Solution to Homework 1

1. (**Practice of Different Methods**) [15]
Solve the following initial-value problems (y: dependent variable)

(a) $\dfrac{dy}{dx} = \dfrac{1}{x^4-1}$, $y(0) = 1$. [5]

(b) $\dfrac{dy}{dx} = \dfrac{x^3}{(2y+1)}$, $y(2) = 1$. [5]

(c) $(x^2-1)\dfrac{dy}{dx} = xy + 1$, $y(0) = 1$. [5]

Solution.

(a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{2}\left\{\frac{1}{x^2-1} - \frac{1}{x^2+1}\right\} \\ &= \frac{\frac{1}{4}}{x-1} - \frac{\frac{1}{4}}{x+1} - \frac{\frac{1}{2}}{x^2+1}\end{aligned}$$

Hence,

$$y = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1} x + c.$$

Plug in the initial condition $x = 0, y = 1$, we get $c = 1$.

$$\implies y = \boxed{\frac{1}{4}\ln(1-x) - \frac{1}{4}\ln(x+1) - \frac{1}{2}\tan^{-1} x + 1}.$$

Interval of definition: $\boxed{x \in (-1, 1)}$.

(b)

$$\frac{dy}{dx} = \frac{x^3}{(2y+1)} \implies (2y+1)dy = x^3 dx \implies y^2 + y = \frac{1}{4}x^4 + c$$

Plug in the initial condition $x = 2, y = 1$, we get $c = -2$.

$$\implies y^2 + y = \frac{1}{4}x^4 - 2 \implies \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}\left(x^4 - 7\right) \implies y = \frac{1}{2}\left(-1 \pm \sqrt{x^4-7}\right)$$

Plug in the initial condition $x = 2, y = 1$, we know that we have to choose

$$\boxed{y = \frac{1}{2}\left(-1 + \sqrt{x^4 - 7}\right)}.$$

Interval of definition: $\boxed{x \in \left(7^{\frac{1}{4}}, \infty\right)}$.

(c)

$$(x^2 - 1)\frac{dy}{dx} = xy + 1 \implies \frac{dy}{dx} = \frac{xy+1}{x^2-1} = \frac{x}{x^2-1}y + \frac{1}{x^2-1},\ x \neq \pm 1.$$

We shall introduce an integrating factor $\mu(x)$ to solve this linear equation, which has to satisfy the following auxiliary DE:

$$\frac{d\mu}{dx} = -\frac{x}{x^2-1}\mu \implies \frac{d\mu}{\mu} = -\frac{x}{x^2-1}dx = -\frac{1}{2}\frac{d(x^2)}{x^2-1} \implies \ln|\mu| = -\frac{1}{2}\ln|x^2-1|.$$

Based on the initial condition $x = 0$, we choose the domain of x to be $x \in (-1, 1)$ and hence we get an integrating factor

$$\mu = \frac{1}{\sqrt{1-x^2}}.$$

Finally, plug in the integrating factor and we get

$$\frac{d(\mu y)}{dx} = \mu\left(\frac{x}{x^2-1}y + \frac{1}{x^2-1}\right) + y\left(-\frac{x}{x^2-1}\mu\right) = \frac{\mu}{x^2-1} = -\frac{1}{(\sqrt{1-x^2})^3}.$$

To solve μy, we need to compute the following integral:

$$\int -\frac{1}{(\sqrt{1-x^2})^3}dx \overset{x=\sin\theta}{=} \int \frac{-\cos\theta}{\cos^3\theta}d\theta = \int -\sec^2\theta d\theta = -\tan\theta + c$$
$$= -\frac{x}{\sqrt{1-x^2}} + c.$$

Hence,

$$\mu y = \frac{1}{\sqrt{1-x^2}}y = -\frac{x}{\sqrt{1-x^2}} + c.$$

Plug in the initial condition $x = 0, y = 1$ we get $c = 1$.

$$\implies \boxed{y = -x + \sqrt{1-x^2}}.$$

Singular points $x = \pm 1$ cannot be added back to the interval of definition, because

$$\frac{dy}{dx} = -1 - \frac{x}{\sqrt{1-x^2}}$$

is not defined at the singular points. Interval of definition: $\boxed{x \in (-1, 1)}$.

