_1 x}\right)^* + \left(C_3 - C_4^*\right) e^{m_3 x} + \left(C_4 - C_3^*\right)\left(e^{m_3 x}\right)^* \\ &\quad = \left(C_1 - C_2^*\right) e^{m_1 x} + \left(C_2 - C_1^*\right) e^{m_2 x} + \left(C_3 - C_4^*\right) e^{m_3 x} + \left(C_4 - C_3^*\right) e^{m_4 x} = 0 \\ &\iff \boxed{C_1 = C_2^*, C_3 = C_4^*} \quad \text{since } \{e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, e^{m_4 x}\} \text{ are linearly independent.} \end{aligned}$$

(c) Plug in the initial condition, we have the following system of linear equations:

$$\begin{cases} C_1 e^{m_1 x_0} + C_2 e^{m_2 x_0} + C_3 e^{m_3 x_0} + C_4 e^{m_4 x_0} = 1 \\ C_1 m_1 e^{m_1 x_0} + C_2 m_2 e^{m_2 x_0} + C_3 m_3 e^{m_3 x_0} + C_4 m_4 e^{m_4 x_0} = r \\ C_1 m_1^2 e^{m_1 x_0} + C_2 m_2^2 e^{m_2 x_0} + C_3 m_3^2 e^{m_3 x_0} + C_4 m_4^2 e^{m_4 x_0} = r^2 \\ C_1 m_1^3 e^{m_1 x_0} + C_2 m_2^3 e^{m_2 x_0} + C_3 m_3^3 e^{m_3 x_0} + C_4 m_4^3 e^{m_4 x_0} = r^3 \end{cases}$$

$$\implies C_j = \frac{\Delta_j}{\Delta},\ j = 1, 2, 3, 4,$$

where

$$\Delta = \begin{vmatrix} e^{m_1 x_0} & e^{m_2 x_0} & e^{m_3 x_0} & e^{m_4 x_0} \\ m_1 e^{m_1 x_0} & m_2 e^{m_2 x_0} & m_3 e^{m_3 x_0} & m_4 e^{m_4 x_0} \\ m_1^2 e^{m_1 x_0} & m_2^2 e^{m_2 x_0} & m_3^2 e^{m_3 x_0} & m_4^2 e^{m_4 x_0} \\ m_1^3 e^{m_1 x_0} & m_2^3 e^{m_2 x_0} & m_3^3 e^{m_3 x_0} & m_4^3 e^{m_4 x_0} \end{vmatrix} = e^{(\sum_{i=1}^4 m_i) x_0} \begin{vmatrix} 1 & 1 & 1 & 1 \\ m_1 & m_2 & m_3 & m_4 \\ m_1^2 & m_2^2 & m_3^2 & m_4^2 \\ m_1^3 & m_2^3 & m_3^3 & m_4^3 \end{vmatrix}$$

$$= e^{(\sum_{i=1}^4 m_i) x_0} (m_2 - m_1)(m_3 - m_1)(m_4 - m_1)(m_3 - m_2)(m_4 - m_2)(m_4 - m_3)$$

$$\Delta_1 = \begin{vmatrix} 1 & e^{m_2 x_0} & e^{m_3 x_0} & e^{m_4 x_0} \\ r & m_2 e^{m_2 x_0} & m_3 e^{m_3 x_0} & m_4 e^{m_4 x_0} \\ r^2 & m_2^2 e^{m_2 x_0} & m_3^2 e^{m_3 x_0} & m_4^2 e^{m_4 x_0} \\ r^3 & m_2^3 e^{m_2 x_0} & m_3^3 e^{m_3 x_0} & m_4^3 e^{m_4 x_0} \end{vmatrix} = e^{(\sum_{i=2}^4 m_i) x_0} \begin{vmatrix} 1 & 1 & 1 & 1 \\ r & m_2 & m_3 & m_4 \\ r^2 & m_2^2 & m_3^2 & m_4^2 \\ r^3 & m_2^3 & m_3^3 & m_4^3 \end{vmatrix}$$

