Homework 4

Due: 1/3, 18:00

1. (**Laplace Transform of Periodic Extension**) [10]
A function $f(t) = e^{at}$ for $0 \le t < T$, and $f(t) = f(t-T)$ for $t \ge T$.
Evaluate $\mathscr{L}\{f(t)\}$.

2. (**Inverse Laplace Transform**) [15]
Evaluate
$$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\tanh\left(\frac{s}{2}\right)\right\}.$$
Hint 1: $\tanh\left(\frac{s}{2}\right) = \frac{1-e^{-s}}{1+e^{-s}} = \frac{\left(1-e^{-s}\right)^2}{1-e^{-2s}}$.
Hint 2: $\mathscr{L}\{(f*f)(t)\} = \{F(s)\}^2$.

3. (**Fourier Series Expansion**) [15]
Expand $f(x) = xe^{-x}$, $0 < x < \pi$,
(a) in a Fourier cosine series. [5]
(b) in a Fourier sine series. [5]
(c) in a Fourier series. [5]

4. (**Wave Equation**) [10]
Solve the following boundary-value problem:
$$\begin{array}{rl} \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad 0 < x < \frac{\pi}{2}, \quad t > 0 \\ \text{subject to}: & u(0,t) = 0, \quad u_x\left(\frac{\pi}{2}, t\right) = 0, \quad t > 0 \\ & u(x,0) = 1, \quad u_t(x,0) = 0, \quad 0 < x < \frac{\pi}{2} \end{array}$$
Hint. See Example 2 on Page 434 of the textbook. There is a reason why we pick $u(x,0) = 1$.

5. (**Laplace's Equation**) [10]
Solve the following boundary-value problem:
$$\begin{array}{rl} \text{Solve } u(x,y): & u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b \\ \text{subject to}: & u_x(0,y) = u(0,y), \quad u_x(a,y) = G(y), \quad 0 < y < b \\ & u(x,0) = 0, \quad u(x,b) = 0, \quad 0 < x < a \end{array}$$

6. (**Wave Equation on an Infinite String**) [10]

Solve the following boundary-value problem:

$$\boxed{\begin{array}{rl} \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad -\infty < x < \infty, \quad t > 0 \\ \text{subject to}: & u(\pm\infty, t) = u_x(\pm\infty, t) = 0, \quad t > 0 \\ & u(x,0) = e^{-|x|}, \quad u_t(x,0) = 0, \quad -\infty < x < \infty \end{array}}$$

7. (**Laplace's Equation on a Semi-Infinite Plate**) [15]

Solve the following boundary-value problem:

$$\boxed{\begin{array}{rl} \text{Solve } u(x,y): & u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad y > 0 \\ \text{subject to}: & u(0,y) = ye^{-y}, \quad u(\pi, y) = 0, \quad y > 0 \\ & u(x,0) = xe^{-x}, \quad u(x,\infty) = u_y(x,\infty) = 0, \quad 0 < x < \pi \end{array}}$$