2. (**Discontinuous Coefficients**) [10]

Solve

$$\frac{dy}{dx} + P(x)y = x$$

subject to $y(0) = 0$, where $P(x) = \begin{cases} 1, & x \geq 0 \\ -1 & x < 0 \end{cases}$.

Solution.

We first solve for $x \geq 0$:

$$\frac{dy}{dx} + y = x \implies \frac{dy}{dx} = x - y.$$

Use the following substitution: $u := x - y \implies y = x - u$. We then get

$$\frac{dy}{dx} = 1 - \frac{du}{dx} = u \implies \frac{du}{dx} = 1 - u \implies \frac{du}{1-u} = dx,\ u \neq 1 \implies -\ln|1-u| = x + c,\ u \neq 1.$$

Plug in the initial condition $x = 0, y = 0 \implies u = 0$, we get $c = 0$. Hence,

$$1 - u = 1 - x + y = e^{-x} \implies y = x + e^{-x} - 1,\ x \geq 0.$$

For $x < 0$:

$$\frac{dy}{dx} - y = x \implies \frac{dy}{dx} = x + y.$$

Use the following substitution: $v := x + y \implies y = v - x$. We then get

$$\frac{dy}{dx} = \frac{dv}{dx} - 1 = v \implies \frac{dv}{dx} = 1 + v \implies \frac{dv}{1+v} = dx,\ v \neq -1 \implies \ln|1+v| = x + c,\ v \neq -1.$$

Plug in the boundary condition $x \uparrow 0, y \to 0 \implies v \to 0$, we get $c = 0$. Hence,

$$1 + v = 1 + x + y = e^{x} \implies y = -x + e^{x} - 1,\ x < 0.$$

Therefore, the final answer

$$\boxed{y = \begin{cases} x + e^{-x} - 1, & x \geq 0 \\ -x + e^{x} - 1, & x < 0 \end{cases} = |x| + e^{-|x|} - 1}.$$

Interval of definition: $\boxed{x \in \mathbb{R}}$.

3. (**Nonlinear ODE Made Linear**) [10]

Solve

$$\frac{dy}{dx} = 1 + xe^{-y}$$

subject to $y(0) = 0$.

Solution.

First we manipulate the original DE as follows:

$$\frac{dy}{dx} = 1 + xe^{-y} \implies e^y \frac{dy}{dx} = e^y + x \implies \frac{d}{dx} e^y = e^y + x,$$

since $d(e^y) = e^y dy$. Hence, we can use the result in Problem 2 (the solution to $\frac{dy}{dx} = x + y$ is $y = -x + ce^x - 1$) to get

$$e^y = -x + ce^x - 1.$$

Plug in the initial condition $x = 0, y = 0$, we get $c = 2$. Hence,

$$e^y = -x + 2e^x - 1 \implies \boxed{y = \ln(-x + 2e^x - 1)}.$$

Note that the function $2e^x - x - 1$ is minimized at $x = -\ln 2$, that is, $2e^x - 1 = 0$, by studying its derivative. Hence $2e^x - x - 1 \geq \ln 2 > 0$, and the interval of definition of the solution is $\boxed{x \in \mathbb{R}}$.

4. (**Singular Points, Interval of Definition, and Initial Conditions**) [11]
skip (1) Solve

$$x(x-1)\frac{dy}{dx} = x + y$$

subject to

(a) $y(2) = 1$ [2]

(b) $y(-1) = 1$ [2]

(c) $y(1/2) = 1$ [2]

(2) Identify the singular points that cannot be included into the interval of definition. [5]

Solution.