$$= e^{(\sum_{i=2}^4 m_i) x_0} (m_2 - r)(m_3 - r)(m_4 - r)(m_3 - m_2)(m_4 - m_2)(m_4 - m_3)$$

$$\Delta_3 = \begin{vmatrix} e^{m_1 x_0} & e^{m_2 x_0} & 1 & e^{m_4 x_0} \\ m_1 e^{m_1 x_0} & m_2 e^{m_2 x_0} & r & m_4 e^{m_4 x_0} \\ m_1^2 e^{m_1 x_0} & m_2^2 e^{m_2 x_0} & r^2 & m_4^2 e^{m_4 x_0} \\ m_1^3 e^{m_1 x_0} & m_2^3 e^{m_2 x_0} & r^3 & m_4^3 e^{m_4 x_0} \end{vmatrix} = e^{(m_1 + m_2 + m_4) x_0} \begin{vmatrix} 1 & 1 & 1 & 1 \\ m_1 & m_2 & r & m_4 \\ m_1^2 & m_2^2 & r^2 & m_4^2 \\ m_1^3 & m_2^3 & r^3 & m_4^3 \end{vmatrix}$$

$$= e^{(m_1 + m_2 + m_4) x_0} (m_2 - m_1)(r - m_1)(m_4 - m_1)(r - m_2)(m_4 - m_2)(m_4 - r)$$

Hence, $\boxed{C_2 = C_1^*, C_4 = C_3^*}$, and

$$\begin{aligned} C_1 = \frac{\Delta_1}{\Delta} &= e^{-m_1 x_0} \frac{(m_2 - r)(m_3 - r)(m_4 - r)}{(m_2 - m_1)(m_3 - m_1)(m_4 - m_1)} \\ &= e^{(1-i) x_0} \frac{(r^4 + 4)}{(m_1 - r)(2i)(2)(2 - 2i)} \\ &= \boxed{e^{(1-i) x_0} \frac{(r^4 + 4)}{8(-1 + i - r)(1 + i)}} \\ C_3 = \frac{\Delta_3}{\Delta} &= e^{-m_3 x_0} \frac{(r - m_1)(r - m_2)(m_4 - r)}{(m_3 - m_1)(m_3 - m_2)(m_4 - m_3)} \\ &= e^{(-1-i) x_0} \frac{(r^4 + 4)}{(m_3 - r)(2)(2 + 2i)(-2i)} \\ &= \boxed{e^{(-1-i) x_0} \frac{(r^4 + 4)}{8(1 + i - r)(1 - i)}} \end{aligned}$$

From here it is then easy to find the real-valued solution of the original IVP, for a given x_0 and r. Below is how to do it, but if you do not manage to work it out since we do not provide explicit x_0 and r here, it is fine.

Let $C_1 = \alpha_1 + i\beta_1$, $C_2 = \alpha_1 - i\beta_1$, $C_3 = \alpha_2 + i\beta_2$, $C_4 = \alpha_2 - i\beta_2$. Then,

$$\begin{aligned}
C_1 e^{m_1 x} + C_2 e^{m_2 x} &= 2\mathrm{Re}\left\{C_1 e^{m_1 x}\right\} = 2\mathrm{Re}\left\{(\alpha_1 + i\beta_1)e^{-x}(\cos x + i\sin x)\right\} \\
&= 2e^{-x}\left(\alpha_1 \cos x - \beta_1 \sin x\right) \\
C_3 e^{m_3 x} + C_4 e^{m_4 x} &= 2\mathrm{Re}\left\{C_3 e^{m_3 x}\right\} = 2\mathrm{Re}\left\{(\alpha_2 + i\beta_2)e^{x}(\cos x + i\sin x)\right\} \\
&= 2e^{x}\left(\alpha_2 \cos x - \beta_2 \sin x\right)
\end{aligned}$$