(1) First we manipulate the equation as follows:

$$x(x-1)\frac{dy}{dx} = x + y \implies \frac{dy}{dx} = \frac{x+y}{x(x-1)} = \frac{y}{x(x-1)} + \frac{1}{x-1},\ x \neq 0, 1.$$

We find an integrating factor $\mu(x)$ by solving the following auxiliary DE:

$$\frac{d\mu}{dx} = -\frac{\mu}{x(x-1)} \implies \frac{d\mu}{\mu} = -\frac{dx}{x(x-1)} = \left(\frac{1}{x} - \frac{1}{x-1}\right)dx \implies \ln|\mu| = \ln|x| - \ln|x-1|.$$

Depending on the initial point of x, we shall choose different interval of definition and consequently influence how we remove the absolute values on the right hand side of the last equality.

(a) $y(2) = 1$:

The interval of definition shall lie inside $(1, \infty)$. Hence the signs of x and $x - 1$ are the same, and we get an integrating factor

$$\mu(x) = \frac{x}{x-1}.$$

Plug it back, we get

$$\frac{d(\mu y)}{dx} = \mu\frac{1}{x-1} = \frac{x}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2} \implies \mu y = \ln|x-1| - \frac{1}{x-1} + c.$$

Plug in the initial condition, we get $c = 3$, and hence

$$y = \frac{x-1}{x}\left\{\ln|x-1| - \frac{1}{x-1} + 3\right\} = \boxed{\frac{x-1}{x}\ln(x-1) + \frac{3x-4}{x}}$$

The singular point $x = 1$ cannot be added back to the interval of definition which is $\boxed{x \in (1, \infty)}$.

(b) $y(-1) = 1$:

The interval of definition shall lie inside $(-\infty, 0)$. Hence the signs of x and $x - 1$ are the same, and we get an integrating factor

$$\mu(x) = \frac{x}{x-1}.$$

Plug it back, we get

$$\frac{d(\mu y)}{dx} = \mu \frac{1}{x-1} = \frac{x}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2} \implies \mu y = \ln|x-1| - \frac{1}{x-1} + c.$$

Plug in the initial condition, we get $c = -\ln 2$, and hence

$$y = \frac{x-1}{x}\left\{\ln|x-1| - \frac{1}{x-1} - \ln 2\right\} = \boxed{\frac{x-1}{x}\ln\left(\frac{1-x}{2}\right) - \frac{1}{x}}$$

The singular point $x = 0$ cannot be added back to the interval of definition which is $\boxed{x \in (-\infty, 0)}$.

(c) $y(1/2) = 1$:

The interval of definition shall lie inside $(0, 1)$. Hence the signs of x and $x - 1$ are different, and we get an integrating factor

$$\mu(x) = \frac{x}{1-x}.$$

Plug it back, we get

$$\frac{d(\mu y)}{dx} = \mu \frac{1}{x-1} = \frac{-x}{(1-x)^2} = \frac{1}{1-x} - \frac{1}{(1-x)^2} \implies \mu y = \ln|1-x| - \frac{1}{1-x} + c.$$

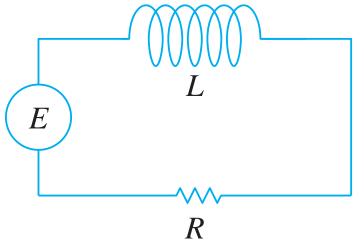
Plug in the initial condition, we get $c = 3 + \ln 2$, and hence

$$y = \frac{1-x}{x}\left\{\ln|1-x| - \frac{1}{1-x} + 3 + \ln 2\right\} = \boxed{\frac{1-x}{x}\ln(2(1-x)) + \frac{2-3x}{x}}$$

The singular points $x = 0$ and $x = 1$ cannot be added back to the interval of definition which is $\boxed{x \in (0, 1)}$.

(2) As seen in the above discussion, in all three cases, none of the singular points can be added back to the interval of definition.

5. (**LR Circuit with AC Power**) [10]



Consider the above LR circuit, where $E(t) = 10\sin(t)$ volts, $R = 10$ ohms, $L = 0.5$ henry, and initial current $i(0) = 0$.
Find $i(t)$.