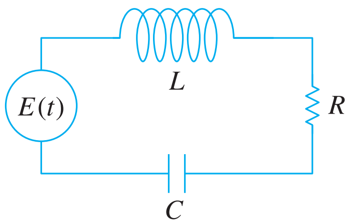
Hence,

$$y = 2e^{-x}\left(\alpha_1 \cos x - \beta_1 \sin x\right) + 2e^{x}\left(\alpha_2 \cos x - \beta_2 \sin x\right),$$

where

$$\begin{aligned}
\alpha_1 &= \mathrm{Re}\left\{e^{(1-i)x_0}\frac{(r^4+4)}{8(-1+i-r)(1+i)}\right\} = \frac{r^4+4}{8}e^{x_0}\mathrm{Re}\left\{\frac{\cos x_0 - i\sin x_0}{-(r+2)-ir}\right\} \\
&= \boxed{\frac{r^4+4}{8}e^{x_0}\frac{-(r+2)\cos x_0 + r\sin x_0}{2r^2+4r+4}} \\
\beta_1 &= \mathrm{Im}\left\{e^{(1-i)x_0}\frac{(r^4+4)}{8(-1+i-r)(1+i)}\right\} = \frac{r^4+4}{8}e^{x_0}\mathrm{Im}\left\{\frac{\cos x_0 - i\sin x_0}{-(r+2)-ir}\right\} \\
&= \boxed{\frac{r^4+4}{8}e^{x_0}\frac{r\cos x_0 + (r+2)\sin x_0}{2r^2+4r+4}} \\
\alpha_2 &= \mathrm{Re}\left\{e^{(-1-i)x_0}\frac{(r^4+4)}{8(1+i-r)(1-i)}\right\} = \frac{r^4+4}{8}e^{-x_0}\mathrm{Re}\left\{\frac{\cos x_0 - i\sin x_0}{-(r-2)-ir}\right\} \\
&= \boxed{\frac{r^4+4}{8}e^{-x_0}\frac{-(r-2)\cos x_0 - r\sin x_0}{2r^2-4r+4}} \\
\beta_2 &= \mathrm{Im}\left\{e^{(-1-i)x_0}\frac{(r^4+4)}{8(1+i-r)(1-i)}\right\} = \frac{r^4+4}{8}e^{-x_0}\mathrm{Im}\left\{\frac{\cos x_0 - i\sin x_0}{-(r-2)-ir}\right\} \\
&= \boxed{\frac{r^4+4}{8}e^{-x_0}\frac{-r\cos x_0 - (r-2)\sin x_0}{2r^2-4r+4}}
\end{aligned}$$

5. (**Method of Undetermined Coefficients**) [10]



Consider the above LRC series circuit. Recall from Chapter 1 that the voltage drop across the three elements are $L\frac{dI}{dt}$, IR, and $\frac{q}{C}$ respectively. Using the fact that $I = \frac{dq}{dt}$ and Kirchhoff's Law, we have

$$Lq'' + Rq' + q/C = E(t).$$

Suppose $L = 0.25$, $R = 1$, $C = 0.8$, $E(t) = e^{-t}\sin 10t + 2e^{-2t}\cos t$, $q(0) = q_0$, $I(0) = 0$. Find the current $I(t)$.

Solution.

The second order differential equation is shown below:

$$q'' + 4q' + 5q = 4e^{-t}\sin 10t + 8e^{-2t}\cos t.$$

First we solve the complementary solution $q_c(t)$. Since the polynomial $L := D^2 + 4D + 5$ has two complex roots $-2 \pm i$, we know that the complementary solution

$$q_c(t) = C_1 e^{(-2+i)t} + C_1^* e^{(-2-i)t} = 2\text{Re}\left\{C_1 e^{(-2+i)t}\right\}.$$

Next we find the particular solution, using the annihilator approach and the superposition principle of nonhomogeneous equations. Note that $4e^{-t}\sin 10t + 8e^{-2t}\cos t = 4g_1(t) + 8g_2(t)$, where

$$g_1(t) = e^{-t}\sin 10t = \text{Im}\left\{e^{(-1+10i)t}\right\},\ g_2(t) = e^{-2t}\cos t = \text{Re}\left\{e^{(-2+i)t}\right\}.$$