Solution.

Current $i(t)$ satisfies the following DE:

$$L\frac{di}{dt} + Ri = E(t) \implies \frac{1}{2}\frac{di}{dt} + 10i = 10\sin t \implies \frac{di}{dt} + 20i = 20\sin t$$

Let $\mu(t)$ be the integrating factor to be found, which satisfies

$$\frac{d\mu}{dt} = 20\mu.$$

We can find one integrating factor $\mu(t) = e^{20t}$. Plug in $\mu(t) = e^{20t}$ and $i(0) = 0$, we get

$$\mu i = \int 20e^{20t}\sin t\, dt = \left(\frac{400}{401}\sin t - \frac{20}{401}\cos t\right)e^{20t} + c.$$

Note that in the above we use integration by parts to derive the indefinite integral.
Plug in the initial condition $t = 0, i = 0, \mu = 1$, we get $c = \frac{20}{401}$. Hence the current is

$$\boxed{i(t) = \frac{400}{401}\sin t - \frac{20}{401}\cos t + \frac{20}{401}e^{-20t}}.$$

Interval of definition: $\boxed{t \in [0, \infty)}$ or $\boxed{t \in \mathbb{R}}$.

6. (**Gompertz Differential Equation**) [15]

English Mathematician B. Gompertz (1779 – 1865) proposed the following equation to model population dynamics:

$$\frac{dP}{dt} = P(a - b\ln P).$$

Suppose the initial population $P(0) = P_0$.

(a) Find $P(t)$. [5]

(b) Find the capacity of population, that is, $P(\infty)$. [5]

(c) Find the threshold of population beyond which its growth rate decreases as population grows. [5]

Solution.

(a) The DE is separable and can be solved as follows:

$$\frac{dP}{dt} = P(a - b\ln P) \implies dt = \frac{dP}{P(a-b\ln P)} = \frac{d(\ln P)}{a - b\ln P}$$
$$\implies t = -\frac{1}{b}\ln|a - b\ln P| + c.$$

Plug in the initial condition $P(0) = P_0$, we get $c = \frac{1}{b}\ln|a - b\ln P_0|$. Hence

$$t = -\frac{1}{b}\ln|a - b\ln P| + \frac{1}{b}\ln|a - b\ln P_0| = \frac{1}{b}\ln\frac{a - b\ln P_0}{a - b\ln P}$$
$$\implies \ln P(t) = \frac{a}{b}\left(1 - e^{-bt}\right) + e^{-bt}\ln P_0$$
$$\implies \boxed{P(t) = \exp\left(\frac{a}{b}\left(1 - e^{-bt}\right)\right) P_0^{e^{-bt}}}.$$

Interval of definition: $\boxed{t \in [0,\infty)}$ or $\boxed{t \in \mathbb{R}}$.

(b)

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \exp\left(\frac{a}{b}\left(1 - e^{-bt}\right)\right) P_0^{e^{-bt}} = \boxed{e^{\frac{a}{b}}}.$$

(c) With the assumption that $P_0 < e^{\frac{a}{b}}$, we can easily see that $\ln P(t)$ is an increasing function, and hence so is $P(t)$, which implies that $P(t) < P(\infty) = e^{\frac{a}{b}}$.
Therefore,

$$\frac{dP}{dt} = P(a - b\ln P) > 0,\ \forall\, t.$$

To find the saddle point, we focus on the second derivative of P:

$$\frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{dP}{dt}\frac{d}{dP}P(a - b\ln P) = P(a - b\ln P)\left(a - b\ln P - \frac{bP}{P}\right)$$

Note that the saddle point $P^* \in (0, e^{\frac{a}{b}})$ will make $\frac{d^2P}{dt^2} = 0$. Hence,

$$0 = a - b\ln P^* - \frac{bP^*}{P^*} = a - b - b\ln P^* \implies \boxed{P^* = e^{\frac{a-b}{b}}}.$$