Let $q_{p,1}$ be a particular solution of $q'' + 4q' + 5q = e^{(-1+10i)t}$, and $q_{p,2}$ be a particular solution of $q'' + 4q' + 5q = e^{(-2+i)t}$, then a particular solution q_p of the original DE is

$$q_p(t) = 4\text{Im}\left\{q_{p,1}(t)\right\} + 8\text{Re}\left\{q_{p,2}(t)\right\}.$$

Solve $q_{p,1}(t)$: $q'' + 4q' + 5q = e^{(-1+10i)t}$, and hence $q_{p,1} = B_1 e^{(-1+10i)t}$. We find that

$$B_1 = \frac{1}{(-1+10i)^2 + 4(-1+10i) + 5} = \frac{1}{-99 - 20i - 4 + 40i + 5} = \frac{1}{-98 + 20i}.$$

Solve $q_{p,2}(t)$: $q'' + 4q' + 5q = e^{(-2+i)t}$, and hence $q_{p,2} = B_2 t e^{(-2+i)t}$. We find that

$$B_2 = \frac{1}{2(-2+i) + 4} = \frac{1}{2i}.$$

The general solution of the original DE can then be represented as follows:

$$q(t) = 4\text{Im}\left\{B_1 e^{(-1+10i)t}\right\} + 8\text{Re}\left\{B_2 t e^{(-2+i)t}\right\} + 2\text{Re}\left\{C_1 e^{(-2+i)t}\right\},$$

where $B_1 = \frac{1}{-98+20i}$, $B_2 = \frac{1}{2i}$, and C_1 will be determined by the initial conditions. With the initial condition we have

$$\begin{aligned} q_0 &= 4\text{Im}\{B_1\} + 2\text{Re}\{C_1\} = \frac{-20}{2501} + 2a \\ 0 &= 4\text{Im}\{B_1(-1+10i)\} + 8\text{Re}\{B_2\} + 2\text{Re}\{C_1(-2+i)\} \\ &= \frac{-960}{2501} - 2(2a+b), \end{aligned}$$

where $a = \text{Re}\{C_1\}$, $b = \text{Im}\{C_1\}$. Solve the above we get

$$a = \frac{q_0}{2} + \frac{10}{2501}, \ b = -q_0 - \frac{500}{2501}.$$

Finally,

$$\begin{aligned} I(t) &= q'(t) \\ &= 4\text{Im}\left\{B_1(-1+10i)e^{(-1+10i)t}\right\} + 8\text{Re}\left\{B_2\left((-2+i)t+1\right)e^{(-2+i)t}\right\} \\ &\quad + 2\text{Re}\left\{C_1(-2+i)e^{(-2+i)t}\right\} \\ &= 4e^{-t}\text{Im}\left\{\frac{-1+10i}{-98+20i}\left(\cos 10t + i\sin 10t\right)\right\} + 8e^{-2t}\text{Re}\left\{\frac{(-2+i)t+1}{2i}\left(\cos t + i\sin t\right)\right\} \\ &\quad + 2e^{-2t}\text{Re}\left\{(a+ib)(-2+i)\left(\cos t + i\sin t\right)\right\} \\ &= \frac{298}{2501}e^{-t}\sin 10t - \frac{960}{2501}e^{-t}\cos 10t + 4te^{-2t}\cos t - (8t-4)e^{-2t}\sin t \\ &\quad - \frac{960}{2501}e^{-2t}\cos t - \left(5q_0 + \frac{2020}{2501}\right)e^{-2t}\sin t \\ &= \boxed{\frac{298}{2501}e^{-t}\sin 10t - \frac{960}{2501}e^{-t}\cos 10t + \left(4t + \frac{960}{2501}\right)e^{-2t}\cos t - \left(8t + 5q_0 - \frac{7984}{2501}\right)e^{-2t}\sin t}. \end{aligned}$$

6. (**Variation of Parameters**) [10]
Find the general solution of the following DE:

$$y'' + y' - 2y = xe^{x^2}.$$

Solution.

First we solve the complementary solution y_c. Since the polynomial
$L := D^2 + D - 2 = (D+2)(D-1)$ has two roots $\{-2, 1\}$, we have

$$y_c(x) = c_1 e^{-2x} + c_2 e^x,$$

where $f_1(x) := e^{-2x}$ and $f_2(x) := e^x$ are two linearly independent solutions of the linear homogeneous equation.

To find a particular solution y_p, let $y_p := u_1 f_1 + u_2 f_2$. $u_1{}' = \frac{W_1}{W}$, and $u_2{}' = \frac{W_2}{W}$, where

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1{}' & f_2{}' \end{vmatrix} = e^{-x}(1 - (-2)) = 3e^{-x}$$

$$W_1 = \begin{vmatrix} 0 & f_2 \\ xe^{x^2} & f_2{}' \end{vmatrix} = -xe^{x^2+x}$$

$$W_2 = \begin{vmatrix} f_1 & 0 \\ f_1{}' & xe^{x^2} \end{vmatrix} = xe^{x^2-2x}$$

Hence,

$$\begin{aligned}
\frac{du_1}{dx} &= \frac{-x}{3} e^{x^2+2x} \\
\implies u_1 &= -\frac{1}{3}\int xe^{x^2} e^{2x} dx = -\frac{1}{6}\int e^{2x} d\left(e^{x^2}\right) = -\frac{1}{6}\left\{e^{x^2+2x} - 2\int e^{x^2+2x} dx\right\} \\
&= -\frac{1}{6}e^{x^2+2x} + \frac{1}{3e}\int e^{(x+1)^2} dx = -\frac{1}{6}e^{x^2+2x} + \frac{1}{3e}F(x+1) \\
\frac{du_2}{dx} &= \frac{x}{3} e^{x^2-x} \\
\implies u_2 &= \frac{1}{3}\int xe^{x^2} e^{-x} dx = \frac{1}{6}\int e^{-x} d\left(e^{x^2}\right) = \frac{1}{6}\left\{e^{x^2-x} + \int e^{x^2-x} dx\right\} \\
&= \frac{1}{6}e^{x^2-x} + \frac{1}{6e^{\frac{1}{4}}}\int e^{\left(x-\frac{1}{2}\right)^2} dx = \frac{1}{6}e^{x^2-x} + \frac{1}{6e^{\frac{1}{4}}}F\left(x - \frac{1}{2}\right)
\end{aligned}$$

where $F(t) := \int e^{t^2} dt$.
We obtain a particular solution

$$\begin{aligned}
y_p = u_1 f_1 + u_2 f_2 &= -\frac{1}{6}e^{x^2} + \frac{1}{3}e^{-2x-1}F(x+1) + \frac{1}{6}e^{x^2} + \frac{1}{6}e^{x-\frac{1}{4}}F\left(x - \frac{1}{2}\right) \\
&= \frac{1}{3}e^{-2x-1}F(x+1) + \frac{1}{6}e^{x-\frac{1}{4}}F\left(x - \frac{1}{2}\right),
\end{aligned}$$

and therefore

$$\boxed{y(x) = c_1 e^{-2x} + c_2 e^x + \frac{1}{3}e^{-2x-1}F(x+1) + \frac{1}{6}e^{x-\frac{1}{4}}F\left(x - \frac{1}{2}\right)}.$$

Bonus. (**Reduction of Order Two Times**) [10]

Consider a homogeneous linear third-order differential equation

$$(x^3 + 3x^2 - 3x + 1)y''' - 3(x^2 + 2x - 1)y'' + 6(x + 1)y' - 6y = 0.$$

(a) Verify that $f_1(x) = x + 1$ and $f_2(x) = x^2 + 1$ are both solutions to the above DE.

(b) Use the substitution $y = f_1(x)u_1(x)$ to convert the original DE into a second-order DE of $v_1 := u_1'$. Write down this DE, and verify that $\left(\frac{f_2(x)}{f_1(x)}\right)'$ is a solution to it.

(c) Use reduction of order to find another linearly independent solution to the derived second-order DE.

(d) From (c) derive a third solution $f_3(x)$ of the original third-order DE so that $\{f_1, f_2, f_2\}$ are linearly independent.

Solution.

(a)

$$\begin{aligned}
&{f_1}' = 1,\ {f_1}'' = 0\\
&\implies (x^3 + 3x^2 - 3x + 1){f_1}''' - 3(x^2 + 2x - 1){f_1}'' + 6(x + 1){f_1}' - 6f_1\\
&\qquad = 6(x + 1) - 6(x + 1) = 0\\
&{f_2}' = 2x,\ {f_2}'' = 2,\ {f_2}''' = 0\\
&\implies (x^3 + 3x^2 - 3x + 1){f_2}''' - 3(x^2 + 2x - 1){f_2}'' + 6(x + 1){f_2}' - 6f_1\\
&\qquad = -6(x^2 + 2x - 1) + 12x(x + 1) - 6(x^2 + 1) = 0.
\end{aligned}$$

(b) Set $y = u_1(x + 1)$, then $y' = {u_1}'(x + 1) + u_1$, $y'' = {u_1}''(x + 1) + 2{u_1}'$, $y''' = {u_1}'''(x + 1) + 3{u_1}''$. Hence, the original DE becomes

$$\left(x^4 + 4x^3 - 2x + 1\right) {u_1}''' + (6 - 12x){u_1}'' + 12{u_1}' = 0.$$

With $v_1 = {u_1}'$, the above DE is

$$\boxed{\left(x^4 + 4x^3 - 2x + 1\right) {v_1}'' + (6 - 12x){v_1}' + 12v_1 = 0}.$$

Plug in $v_1 = \left(\frac{f_2(x)}{f_1(x)}\right)' = 1 - \frac{2}{(x+1)^2}$, we get ${v_1}' = \frac{4}{(x+1)^3}$, ${v_1}'' = \frac{-12}{(x+1)^4}$, and

$$\begin{aligned}
&\left(x^4 + 4x^3 - 2x + 1\right) {v_1}'' + (6 - 12x){v_1}' + 12v_1\\
&= -12\frac{x^4 + 4x^3 - 2x + 1}{(x + 1)^4} + 24\frac{1 - 2x}{(x + 1)^3} + 12 - 24\frac{1}{(x + 1)^2} = 0.
\end{aligned}$$

(c) Set $v_1 = u_2\left(1 - \frac{2}{(x+1)^2}\right)$, then ${v_1}' = \left(1 - \frac{2}{(x+1)^2}\right){u_2}' + \frac{4}{(x+1)^3}u_2$, ${v_2}'' = \left(1 - \frac{2}{(x+1)^2}\right){u_2}'' + \frac{8}{(x+1)^3}{u_2}' - \frac{12}{(x+1)^4}u_2$, and the above DE becomes

$$(x + 1)\left(x^2 + 2x - 1\right)\left(x^4 + 4x^3 - 2x + 1\right){u_2}'' - 2\left(2x^4 - x^3 - 3x^2 - x - 1\right){u_2}' = 0.$$

With $v_2 = {u_2}'$, the above DE becomes

$$(x+1)\left(x^2+2x-1\right)\left(x^4+4x^3-2x+1\right){v_2}' - 2\left(2x^4-x^3-3x^2-x-1\right)v_2 = 0.$$

Hence

$$v_2 = \exp\left\{\int \frac{2\left(2x^4-x^3-3x^2-x-1\right)}{(x+1)\left(x^2+2x-1\right)\left(x^4+4x^3-2x+1\right)}dx\right\},$$

and a second solution is

$$\left(1-\frac{2}{(x+1)^2}\right)\int \exp\left\{\int \frac{2\left(2x^4-x^3-3x^2-x-1\right)}{(x+1)\left(x^2+2x-1\right)\left(x^4+4x^3-2x+1\right)}dx\right\}dx.$$

(d) To find f_3, we multiply the integral of the solution found above with f_1 and get

$$(x+1)\int\left\{\left(1-\frac{2}{(x+1)^2}\right)\int \exp\left\{\int \frac{2\left(2x^4-x^3-3x^2-x-1\right)}{(x+1)\left(x^2+2x-1\right)\left(x^4+4x^3-2x+1\right)}dx\right\}dx\right\}dx.$$